Okounkov's conjecture via BPS Lie algebras

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Quivers and algebras

Path algebra

A *quiver* is a pair of (finite) sets Q_1, Q_0 of arrows and vertices, respectively, and morphisms $s, t\colon Q_1 \to Q_0$. For K a field, the path algebra KQ is the K-algebra with basis given by paths in Q including paths $\boldsymbol{e}_i$ of length zero at each $i \in Q_0$. Multiplication is given by concatenation of paths.

- The dimension vector of a KQ-module ρ is the tuple $\dim_Q(\rho) := (\boldsymbol{e}_i \cdot \rho)_{i \in Q_0} \in \mathbb{N}^{Q_0}$.
- The *doubled* quiver $\overline{Q}$ is obtained by adding an arrow a^* for each $a \in Q_1$ and setting $s(a^*) = t(a)$ and $t(a^*) = s(a)$.
- The *preprojective algebra* Π_Q is the quotient $\mathbb{C}\overline{Q}/\langle\sum_{a \in Q_1}[a, a^*]\rangle$

Example

Consider the Jordan quiver: 

. Then $\mathbb{C}\overline{Q} = \mathbb{C}\langle a, a^*\rangle$ and $\Pi_Q = \mathbb{C}\langle a, a^*\rangle/\langle [a, a^*]\rangle \cong \mathbb{C}[a, a^*]$.

Spaces of representations

- Fix a quiver Q and dimension vector $\mathrm{d} \in \mathbb{N}^{Q_0}$. Set
$$\mathrm{Rep}_{\mathrm{d}}(Q) := \prod_{a \in Q_1} \mathrm{Hom}(\mathbb{C}^{\mathrm{d}_{s(a)}}, \mathbb{C}^{\mathrm{d}_{t(a)}})$$
which is acted on by $\mathrm{GL}_{\mathrm{d}} := \prod_{i \in Q_0} \mathrm{GL}_{\mathrm{d}_i}(\mathbb{C})$. Then $\{\mathrm{GL}_{\mathrm{d}}\text{-orbits}\} \overset{1:1}{\Leftrightarrow} \{\mathrm{d}\text{-dimensional } \mathbb{C}Q\text{-modules}\}/\sim^{\mathrm{iso}}$
- $\mathrm{Rep}_{\mathrm{d}}(\overline{Q}) \cong \mathrm{T}^* \mathrm{Rep}_{\mathrm{d}}(Q)$ admits the (co)moment map $\mu_{Q,\mathrm{d}}$ to $\mathfrak{gl}_{\mathrm{d}} := \prod_{i \in Q_0} \mathrm{Mat}_{\mathrm{d}_i \times \mathrm{d}_i}(\mathbb{C})$:
$$\mathrm{Rep}_{\mathrm{d}}(\overline{Q}) \ni \rho \mapsto \sum_{a \in Q_1} [\rho(a), \rho(a^*)] \in \mathfrak{gl}_{\mathrm{d}}$$
- $\mathrm{Rep}_{\mathrm{d}}(\Pi_Q) := \mu_{Q,\mathrm{d}}^{-1}(0) \subset \mathrm{Rep}_{\mathrm{d}}(\overline{Q})$ is the subspace of Π_Q-modules.
- We *could* consider the *stack* $\mathfrak{M}_{\mathrm{d}}(\Pi_Q) = \mu_{Q,\mathrm{d}}^{-1}(0)/\mathrm{GL}_{\mathrm{d}}$. It is highly singular, and highly stacky. E.g. for Q the Jordan quiver $\mathrm{Rep}_d(\Pi_Q)$ is the stack of pairs of commuting $d \times d$-matrices, which at $(0_{d\times d}, 0_{d \times d})$, is very singular, with stabilizer GL_d.

Nakajima quiver varieties

Framed quiver

Let $\mathsf{f} \in \mathbb{N}^{Q_0}$ be a "framing" dimension vector. We define Q_{f} by setting

$$(Q_{\mathsf{f}})_0 \coloneqq Q_0 \coprod \{\infty\}; \qquad (Q_{\mathsf{f}})_1 \coloneqq Q_1 \coprod \{r_{i,m} | i \in Q_0, 1 \leq m \leq \mathsf{f}_i\}.$$

$s(r_{i,m}) = \infty$ and $t(r_{i,m}) = i$.

