Telescope conjecture via homological residue fields with applications to schemes

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Outline

1. Telescope Conjecture via homological residue fields

2. Applications to $\mathcal{D}(X)$

Telescope Conjecture via homological residue fields

Big tt-categories

$(\mathcal{T}, \otimes, 1)$ - a rigidly-compactly generated tensor triangulated category, a.k.a. a big tt-category. This means that:

- $\mathcal{T}$ is a triangulated category with all (co)products,
- $- \otimes -$ is a symmetric monoidal product on $\mathcal{T}$ with unit 1, compatible with the triangulated structure.
- $(\mathcal{T}^c, \otimes, 1)$ - the full subcategory of compact objects is a small tt-subcategory generating $\mathcal{T}$.
- $- \otimes -$ is closed, so $\mathcal{T}$ has an internal Hom functor $[-,-]$.
- Every compact object is rigid, meaning that $[x, 1] \otimes Y \cong [x, Y]$ for all $x \in \mathcal{T}^c$, $Y \in \mathcal{T}$.

Examples:

- $(\mathcal{D}(X), \otimes^{\mathbf{L}}_X, \mathcal{O}_X)$, the derived category of a quasi-compact & quasi-separated scheme
- $(\mathcal{SH}, \wedge, S)$, the stable homotopy category of spectra
- $(\text{stMod-}kG, \otimes_k, k)$, the stable module category of a finite group G over field k

Balmer spectrum

- A thick $\otimes$-ideal is a thick subcategory $\mathcal{S}$ closed under $X \otimes -$ for any X. It is prime if $X \otimes Y \in \mathcal{S}$ implies $X \in \mathcal{S}$ or $Y \in \mathcal{S}$.
- $\operatorname{Spec} \mathcal{T}^c$ is the set of all prime thick $\otimes$-ideals in $\mathcal{T}^c$, topologized by the base of closed sets of the form $\operatorname{supp}(x) = \{\mathfrak{p} \in \operatorname{Spec} \mathcal{T}^c \mid x \notin \mathfrak{p}\}$ with $x \in \mathcal{T}^c$.
- A subset V of $\operatorname{Spec} \mathcal{T}^c$ is called Thomason if it is a union of closed sets with quasi-compact complements.

Theorem (Balmer '05)

$$\left\{ \begin{array}{c} \textit{Thomason subsets} \\ \textit{of } \operatorname{Spec} \mathcal{T}^c \end{array} \right\} \overset{1-1}{\longleftrightarrow} \left\{ \begin{array}{c} \textit{Thick } \otimes\textit{-ideals} \\ \textit{in } \mathcal{T}^c \end{array} \right\}$$

$$V \mapsto \mathcal{K}_V = \{x \in \mathcal{T}^c \mid \operatorname{supp}(x) \subseteq V\}.$$

Examples:

- $\operatorname{Spec} \mathcal{SH}^c$, Devinatz-Hopkins-Smith '88
- $\operatorname{Spec} \mathcal{D}(X)^c = X$, Thomason '97
- $\operatorname{Spec} \operatorname{stMod}\text{-}kG^c = \mathcal{V}_G(k)$, Benson-Carlson-Rickard '97

Abstract model theory of a big tt-category

[Krause '00, Beligiannis '00, Wagstaffe '21, Wagstaffe-Prest '23]

