Namikawa-Weyl groups of quiver varieties

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This talk is based on my 2018 master's thesis, supervised by Raf Bocklandt and Eric Opdam.

We almost solved the problem of Namikawa-Weyl groups for quiver varieties.

However some technical problems were left.
Get in touch!

Goals

Quest 1: Why quiver varieties?

- Representation theory
- Hands-on definition
- Kleinian singularities

Quest 2: What are Namikawa-Weyl groups?

- Classical Weyl-groups
- Symplectic singularities
- Poisson deformations
- Namikawa-Weyl groups

Quest 3: Namikawa-Weyl groups of quiver varieties?

- Basic construction
- Remaining problems

Quest 1: Why quiver varieties?

- Given algebra $A = \mathbb{C}Q/I$
- Capture its representations in a moduli space

$$\mathcal{M} := \mathrm{Rep}(A)/\sim .$$

- Sometimes $\mathcal{M}$ can be defined as algebraic variety
- In that case, it is typically singular
- $\rightarrow$ Interesting for deformation people!

Quest 1: Why quiver varieties?

Quiver varieties: from CY2 categories

- Given cyclic A_∞-category $\mathcal{C}$ of degree 2
- Assume $X_1, \dots, X_k$ are generators
- Assume $\mathrm{Ext}^*(X_i, X_j) \cong \underbrace{\mathbb{C}\,\mathrm{id}}_{\deg 0} \oplus \underbrace{V_{ij} \oplus V_{ji}^*}_{\deg 1} \oplus \underbrace{\mathbb{C}\,\mathrm{id}^*}_{\deg 2}$
- Then $\mu^{\geq 3} = 0$ on the generators
- But $\mu^2(a, a^*) = \mathrm{id}^*$ etc
- Thus $\mathcal{C}$ is (almost) derived equivalent to $\mathrm{Rep}(\Pi)$, where

$$\Pi = \frac{\mathbb{C}\overline{Q}}{\left(\sum_{a \in Q_1} aa^* - a^*a\right)}$$

- $\to$ Interesting algebra! What are its reps?

Quest 1: Why quiver varieties?

Construction of GIT quotient:

- Let X affine variety with action of reductive group G
- Define

$$X /\!\!/ G := \mathrm{Spec}(\mathbb{C}[X]^G).$$

- Fact: The points of $X /\!\!/ G$ are $\overset{1:1}{\longleftrightarrow}$ closed G-orbits
- Example:

$$\mathbb{C}^* \text{ act on } \mathbb{C} \text{ by multiplication}, \quad \mathbb{C}[X]^{\mathbb{C}^*} = \mathbb{C}, \quad \mathrm{Spec}(\mathbb{C}) = \mathrm{pt}.$$

Quest 1: Why quiver varieties?

Construction of quiver varieties (I)

- Start with quiver Q, dimension vector $\alpha \in \mathbb{N}^{Q_0}$
- Take double quiver $\overline{Q}$ of Q
- Representation space $\mathrm{Rep}(\overline{Q}, \alpha) := \bigoplus_{a \in Q_1} \mathbb{C}^{\alpha_{h(a)}, \alpha_{t(a)}} \oplus \bigoplus_{a \in Q_1} \mathbb{C}^{\alpha_{t(a)}, \alpha_{h(a)}}$.
- The group $\mathrm{GL}_\alpha := \prod_{v \in Q_0} \mathrm{GL}_{\alpha_v}$ acts on $\mathrm{Rep}(\overline{Q}, \alpha)$ by "conjugation":

$$(g.\rho)(a) = g_{h(a)} \rho(a) g_{t(a)}^{-1} \in \mathbb{C}^{\alpha_{h(a)}, \alpha_{t(a)}}.$$

Quest 1: Why quiver varieties?

Construction of quiver varieties (II)

- Define subspace $\mathrm{Rep}(\Pi_Q, \alpha) \subseteq \mathrm{Rep}(\overline{Q}, \alpha)$ as those ρ with

$$\sum_{h(a)=v} \rho(a)\rho(a^*) - \sum_{t(a)=v} \rho(a^*)\rho(a) = 0.$$

- Define quiver variety as

$$\mathcal{M}(Q, \alpha) := \mathrm{Rep}(\Pi_Q, \alpha) /\!\!/ \mathrm{GL}_\alpha \,.$$

Quest 1: Why quiver varieties?