Definition

We define $\mathrm{N}_Q(\mathsf{f}, \mathsf{d}) \coloneqq \mu^{-1}_{Q_{\mathsf{f}},(\mathsf{d},1)}(0)^{\mathrm{st}} / \mathrm{GL}_{\mathsf{d}}$. The "st" means we only consider the stable locus: those $\Pi_{Q_{\mathsf{f}}}$ modules ρ that are generated by $\boldsymbol{e}_\infty \cdot \rho \cong \mathbb{C}$.

Key fact(s): $\mathrm{N}_Q(\mathsf{f}, \mathsf{d})$ is a *smooth variety*.

Example

Let Q be the Jordan quiver, set $\mathsf{f} = 1$. Then $\overline{Q_{\mathsf{f}}}$ is the ADHM quiver, and $\mathrm{N}_Q(\mathsf{f}, \mathsf{d}) \cong \mathrm{Hilb}_{\mathsf{d}}(\mathbb{A}^2)$.

Some geometric representation theory

Fix a quiver Q and $\mathsf{f} \in \mathbb{N}^{Q_0}$. Set $\mathbb{M}_{Q,\mathsf{f}} := \bigoplus_{\mathsf{d}\in\mathbb{N}^{Q_0}} \mathrm{H}(\mathrm{N}_Q(\mathsf{f},\mathsf{d}),\mathbb{Q})$

Theorem (Grojnowski, Nakajima)

Let $Q = \bullet\circlearrowleft$ be the Jordan quiver, and set $\mathsf{f} = 1$. We've seen that $\mathrm{N}_Q(1,\mathsf{d}) \cong \mathrm{Hilb}_{\mathsf{d}}(\mathbb{A}^2)$. The $\mathbb{N} = \mathbb{N}^{Q_0}$-graded vector space $\mathbb{M}_{Q,1}$ carries an action of an infinite-dimensional Heisenberg algebra heis_∞, and is an irreducible lowest weight module.

Given Q a quiver *without loops* we may define the Kac–Moody Lie algebra $\mathfrak{g}_Q \cong \mathfrak{n}_Q^- \oplus \mathfrak{h}_Q \oplus \mathfrak{n}_Q^+$. The positive part $\mathfrak{n}_Q^+$ is free Lie algebra generated by one (Chevalley) generator for each $i \in Q_0$, subject to the Serre relations.

Theorem (Nakajima)

Let Q be a quiver without loops. Then $\mathbb{M}_{Q,\mathsf{f}}$ is a $\mathfrak{g}_Q$-module, and $\mathbb{M}^{\text{lowest}}_{Q,\mathsf{f}} := \bigoplus_{\mathsf{d}\in\mathbb{N}^{Q_0}} \mathrm{H}^{\text{lowest}}(\mathrm{N}_Q(\mathsf{f},\mathsf{d}),\mathbb{Q})$ is an irreducible lowest weight module, with lowest weight dependent on f.

Deformations

- Let $\mathrm{N}^0_Q(\mathrm{f},\mathrm{d})$ be the affinization of $\mathrm{N}_Q(\mathrm{f},\mathrm{d})$. E.g. if Q is the Jordan quiver and $\mathrm{f}=1$ then $\mathrm{N}^0_Q(\mathrm{f},\mathrm{d})=\mathrm{Sym}^d(\mathbb{A}^2)$. In general the morphism $\pi\colon \mathrm{N}_Q(\mathrm{f},\mathrm{d})\to\mathrm{N}^0_Q(\mathrm{f},\mathrm{d})$ is a resolution of singularities.
- This morphism admits a "universal deformation":

$$
\begin{array}{ccc}
\mathrm{N}_Q(\mathrm{f},\mathrm{d}) & \hookrightarrow & \tilde{\mathrm{N}}_Q(\mathrm{f},\mathrm{d}) \\
\downarrow{\scriptstyle\pi} & \lrcorner & \downarrow{\scriptstyle\tilde{\pi}} \\
\mathrm{N}^0_Q(\mathrm{f},\mathrm{d}) & \hookrightarrow & \tilde{\mathrm{N}}^0_Q(\mathrm{f},\mathrm{d}) \\
\downarrow & \lrcorner & \downarrow \\
\{0\} & \hookrightarrow & \mathbb{A}^{Q_0}
\end{array}
$$