- Let $\mathcal{A} = \text{Mod-}\mathcal{T}^c$ be the Grothendieck category of additive functors $(\mathcal{T}^c)^{\text{op}} \to \text{Mod-}\mathbb{Z}$.
- The restricted Yoneda functor $\mathbf{y} : \mathcal{T} \to \mathcal{A}$ is given by $X \mapsto \text{Hom}_{\mathcal{T}}(-, X)_{\mathcal{T}^c}$.
- The tensor product $- \otimes -$ extends to a unique tensor structure on $\mathcal{A}$ so that $\mathbf{y}(X \otimes Y) = \mathbf{y}X \otimes \mathbf{y}Y$ for $X, Y \in \mathcal{T}$.
- A triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{+}$ is pure if $\mathbf{y}$ takes it to a short exact sequence in $\mathcal{A}$. Then f is called a pure monomorphism and g a pure epimorphism in $\mathcal{T}$. An object $X \in \mathcal{T}$ is pure-injective if $\mathbf{y}X$ is injective in $\mathcal{A}$.
- A subcategory $\mathcal{D}$ of $\mathcal{T}$ is definable if it is of the form $\Phi^{\perp_0} = \{X \in \mathcal{T} \mid \text{Hom}_{\mathcal{T}}(f, X) = 0\}$ for a set Φ of morphisms in $\mathcal{T}^c$. If $\mathcal{T}$ has a model, definable subcategories are precisely those closed under products, pure monomorphisms, and pure epimorphisms (and coproducts) [Laking '20].

Definable $\otimes$-ideals

A thick $\otimes$-ideal $\mathcal{L}$ in $\mathcal{T}$ is called:

- localizing, if $\mathcal{L}$ is closed under coproducts,
- smashing if both $\mathcal{L}$ and $\mathcal{L}^{\perp} = \{X \in \mathcal{T} \mid \mathrm{Hom}_{\mathcal{T}}(\mathcal{L}, X) = 0\}$ are localizing,
- definable, if it is definable in the previous sense.

Theorem (Krause '00, Wagstaffe '21, Nicolás '08, Balmer-Favi '11)

The following collections are sets and are in mutual bijections:

1. *smashing $\otimes$-ideals $\mathcal{L}$ of $\mathcal{T}$,*
2. *Bousfield localizations of $\mathcal{T}$ of the form $- \otimes F$, up to $\cong$,*
3. *semiorthogonal $\otimes$-triples $(\mathcal{L}, \mathcal{D}, \mathcal{C})$ in $\mathcal{T}$,*
4. *$\otimes$-compatible recollements in $\mathcal{T}$,*
5. *definable $\otimes$-ideals $\mathcal{D}$ of $\mathcal{T}$,*
6. *idempotent saturated Σ-stable ideals Φ in $\mathcal{T}^c$.*

Telescope Conjecture

A definable $\otimes$-ideal $\mathcal{D}$ is compactly generated if there is a subset $\mathcal{S}$ of objects (= identity morphisms) of $\mathcal{T}^c$ such that $\mathcal{D} = \mathcal{S}^\perp$.

Proposition

$$\left\{ \begin{array}{c} \textit{Thomason subsets} \\ \textit{of } \operatorname{Spec} \mathcal{T}^c \end{array} \right\} \xleftrightarrow{1-1} \left\{ \begin{array}{c} \textit{Compactly generated} \\ \textit{definable } \otimes\textit{-ideals } \mathcal{T} \end{array} \right\}$$

$$V \mapsto \mathcal{T}_V = \mathcal{K}_V{}^\perp.$$

The Telescope Conjecture (TC) is the assertion "Every definable $\otimes$-ideal in $\mathcal{T}$ is compactly generated".

- In $\mathcal{SH}$, (TC) been an open question formulated by Ravenel in 1984 and answered in the negative last year by Burklund, Hahn, Levy, and Schlank.
- (TC) holds in stMod-kG, as proved by Benson, Iyengar, and Krause in 2011.
- In $\mathcal{D}(X)$, (TC) holds for noetherian X (Neeman '92, Alonso, Jeremías, Souto '04), but can fail in general (Keller '94).

Homological residue fields

[Balmer-Krause-Stevenson '19, Balmer '20]