Quiver varieties are well-defined and have symplectic structure:

- GL_α-action restricts to $\mathrm{Rep}(\Pi_Q, \alpha)$
- GL_α-orbits $\stackrel{1:1}{\longleftrightarrow}$ representations of the preprojective algebra

$$\Pi_Q := \frac{\mathbb{C}\overline{Q}}{\left(\sum_{a \in Q_1} (aa^* - a^* a)\right)}.$$

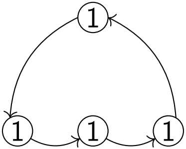
- Π_Q is Calabi-Yau of dimension 2
- Points of $\mathcal{M}(Q, \alpha)$ are $\stackrel{1:1}{\longleftrightarrow}$ semisimple representations of Π_Q
- $\mathrm{Rep}(\overline{Q}, \alpha)$ is a symplectic vector space with

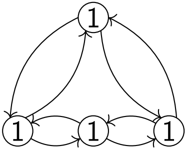
$$\omega(\rho, \sigma) := \sum_{a \in Q_1} \mathrm{tr}(\sigma(a)\rho(a^*) - \sigma(a^*)\rho(a)).$$

- $\mathcal{M}(Q, \alpha)^{\mathrm{reg}}$ inherits symplectic structure
- $\mathcal{M}(Q, \alpha)$ has a $\mathbb{C}^*$-action given by scaling

Quest 1: Why quiver varieties?

Example: extended Dynkin quiver setting $\tilde{A}_3$

$(Q, \alpha) =$ 

$(\overline{Q}, \alpha) =$ 

- Invariant ring

$$\mathbb{C}[\mathrm{Rep}(\Pi_Q, \alpha)] = \frac{\mathbb{C}[A, A^*, B, B^*, C, C^*, D, D^*]}{AA^* = BB^* = CC^* = DD^*},$$

$$\begin{aligned}\mathbb{C}[\mathrm{Rep}(\Pi_Q, \alpha)]^{\mathrm{GL}_\alpha} &= \mathbb{C}[AA^*, ABCD, A^*B^*C^*D^*] \\ &\cong \mathbb{C}[U, V, W]/(U^4 - VW).\end{aligned}$$

- These are the Kleinian singularities!

$$A_n : xy - z^{n+1} = 0, \qquad D_n : x^2 + y^2 z + z^{n-1} = 0,$$

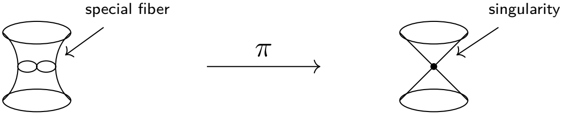
$$E_6 : x^2 + y^3 + z^4 = 0, \qquad E_7 : x^2 + y^3 + yz^3 = 0,$$

$$E_8 : x^2 + y^3 + z^5.$$

Quest 1: Why quiver varieties?

Some heuristics on Kleinian singularities:

- Let K be A_n, D_n or $E_{6/7/8}$ Kleinian singularity
- There is a symplectic resolution $\pi : \tilde{K} \to K$
- The special fiber $\pi^{-1}(0)$ consists of n intersecting $\mathbb{P}^1$'s



- The higher-dimensional variety $\mathbb{C}^{2m} \times K$ has symplectic resolution $\mathbb{C}^{2m} \times \tilde{K}$

Quest 2: What are Namikawa-Weyl groups?

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Classical ADE Weyl groups:

- Given one ADE type (A_n, D_n, $E_{6/7/8}$)
- Cartan pairing $(-,-)$ on $\mathbb{C}^{Q_0}$
- Reflections $s_i : \mathbb{C}^{Q_0} \to \mathbb{C}^{Q_0}$ given by

$$s_i(v) = v - (v, e_i)e_i.$$

- Weyl group $W := \langle s_i \rangle_{i \in Q_0} \subseteq \mathrm{GL}(\mathbb{C}^{Q_0})$
- Example A_2:

$$Q = \quad \bullet \longrightarrow \bullet \qquad (-,-) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

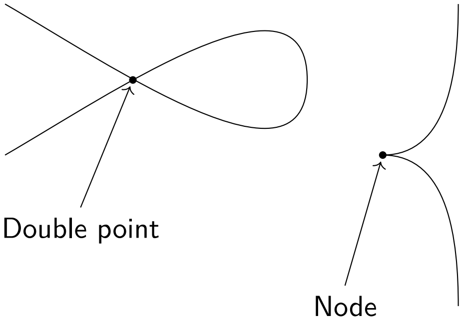
$$s_1 = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$$

$$W \cong S_3 = \{1, (12), (13), (23), (123), (132)\}.$$

Quest 2: What are Namikawa-Weyl groups?

Why symplectic singularities?

- Singularities in dimension 1 are easy:



- We instead want singularities in dimension ≥ 2!

Quest 2: What are Namikawa-Weyl groups?