- For generic $x\in\mathbb{A}^{Q_0}$ the morphism $\tilde{\pi}_x\colon \tilde{\mathrm{N}}_Q(\mathrm{f},\mathrm{d})_x\to\tilde{\mathrm{N}}^0_Q(\mathrm{f},\mathrm{d})_x$ obtained by base change along $\{x\}\hookrightarrow\mathbb{A}^{Q_0}$ is an isomorphism of affine varieties.

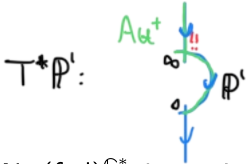
Torus action

Let $\mathsf{f} = \mathsf{f}' + \mathsf{f}''$ be a decomposition of $\mathsf{f} \in \mathbb{N}^{Q_0}$, with $\mathsf{f}', \mathsf{f}'' \in \mathbb{N}^{Q_0} \setminus \{0\}$. We let $\mathbb{C}^*$ act on quiver varieties by scaling $r_{i,1}, \ldots, r_{i,f'_i}$ with weight ± 1, $r^*_{i,1}, \ldots, r^*_{i,f'_i}$ with weight ∓ 1, and leaving all other arrows invariant.

Proposition

There is an identification $\mathrm{N}_Q(\mathsf{f}, \mathsf{d})^{\mathbb{C}^*} = \coprod_{\mathsf{d}'+\mathsf{d}''=\mathsf{d}} \mathrm{N}_Q(\mathsf{f}', \mathsf{d}') \times \mathrm{N}_Q(\mathsf{f}'', \mathsf{d}'')$ given by taking direct sums.

- Define $\mathrm{Att}^{\pm} \subset \mathrm{N}_Q(\mathsf{f}, \mathsf{d})$ to be the subset of ρ for which $\lim_{t \mapsto 0} t \cdot \rho$ exists.
- NB: the morphism $\lim_{t \mapsto 0}(t \cdot -) \colon \mathrm{Att}^{\pm} \to \mathrm{N}_Q(\mathsf{f}, \mathsf{d})^{\mathbb{C}^*}$ might not be continuous!!
- But the morphism $\lim_{t \mapsto 0}(t \cdot -) \colon \mathrm{Att}^{\pm}_x \to \mathrm{N}_Q(\mathsf{f}, \mathsf{d})^{\mathbb{C}^*}_x$ *is* continuous, for generic $x \in \mathbb{A}^{Q_0}$.



Stable envelopes

- For generic x we consider the closed embedding(s)

$$\begin{aligned}\mathrm{Att}_x^{\pm} &\to \tilde{\mathrm{N}}_Q(\mathrm{f},\mathrm{d})_x^{\mathbb{C}^*} \times \tilde{\mathrm{N}}_Q(\mathrm{f},\mathrm{d})_x \\ \rho &\mapsto (\lim_{t\mapsto 0} t\cdot\rho, \rho)\end{aligned}$$

- Maulik and Okounkov define $\mathcal{L}^{\pm} = \lim_{x\mapsto 0}[\mathrm{Att}_x^{\pm}] \in \mathrm{H}_{\mathbb{C}^*\times T}(\mathrm{N}_Q(\mathrm{f},\mathrm{d})^{\mathbb{C}^*} \times \mathrm{N}_Q(\mathrm{f},\mathrm{d}), \mathbb{Q})$ (T is a choice of extra torus acting by scaling arrows of $\overline{Q_{\mathrm{f}}}$)
- The two morphisms defined by these correspondences

$$\mathrm{Stab}^{\pm}\colon\ \mathrm{H}_{\mathbb{C}^*\times T}(\mathrm{N}_Q(\mathrm{f},\mathrm{d})^{\mathbb{C}^*}, \mathbb{Q}) \to \mathrm{H}_{\mathbb{C}^*\times T}(\mathrm{N}_Q(\mathrm{f},\mathrm{d}), \mathbb{Q})$$

  become invertible after tensoring with $\mathrm{Frac}(\mathrm{H}_{\mathbb{C}^*}) = \mathbb{Q}(a)$, where $\mathrm{H}_{\mathbb{C}^*} = \mathbb{Q}[a]$ is the $\mathbb{C}^*$-equivariant cohomology of a point.