- $\mathcal{A} = \text{Mod-}\mathcal{T}^c$ is a locally coherent category, so that the subcategory $\text{fp}(\mathcal{A})$ of finitely presentable objects is abelian.
- Let $\text{Spec}^{\text{h}}(\mathcal{T})$ be the set of all maximal proper $\otimes$-closed Serre subcategories of $\text{fp}(\mathcal{A})$, the homological spectrum.
- For each $\mathcal{B} \in \text{Spec}^{\text{h}}(\mathcal{T})$, we have the cohomological functor $\mathbf{y}_{\mathcal{B}} : \mathcal{T} \to \mathcal{A}_{\mathcal{B}}$ obtained by composing $\mathbf{y}$ with the Gabriel localization $\mathcal{A} \to \mathcal{A}_{\mathcal{B}} := \mathcal{A}/\varinjlim \mathcal{B}$.
- Consider the injective envelope $\mathbf{y}_{\mathcal{B}}(1) \to \overline{E_{\mathcal{B}}}$ in $\mathcal{A}_{\mathcal{B}}$. Then there is a unique and pure-injective object $E_{\mathcal{B}}$ such that $\mathbf{y}(E_{\mathcal{B}})$ is equal to the image of $\overline{E_{\mathcal{B}}}$ in $\mathcal{A}$. We call $E_{\mathcal{B}}$ the homological residue field object over $\mathcal{B} \in \text{Spec}^{\text{h}}(\mathcal{T})$.

Homological residue fields cont'd

- We define the homological support of an object $X \in \mathcal{T}$ as $\mathrm{supp}^{\mathrm{h}}(X) = \{\mathcal{B} \in \mathrm{Spec}^{\mathrm{h}}(\mathcal{T}) \mid [X, E_{\mathcal{B}}] \neq 0\}$.
- Then $\mathrm{Spec}^{\mathrm{h}}(\mathcal{T})$ is topoligized by a base of closed sets of the form $\mathrm{supp}^{\mathrm{h}}(x)$ for all $x \in \mathcal{T}^{\mathrm{c}}$.
- There is a natural continuous map $\varphi : \mathrm{Spec}^{\mathrm{h}}(\mathcal{T}) \to \mathrm{Spec}\,\mathcal{T}^{\mathrm{c}}$ from the homological to the Balmer spectrum defined by $\varphi(\mathcal{B}) = \mathbf{y}^{-1}(\mathcal{B})$.
- The map φ is always surjective.
- The injectivity of φ is known as the "Nerves of Steel Conjecture". It has been checked for all standard examples including $\mathcal{D}(X)$, $\mathcal{SH}$, and stMod-kG. Studied in e.g. [Barthel-Heard-Sanders '21, Bird-Williamson '23]

Examples: [Balmer-Cameron '21]

- In $\mathcal{D}(X)$: Standard residue field sheafs $k(x)$.
- In $\mathcal{SH}$: Morava K-theory spectra.
- In stMod-kG: π-points

Locality of (TC)

Balmer and Favi proved that (TC) is affine-local in the following sense.

Theorem (Balmer-Favi '11)

Let $\mathcal{D}$ be a definable $\otimes$-ideal of $\mathcal{T}$ and let $\operatorname{Spec}\mathcal{T}^c = \bigcup_{i=1}^n U_i$ be a cover by open quasi-compact sets. TFAE:

(i) *$\mathcal{D}$ is compactly generated in $\mathcal{T}$,*

(ii) *$\mathcal{D} \cap \mathcal{T}_{U_i^c}$ is compactly generated in $\mathcal{T}_{U_i^c}$ for all $i = 1, \ldots, n$.*

In the case of $\mathcal{D}(X)$, we know that (TC) is even stalk-local.

Theorem (H-Hu-Zhu '21)

Let X be a quasi-compact and quasi-separated scheme and $\mathcal{D}$ a definable $\otimes$-ideal in $\mathcal{D}(X)$. TFAE:

(i) *$\mathcal{D}$ is compactly generated in $\mathcal{D}(X)$,*

(ii) *$\mathcal{D} \cap \mathcal{D}(\mathcal{O}_{X,x})$ is compactly generated in $\mathcal{D}(\mathcal{O}_{X,x})$ for all (closed) points $x \in X$.*

Stalk-locality of (TC)

We say that $\mathcal{T}$ satisfies a Stalk Locality Principle (SLP) if a definable $\otimes$-ideal $\mathcal{D}$ is compactly generated provided that $\mathcal{D} \cap \mathcal{T}_{\mathfrak{p}}$ is compactly generated in the stalk tt-category $\mathcal{T}_{\mathfrak{p}} = \mathcal{T}/\mathrm{Loc}_{\otimes}(\mathfrak{p})$ for all (closed) points $\mathfrak{p} \in \mathrm{Spec}\,\mathcal{T}^c$.