A symplectic singularity is:

- Complex algebraic variety X
- Complex symplectic form ω on X^{reg}

such that:

- ω is holomorphic
- $\exists$ resolution of singularities $\pi : Y \to X$ such that $\pi^*\omega$ extends to a smooth 2-form on Y

e.g. quiver varieties, symplectic quotients, coadjoint orbits, ...

Quest 2: What are Namikawa-Weyl groups?

- Deformations of algebras, varieties, schemes, ...
- Over algebraic rings, local rings, over base schemes, ...
- Formal deformation theory = Functors of Artin rings
- Natural question: What is a "deformation of (X, ω)"?

Quest 2: What are Namikawa-Weyl groups?

How to define deformations of (X, ω):

- Let (X, ω) affine symplectic singularity
- Poisson bracket $\{-,-\}$ on $\mathbb{C}[X]$ determined by (X, ω)
- Standard symplectic manifold $(X, \omega) = (\mathbb{C}^{2d}, \omega_{std})$

$$\{f, g\} = \sum_{i=1}^{d} \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i}.$$

- This turns $\mathbb{C}[X]$ into a Poisson algebra:
- Poisson algebra $=$ commutative algebra $+$ bracket (satisfying Jacobi and Leibniz rule)
- Deformation of (X, ω) $:=$ deformation of $(\mathbb{C}[X], \{-,-\})$

Quest 2: What are Namikawa-Weyl groups?

Yoshinori Namikawa investigated the Poisson deformation theory of (X, ω) and found that:

- Assume X is affine symplectic singularity
- Assume X has a good $\mathbb{C}^\times$-action
- Assume $\pi : Y \to X$ is a symplectic resolution
- Then $\exists$ universal Poisson deformations $\mathcal{X}, \mathcal{Y}$
- (They are Poisson schemes)

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & \mathcal{Y} \\ \downarrow & & \downarrow \\ \mathbb{C}^d & \longrightarrow & \mathbb{C}^d \end{array}$$

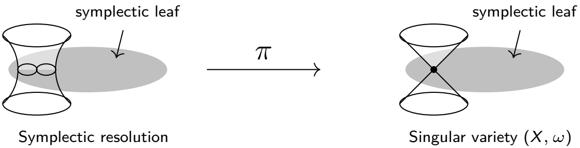
with the following properties:

- Dimension $d = \dim \mathrm{HP}^2(X, \omega)$
- The map $\pi : \mathbb{C}^d \to \mathbb{C}^d$ is a Galois covering (ramified at 0)
- Simply speaking, π is a quotient map $\mathbb{C}^d \to \mathbb{C}^d/W$
- $W = \mathrm{Gal}(\pi)$ is called the Namikawa-Weyl group

Quest 2: What are Namikawa-Weyl groups?

How to compute the Namikawa-Weyl group?

- (X, ω) decomposes into even-dimensional symplectic leaves
- The codimension-0 leaf is simply X^{reg}
- Around every codimension-2 leaf, X looks like $\mathbb{C}^{\dim X - 2} \times K$
- Pick a symplectic resolution $\pi : Y \to X$



- Is the associated Dynkin automorphism D trivial or not?
- Namikawa-Weyl group is $W_X = \prod_{\text{leaves } S} (W_S)^D$

Quest 2: What are Namikawa-Weyl groups?

Trivial example:

- Let (X, ω) be a Kleinian singularity
- Symplectic resolution $\pi : \tilde{X} \to X$
- There is only one codimension-2 leaf and it's a point
- Conclusion: Namikawa-Weyl group = classical Weyl group

Quest 3: Namikawa-Weyl groups of quiver varieties?

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Let's look at the case $X = \mathcal{M}(Q, \alpha)$. Bellamy and Schedler identified the codimension 2 strata:

Given an isotropic decomposition with affine Dynkin quiver Q'', let Q''_f be the finite part, which is a Dynkin diagram.

Theorem 1.20. *Let $\alpha \in \Sigma_{\lambda,\theta}$ be imaginary. Then the codimension two strata of $\mathfrak{M}_\lambda(\alpha, \theta)$ are in bijection with the isotropic decompositions of α. The singularity along each such stratum is étale-equivalent to the du Val singularity of the type A_n, D_n, E_n corresponding to Q''_f.*

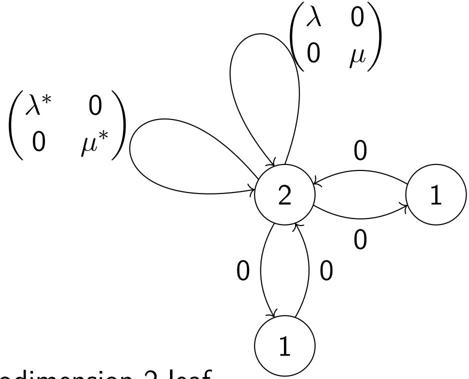
As a consequence, for $\lambda = 0 = \theta$, by [45, Theorem 1.1] the Namikawa Weyl group is a product over all isotropic decompositions B of a group W_B. This group W_B is either the Weyl group of the corresponding Dynkin diagram Q''_f, or else the centralizer therein of an automorphism of this diagram, corresponding to the monodromy around the fiber over a point of the stratum under a crepant resolution of the complement of the codimension > 2 strata.