R-matrices

Definition

For $\mathsf{f}', \mathsf{f}'' \in \mathbb{N}^{Q_0}$ the *R-matrix* is defined as

$$R(a) = (\mathrm{Stab}^-)^{-1} \circ \mathrm{Stab}^+ \in \mathrm{End}_{\mathrm{H}_T}(\mathbb{M}_{Q,\mathsf{f}'} \otimes_{\mathrm{H}_T} \mathbb{M}_{Q,\mathsf{f}''}) \otimes \mathbb{Q}(a)$$

Basic properties

- Expanding in powers of a^{-1}

$$R(a) = \mathrm{id} + \hbar a^{-1} \mathsf{r} + \hbar O(a^{-2})$$

  where $\hbar$ is the T-weight of the symplectic form on $\mathrm{N}_Q(\mathsf{f}, \mathsf{d})$ and $\mathsf{r} \in \mathrm{End}_{\mathrm{H}_T}(\mathbb{M}_{Q,\mathsf{f}'} \otimes_{\mathrm{H}_T} \mathbb{M}_{Q,\mathsf{f}''})$ is the "classical r-matrix". So to get an interesting R-matrix we *must* pick nontrivial T.
- The R-matrix satisfies the Yang-Baxter equation $R_{12}(a_1)R_{13}(a_1 + a_2)R_{23}(a_2) = R_{23}(a_2)R_{13}(a_1 + a_2)R_{12}(a_1)$, the fundamental relation in integrable systems, responsible for producing e.g. knot invariants out of quantum groups.

Yangians

- Given $g \in \mathrm{End}_{\mathrm{H}_T}(\mathbb{M}_{Q,\mathrm{f}'})[a]$, written as $\langle g_1|\otimes|g_2\rangle$ with $g_2 \in \mathbb{M}_{Q,\mathrm{f}'}[a]$ and g_1 in the dual $\mathbb{M}^{\vee}_{Q,\mathrm{f}'}$, we define

$$\boldsymbol{E}_{\mathrm{f}''}(g) = \mathrm{Res}_a((\langle g_1|\otimes -) \circ R \circ (|g_2\rangle \otimes -)) \in \mathrm{End}(\mathbb{M}_{Q,\mathrm{f}''})$$

- MO define $\boldsymbol{Y}_Q \subset \bigoplus_{\mathrm{f}'' \in \mathbb{N}^{Q_0}} \mathrm{End}(\mathbb{M}_{Q,\mathrm{f}''})$ to be the subalgebra generated by all $\boldsymbol{E}(g) := \bigoplus_{\mathrm{f}'' \in \mathbb{N}^{Q_0}} \boldsymbol{E}_{\mathrm{f}''}(g)$.
- Similarly, they define $\mathfrak{g}_Q^{\mathtt{MO}} \subset \bigoplus_{\mathrm{f}'' \in \mathbb{N}^{Q_0}} \mathrm{End}(\mathbb{M}_{Q,\mathrm{f}''})$ to be vector space generated by $\boldsymbol{E}(g)$ with g constant in a.