- I do not know if every big tt-category $\mathcal{T}$ satisfies (SLP).
- If $\mathcal{T}$ satisfies the Local-To-Global principle then it satisfies (SLP). This is the case for example if $\mathrm{Spec}\,\mathcal{T}^c$ is noetherian space. (so stMod-kG and its compact localizations are OK).
- If the Balmer-Favi-Sanders support theory detects vanishing in $\mathcal{T}$, then all compact localizations $\mathcal{T}_V$ satisfy (SLP). (so $\mathcal{SH}$ and its compact localizations are OK).
- It is not known if Balmer-Favi-Sanders support theory detects vanishing even for the case of $\mathcal{D}(X)$.
- Failure of (SLP) would lead to a spectaculary pathological new way of failing (TC): $\mathrm{Def}_{\otimes}(\coprod_{\mathfrak{p} \in \mathrm{Spec}\,\mathcal{T}^c} 1_{\mathfrak{p}}) \neq \mathcal{T}$.

Main result for big tt-categories

Let $\mathrm{Def}_{\otimes}(X)$ denote the smallest definable $\otimes$-ideal in $\mathcal{T}$ which contains X.

Theorem

Let $\mathcal{T}$ be a big tt-category which satisfies the Nerves of Steel Conjecture and whose each compact localization $\mathcal{T}_V$ satisfies (SLP). TFAE:

(i) *$\mathcal{T}$ satisfies (TC),*

(ii) *for any $\mathfrak{p} \in \operatorname{Spec} \mathcal{T}^c$, we have $\mathrm{Def}_{\otimes}(E_{\mathfrak{p}}) = \mathcal{T}_{\mathfrak{p}}$.*

The proof relies on Balmer's Tensor Nilpotence Theorem for homological residue fields, a common generalization of results of Devinatz, Hopkins, and Smith in $\mathcal{SH}$ and of Thomason in $\mathcal{D}(X)$.

Applications to $\mathcal{D}(X)$

The case of $\mathcal{D}(X)$

Proposition

The Stalk Locality Principle holds for each compact localization $\mathcal{D}(X)_V$ of $\mathcal{D}(X)$.

The proof is very specific to commutative algebra.

Theorem

Let X be a quasi-compact and quasi-separated scheme. TFAE:

(i) $\mathcal{D}(X)$ *satisfies (TC),*

(ii) *for any $x \in X$, we have* $\mathrm{Def}_{\otimes}(k(x)) = \mathcal{D}(\mathcal{O}_{X,x})$.

Note: (TC) holds for all noetherian schemes [Neeman '02, Alonso-Jeremías-Souto '04].

P. Balmer '20: "[...] but who cares about non-noetherian schemes?"

Restricted Telescope Conjecture in $\mathcal{D}(R)$

A restricted version of (TC) has a ring extension interpretation in the case of an affine scheme $X = \mathrm{Spec}(R)$.

- An epimorphism $R \to S$ of rings is called pseudoflat if $\mathrm{Tor}_1^R(S, S) = 0$. It is flat if $\mathrm{Tor}_1^R(M, S) = 0$ for all $M \in \text{Mod-}R$.

Theorem

Let R be a commutative ring. The following are equivalent:

(i) *Every definable $\otimes$-ideal $\mathcal{D}$ in $\mathcal{D}(R)$ which si closed under cohomology is compactly generated. (RTC)*

(ii) *Every pseudoflat ring epimorphism over R is flat.*

Theorem (Angeleri-Hügel, Marks, Šťovíček, Takahashi, and Vitória '20)

Every pseudoflat ring epimorphism over a commutative noetherian ring is flat.