- Let $\alpha = n_1\beta_1 + \ldots + n_k\beta_k$ be an isotropic decomposition
- Then the leaf is $\{S_1^{\oplus n_1} \oplus \ldots \oplus S_k^{\oplus n_k} \mid S_i \in \mathrm{Rep}(\Pi, \beta_i) \text{ simple}\}$

Quest 3: Namikawa-Weyl groups of quiver varieties?

- Example: Quiver setting (Q, α) with isotropic decomposition $\alpha = e_1 + e_2 + e_2 + e_3$

$x = S_1 \oplus S_2 \oplus S_3 \oplus S_4 =$



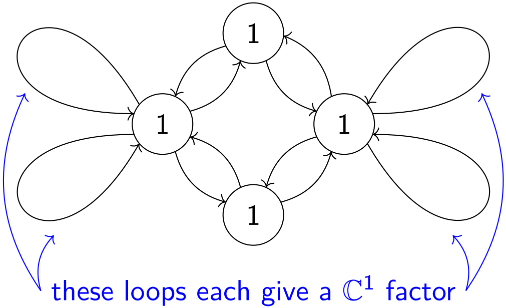
- This element x sits on a codimension 2 leaf

Quest 3: Namikawa-Weyl groups of quiver varieties?

- Local description at $x \in \mathcal{M}(Q, \alpha)$:

$$(\mathcal{M}(Q, \alpha), x) \cong (\mathcal{M}(Q', \alpha'), 0).$$

- Can analyze the singularities this way!
- In the example, the local quiver Q' is:



- The local description is: $\mathbb{C}^4 \times$ Kleinian A_3 singularity.

Quest 3: Namikawa-Weyl groups of quiver varieties?

Definition of Mumford quotient:

- Let X affine variety with action of reductive group G
- Let $\theta : G \to \mathbb{C}^*$ a character
- Define $n\theta : G \to \mathbb{C}^*$ by $(n\theta)(g) = \theta(g)^n$
- A regular function $f : X \to \mathbb{C}$ is a $n\theta$-semiinvariant if

$$f(gx) = (n\theta)(g) \cdot f(x).$$

- Define

$$\mathrm{SI} := \bigoplus_{n \in \mathbb{N}} \mathrm{SI}_{n\theta}, \quad X /\!\!/_\theta G := \mathrm{Proj}(\mathrm{SI}).$$

- Have inclusion $\mathbb{C}[X]^G \subseteq \mathrm{SI}$, inducing a map $X /\!\!/_\theta G \to X /\!\!/ G$

Quest 3: Namikawa-Weyl groups of quiver varieties?

A resolution of $\mathcal{M}(Q,\alpha)$ can sometimes be constructed as follows:

- Take a stability parameter $\theta \in \mathbb{Z}^{Q_0}$
- Gives a character of GL_α by $\theta(g) := \prod_{i \in Q_0} \det(g_i)^{\theta_i}$.
- Define $\mathcal{M}_\theta(Q,\alpha)$ as Mumford quotient
- Its points are the orbits of θ-polystable representations
- Have semisimplification map:

$$\mathcal{M}_\theta(Q,\alpha) \longrightarrow \mathcal{M}(Q,\alpha)$$

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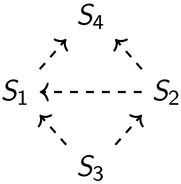
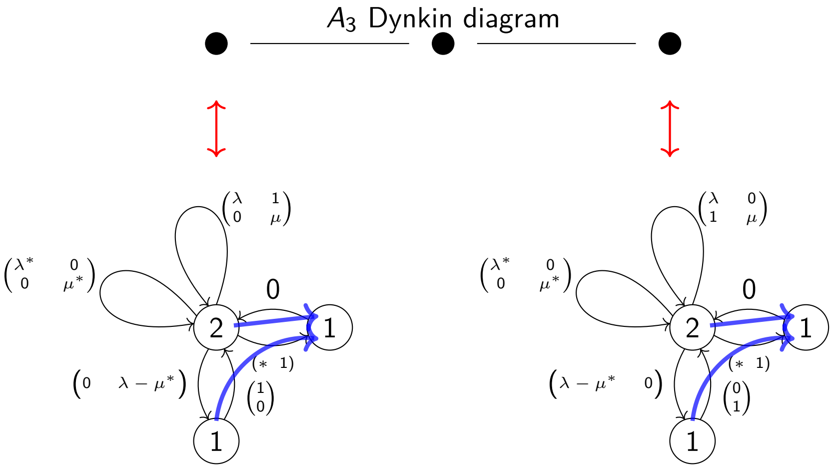