Theorem (Maulik–Okounkov)

- *The $\mathbb{Z}^{Q_0}$-graded H_T-module $\mathfrak{g}_Q^{\mathtt{MO}}$ is closed under commutator.*
- *Each summand $\mathfrak{g}_{Q,\mathrm{d}}^{\mathtt{MO}}$ is free of finite rank.*
- *The morphism $\mathrm{Sym}(\mathfrak{g}_Q^{\mathtt{MO}} \otimes \mathbb{Q}[a]) \to \boldsymbol{Y}_Q$ is an isomorphism.*

A representation theoretic hint

- The morphism

$$\pi\colon \mathsf{N}_Q(\mathsf{f},\mathsf{d}) \to \mathsf{N}^0_Q(\mathsf{f},\mathsf{d})$$

  is a projective morphism from a smooth variety. So the BBDG decomposition theorem applies, and we can write

$$\pi_*\mathbb{Q}_{\mathsf{N}_Q(\mathsf{f},\mathsf{d})}[d] = \mathrm{IC}_{\mathsf{N}^0_Q(\mathsf{f},\mathsf{d})} \oplus \dots$$

  as a direct sum of perverse sheaves ($d = \dim(\mathsf{N}_Q(\mathsf{f},\mathsf{d}))$. In particular. $\mathrm{IH}^*(\mathsf{N}^0_Q(\mathsf{f},\mathsf{d})) \subset \mathbb{M}_{Q,\mathsf{f}}$
- Lowering operators in $\mathfrak{g}_Q^{\mathtt{MO}}$ lift to morphisms of perverse sheaves $\pi_*\mathbb{Q}_{\mathsf{N}_Q(\mathsf{f},\mathsf{d})}[d] \to \pi_*^{\mathbb{C}^*}\mathbb{Q}_{\mathsf{N}_Q^{\mathbb{C}^*}(\mathsf{f},\mathsf{d})}[d^{\mathbb{C}^*}]$.
- So $\mathrm{IH}^*(\mathsf{N}^0_Q(\mathsf{f},\mathsf{d})) \subset \mathbb{M}_{Q,\mathsf{f}}$ is a space of lowest weight vectors for support reasons...

Okounkov's conjecture

Definition-Theorem (Kac)

For any quiver Q and dimension vector $\mathrm{d} \in \mathbb{N}^{Q_0}$ there is a polynomial $\mathrm{a}_{Q,\mathrm{d}}(t) \in \mathbb{Z}[t]$ (the *Kac polynomial*) such that if $q = p^n$ is a prime power,

$$\mathrm{a}_{Q,\mathrm{d}}(q) = \#\left\{ \substack{\text{absolutely indecomposable} \\ \mathrm{d}\text{-dimensional } \mathbb{F}_q Q\text{-modules}} \right\} / \sim^{\text{iso}}$$

Conjecture (Maulik–Okounkov)

$\exists$ *isomorphism of Lie algebras* $\mathfrak{g}_Q^{\mathtt{MO},T} \cong \mathfrak{g}_Q^{\prime\,\mathtt{MO}} \otimes \mathrm{H}_T$ *for* $\mathfrak{g}_Q^{\prime\,\mathtt{MO}}$ *defined over* $\mathbb{Q}$.

Conjecture (Okounkov)

There is an equality $\mathrm{a}_{Q,\mathrm{d}}(t^{-1}) = \sum_{n \in \mathbb{Z}} \dim(\mathfrak{g}_{Q,\mathrm{d}}^{\prime\,\mathtt{MO},n}) t^{n/2}$.

- Maulik–Okounkov proved the conjectures when Q is the Jordan quiver.
- McBreen explicitly described the Yangian in the case Q an ADE Dynkin diagram, his results imply the conjecture for these quivers.

Preprojective CoHA

- Define $\mathcal{A}_{\Pi_Q,\mathrm{d}} := \mathrm{H}^{\mathrm{BM}}(\mathfrak{M}_{\mathrm{d}}(\Pi_Q),\mathbb{Q})$ and $\mathcal{A}_{\Pi_Q} := \bigoplus_{\mathrm{d}\in\mathbb{N}^{Q_0}} \mathcal{A}_{\Pi_Q,\mathrm{d}}$
- We consider the usual correspondence diagram $\mathfrak{M}(\Pi_Q)\times\mathfrak{M}(\Pi_Q) \xleftarrow{\pi_1\times\pi_3} \mathfrak{Exact}(\Pi_Q) \xrightarrow{\pi_2} \mathfrak{M}(\Pi_Q)$ where π_n maps $(\rho_1\to\rho_2\to\rho_3)\mapsto\rho_n$.
- (Schiffmann–Vasserot, Yang–Zhao): pullback along $\pi_1\times\pi_3$ and push forward along π_2 yields a morphism $\mathcal{A}_{\Pi_Q,\mathrm{d}'}\otimes\mathcal{A}_{\Pi_Q,\mathrm{d}''}\to\mathcal{A}_{\Pi_Q,\mathrm{d}'+\mathrm{d}''}$ making $\mathcal{A}_{\Pi_Q}$ into a $\mathbb{N}^{Q_0}$-graded, cohomologically graded algebra.