Separation axioms

By our main Theorem, to understand when (TC) holds in $\mathcal{D}(X)$, we need to understand when $\mathrm{Def}_{\otimes}(k) = \mathcal{D}(R)$ where $(R, \mathfrak{m}, k)$ is a local commutative ring.

- Observation: $\mathrm{Def}_{\otimes}(k) = \mathrm{Def}_{\otimes}(\widehat{R})$, where $\widehat{R} = \varprojlim_{n>0} R/\mathfrak{m}^n$ is the $\mathfrak{m}$-adic completion.
- R is ($\mathfrak{m}$-adically) separated if the natural map $R \to \widehat{R}$ is a monomorphism $\iff \bigcap_{n>0} \mathfrak{m}^n = 0$.
- R is purely separated if the natural map $R \to \widehat{R}$ is a pure monomorphism. Equivalently, each finitely presented R-module F is separated. This holds e.g. if R is complete.
- More generally, R is transfinitely separated if there is an ordinal λ such that $\mathfrak{m}^\lambda = 0$, where recursively $\mathfrak{m}^{\beta+n} = (\bigcap_{\alpha<\beta} \mathfrak{m}^\alpha)^n$, where β is a limit ordinal.
- If moreover for each limit β, the morphism $R/\mathfrak{m}^\beta \to \mathbf{R}\varprojlim_{\alpha<\beta} R/\mathfrak{m}^\alpha$ is a pure monomorphism then R is purely (derived) transfinitely separated.

Necessary condition

Lemma

A local ring R is transfinitely separated if and only if 0 and R are the only idempotent ideals in R (i.e., ideals I such that $I = I^2$).

Lemma

Let I be a an ideal of a commutative ring R. Then the surjective morphism $R \to R/I$ is pseudoflat if and only if I is idemoptent. If $0 \neq I \subseteq J(R)$ then this morphism is not flat.

Corollary

If $\mathcal{D}(X)$ satisfies (TC) then $\mathcal{O}_{X,x}$ is transfinitely separated for any $x \in X$.

Example (Keller '94)

Any local ring with a non-trivial idempotent ideal fails (TC), e.g. $k[x^k \mid k \leq 1]_{(x^k \mid k \leq 1)}$.

Sufficient condition

Proposition

If $\mathcal{O}_{X,x}$ is purely transfinitely separated then $\mathcal{D}(X)$ satisfies (TC).

About proof.

We need to show that $\mathrm{Def}_{\otimes}(k) = \mathcal{D}(R)$ for R purely transfinitely separated. This follows from the assumptions by transfinite induction, because definable $\otimes$-ideals are closed under $\mathbf{R}\varprojlim$ and pure monomorphisms. □

Example

Any 0-dimensional local complete ring R satisfies (TC).

Necessary and sufficient condition

We have the following picture:

$\mathcal{O}_{X,x}$ is purely transfinitely separated for all $x \in X$

$\Downarrow$

$\mathcal{D}(R)$ satisfies (TC)

$\Downarrow$

$\mathcal{O}_{X,x}$ is transfinitely separated for all $x \in X$

We show how these recover some further known cases of (TC) and also that neither of the implications can be conversed in general.

Noetherian stalks

- Any local noetherian ring is purely separated by the Artin-Rees Lemma.
- Then (TC) holds in $\mathcal{D}(X)$ for any quasi-compact quasi-separated scheme X with noetherian stalks [H-Hu-Zhu '21].

Example (Neeman '92, Alonso, Jeremías, Souto '04)

(TC) holds in $\mathcal{D}(X)$ for any noetherian scheme X.

Example (Stevenson '14, Bazzoni-Šťovíček '17)

(TC) holds in $\mathcal{D}(R)$ for R a commutative von Neumann regular ring (every stalk is a field).

Example

(TC) holds in $\mathcal{D}(R)$ for R an almost Dedekind domain (every stalk is a DVR).

Valuation domains

A valuation domain is a commutative domain whose ideals form a chain. A commutative ring R has weak global dimension ≤ 1 if and only if $R_{\mathfrak{m}}$ is a valuation domain for each maximal ideal $\mathfrak{m}$ of R.

Lemma

Let R be of weak global dimension ≤ 1, TFAE:

(i) *$R_{\mathfrak{p}}$ is transfinitely separated for all $\mathfrak{p} \in \operatorname{Spec} R$,*

(ii) *$R_{\mathfrak{p}}$ is purely transfinitely separated for all $\mathfrak{p} \in \operatorname{Spec} R$,*

(iii) *R is strongly discrete ($=$ $R_{\mathfrak{m}}$ has no non-trivial idempotent ideal for each maximal ideal $\mathfrak{m}$).*

Example (Bazzoni-Šťovíček '17)

Let R be of weak global dimension ≤ 1, TFAE:

(i) $\mathcal{D}(R)$ satisfies (TC),

(ii) $\mathcal{D}(R)$ satisfies (RTC),

(iii) R is strongly discrete.

0-dimensional rings

Let $(R, \mathfrak{m}, k)$ be a 0-dimensional local ring, that is, a one-point affine scheme. Then (TC) holds in $\mathcal{D}(R) \iff \mathrm{Def}_{\otimes}(k) = \mathcal{D}(R)$.

Lemma

Any local ring R that is a direct limit $\varinjlim R_i$ of coherent and self-injective rings R_i with flat transition maps is separated if and only if it is purely separated.

Example (Dwyer-Palmieri '08)

The truncated polynomial ring $R[x_1, x_2, x_3, \ldots]/(x_1^{n_1}, x_2^{n_2}, x_3^{n_3}, \ldots)$ is purely separated and thus satisfies (TC).

0-dimensional rings

Lemma

Let R be a 0-dimensional local ring and I its finitely generated ideal. If $\mathcal{D}(R/I)$ satisfies (TC) then so does $\mathcal{D}(R)$.

About proof.

Since R is 0-dimensional, I is nilpotent, and so $R \in \mathrm{Loc}_{\otimes}(R/I)$ in $\mathcal{D}(R)$.
Then $\mathrm{Def}_{\otimes}(k_{R/I}) = \mathcal{D}(R/I)$ implies $\mathrm{Def}_{\otimes}(k_R) = \mathcal{D}(R)$. □

Example (Pure separation is not necessary)

There is a separated 0-dimensional local ring R with elements $y, z \in R$ such that $R/(y)$ is not separated and $R/(y, z)$ is purely separated. Then R is separated, not purely separated but satisfies (TC).

Lemma

Let $(R, \mathfrak{m}, k)$ be a local ring and I its finitely generated ideal such that R/I is not transfinitely separated. Then $\mathrm{Def}_{\otimes}(k) \neq \mathcal{D}(R)$ and so $\mathcal{D}(R)$ fails (TC).

About proof.

Let J lift an idempotent ideal of R/I.

$$\left\{ X \in \mathcal{D}(R) \;\middle|\; \begin{array}{c} \mathrm{Hom}_{\mathcal{D}(R)}(K(I), \Sigma^n X) \xrightarrow{\cdot j} \mathrm{Hom}_{\mathcal{D}(R)}(K(I), \Sigma^n X) \\ \text{is a zero map } \forall j \in J, n \in \mathbb{Z} \end{array} \right\}$$

is a definable $\otimes$-ideal containing k but not R. □

Separated ring failing (TC)

Commutative algebra fact for R local noetherian: A local morphism $R \to S$ is an epimorphism $\iff$ it is surjective.

Example (Lazard '69)

There is a non-surjective epimorphism of local 0-dimensional rings $R \to S$.

Such an example cannot be flat. In Lazard's example, the morphism is not pseudoflat.

Example (Separation is not sufficient)

- There is a separated local ring R with a non-zero-divisor $y \in R$ such that $R/(y)$ is not transfinitely separated.
- In this example, even (RTC) fails. Then there is a local ring epimorphism $f : R \to S$ which is pseudoflat but not surjective.
- However, my construction cannot yield a 0-dimensional example. Is there a 0-dimensional local ring with a non-surjective pseudoflat epimorphic ring extension?

Thank you for your attention!