Theorem (-, Meinhardt)

There is a Lie sub-algebra $\mathfrak{n}^+_{\Pi_Q}\subset\mathcal{A}_{\Pi_Q}$ *and a* $\mathrm{H}_{\mathbb{C}^*}=\mathbb{Q}[a]$*-action on* $\mathcal{A}_{\Pi_Q}$ *such that* $\mathrm{Sym}(\mathfrak{n}^+_{\Pi_Q}\otimes\mathbb{Q}[a])\to\mathcal{A}_{\Pi_Q}$ *is a PBW isomorphism.*

Theorem (-)

There is an equality of characteristic functions $\chi_{t^{1/2}}(\mathfrak{n}^+_{\Pi_Q,\mathrm{d}})=\mathtt{a}_{Q,\mathrm{d}}(t^{-1})$

Conjecture *: There is an isomorphism of Lie algebras $\mathfrak{n}^{\mathtt{MO},+}_Q\cong\mathfrak{n}^+_{\Pi_Q}\otimes\mathrm{H}_T$.

The decomposition theorem

- There is a canonical affinization map $\mathtt{JH}\colon \mathfrak{M}(\Pi_Q) \to \mathcal{M}(\Pi_Q)$, where $\mathcal{M}(\Pi_Q)$ is the coarse moduli space; points of $\mathcal{M}(\Pi_Q)$ are in bijection with semisimple Π_Q-modules.
- One definition of $\mathrm{H}^{\mathrm{BM}}(\mathfrak{M}(\Pi_Q), \mathbb{Q})$ is as the derived global sections of the Verdier dual of the constant sheaf $\mathbb{D}\mathbb{Q}_{\mathfrak{M}(\Pi_Q)}$.
- Factoring the structure morphism $\mathfrak{M}(\Pi_Q) \to \mathrm{pt}$ through $\mathtt{JH}$, we find $\mathrm{H}^{\mathrm{BM}}(\mathfrak{M}(\Pi_Q), \mathbb{Q}) \cong \mathrm{H}(\mathcal{M}(\Pi_Q), \mathtt{JH}_* \mathbb{D}\mathbb{Q}_{\mathfrak{M}(\Pi_Q)})$

Theorem (Decomposition theorem (-))

$\mathtt{JH}_* \mathbb{D}\mathbb{Q}^{(\mathrm{vir})}_{\mathfrak{M}(\Pi_Q)} \cong \bigoplus_{n \in 2\cdot\mathbb{N}} {}^{\mathfrak{p}}\mathcal{H}^n(\mathtt{JH}_* \mathbb{D}\mathbb{Q}^{(\mathrm{vir})}_{\mathfrak{M}(\Pi_Q)})[-n]$. *Setting* ${}^{\mathfrak{p}}\mathcal{A}^0_{\Pi_Q} = \mathrm{H}(\mathcal{M}(\Pi_Q), {}^{\mathfrak{p}}\mathcal{H}^0(\mathtt{JH}_* \mathbb{D}\mathbb{Q}^{(\mathrm{vir})}_{\mathfrak{M}(\Pi_Q)})) \subset \mathcal{A}_{\Pi_Q}$, *we obtain the subalgebra* ${}^{\mathfrak{p}}\mathcal{A}^0_{\Pi_Q} \cong \mathrm{U}(\mathfrak{n}^+_{\Pi_Q})$

In (something like) English, the theorem tells us that the BPS Lie algebra can be lifted to an algebra object in the category of perverse sheaves on the coarse moduli space $\mathcal{M}(\Pi_Q)$.

Structure theorem

Let Q be a quiver and pick $\mathrm{d} \in \mathbb{N}^{Q_0}$ such that there exists a simple d-dimensional Π_Q-module. Then by the decomposition theorem there is a unique summand

$$\mathrm{IH}^*(\mathcal{M}_{\mathrm{d}}(\Pi_Q)) \subset \mathfrak{n}^+_{\Pi_Q,\mathrm{d}}$$

which is *primitive* (for support reasons).

Theorem (-,Hennecart,Schlegel-Mejia)

Assume that Q has no isotropic roots, then $\mathfrak{n}^+_{\Pi_Q}$ is one half of a generalised Kac–Moody Lie algebra $\mathfrak{g}_{\Pi_Q}$, with Chevalley generators given by the above intersection cohomology groups.

(With isotropic roots the statement is just a little more complicated.)

Proposition

There is a natural isomorphism $\mathfrak{n}^+_{\Pi_{Q_{\mathrm{f}}},(\mathrm{d},1)} \cong \mathrm{H}(\mathrm{N}_Q(\mathrm{f},\mathrm{d}),\mathbb{Q})$. Via the isomorphisms $\mathfrak{g}_{\Pi_Q,\bullet} \cong \mathfrak{g}_{\Pi_{Q_{\mathrm{f}}},(\bullet,0)}$ we get a $\mathfrak{g}_{\Pi_Q,\bullet}$-action on $\mathfrak{g}_{\Pi_{Q_{\mathrm{f}}},(\bullet,1)} \cong \mathbb{M}_{Q,\mathrm{f}}$.

The main theorem (with Tommaso Botta)

- For $(x_i)_{i\in Q_0}\in\mathbb{A}^{Q_0}$ we define the deformed stack $\mathfrak{M}_{\mathrm{d}}(\Pi_Q)_x$ in analogy with deformed Nakajima quiver varieties $\mathrm{N}_Q(\mathrm{f},\mathrm{d})_x$.
- For generic x we have (almost) diagonal embedding $\Delta_x\colon\ \mathrm{N}_Q(\mathrm{f},\mathrm{d})_x\hookrightarrow \mathrm{N}_Q(\mathrm{f},\mathrm{d})_x\times\mathfrak{M}_{(\mathrm{d},1)}(\Pi_{Q_{\mathrm{f}}})_x$
- We define the nonabelian stable envelope via the correspondence $\lim_{x\mapsto 0}[\mathrm{N}_Q(\mathrm{f},\mathrm{d})_x]\in \mathrm{H}(\mathrm{N}_Q(\mathrm{f},\mathrm{d}),\mathbb{Q})\otimes \mathrm{H}^{\mathrm{BM}}(\mathfrak{M}_{(\mathrm{d},1)}(\Pi_{Q_{\mathrm{f}}},\mathbb{Q})$

$$\Psi_{\mathrm{f}}\colon\ \mathbb{M}_{Q,\mathrm{f}}\to\bigoplus_{\mathrm{d}\in\mathbb{N}^{Q_0}}\mathcal{A}_{\Pi_{Q_{\mathrm{f}}},(\mathrm{d},1)}$$

(Defined also for T-equivariant versions).

Theorem (Botta,-)

- *The morphism Ψ_{f} induces an isomorphism $\mathbb{M}_{Q,\mathrm{f}}\to\mathfrak{n}^+_{\Pi_{Q_{\mathrm{f}}},(\bullet,1)}$, sending lowest weight vectors to Chevalley raising operators.*
- *Both $\mathfrak{g}_Q^{\mathtt{MO}}$ and $\mathfrak{g}_{\Pi_Q}$ are realised as Lie subalgebras of $\bigoplus_{\mathrm{f}\in\mathbb{N}^{Q_0}}\mathrm{End}(\mathfrak{n}^+_{\Pi_{Q_{\mathrm{f}}},(\bullet,1)})$, and are the same subalgebras $\Rightarrow$*
  **+MO+Okounkov conjectures hold.*

Thank you!

That's it, thanks for listening!!