$$\xrightarrow{\text{semisimplify}} \qquad S_1 \oplus S_2 \oplus S_3 \oplus S_4$$

- This map is often a symplectic resolution

Quest 3: Namikawa-Weyl groups of quiver varieties?

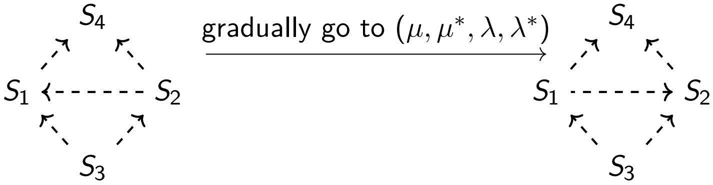
- In the example, picking $\theta = (-1, -1, +3)$ works
- Some representations which lie over the leaf:



Quest 3: Namikawa-Weyl groups of quiver varieties?

Now let's show the monodromy is nontrivial!

- Start at some point $(\lambda, \lambda^*, \mu, \mu^*)$ in the leaf
- Pick a lift in the left-most $\mathbb{P}^1$, i.e. of type



- We end up in the right-most $\mathbb{P}^1$
- Conclusion: Automorphism of this leaf is nontrivial!

Quest 3: Namikawa-Weyl groups of quiver varieties?

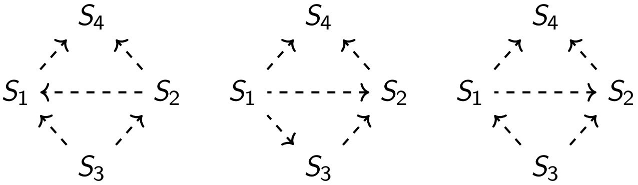
Great! How to do this for other (Q, α)? There are several issues:

- Explicit representations in the fiber are hard to find
- Which θ to choose to make the fiber as simple as possible?
- For some (Q, α), not a single $\mathcal{M}_\theta(Q, \alpha)$ is a resolution
- Does it suffice to find a "local resolution" around every leaf?
- How to construct such a "local resolution" with easy fibers?

Quest 3: Namikawa-Weyl groups of quiver varieties?

An approach to build a “synthetic resolution” which always exists:

1. Let L be a codimension 2 leaf
2. Let $x \in L$ be a point $x = S_1 \oplus \ldots \oplus S_k$
3. $M_x :=$ closed orbits which semisimplify to x and are of shape



4. Define $\pi : \cup M_x \to L$ as semisimplification
5. Make it into algebraic or analytic variety

Thanks for coming!

Appendix: Symplectic quotients

Recall GIT quotient:

- If X affine variety and G acts on X, then

$$X /\!\!/ G := \mathrm{Spec}(\mathbb{C}[X]^G).$$

- Example: V symplectic vector space, $G \leq \mathrm{Sp}(V)$ finite, then

$$V /\!\!/ G \text{ is sometimes a symplectic singularity.}$$

- Example: $G \leq \mathrm{SL}(2, \mathbb{C})$ finite group, then

$$\mathbb{C}^2 /\!\!/ G \text{ is a Kleinian singularity.}$$

Appendix: Symplectic quotients

For instance, we can obtain the Kleinian A_1 singularity as follows:

- Let $G = C_2 = \{1, s\}$ act on $V = \mathbb{C}^2$ by

$$1.(x, y) = (x, y), \quad s.(x, y) = (-x, -y).$$

- On polynomials this translates to

$$1.f = f, \quad s.f = f(-X, -Y).$$

- Thus $\mathbb{C}[V]^G = \mathbb{C}[X^2, Y^2, XY] \cong \mathbb{C}[U, V, W]/(UV - W^2)$.
- This is the A_1 singularity!

Appendix: Symplectic quotients

Bellamy's result on their Namikawa-Weyl groups:

- Let $\Gamma \leq G$ fix a precisely $(\dim(V) - 2)$-dimensional vector space
- Then $\{\text{points fixed by } \Gamma\} \subseteq V$ is a codimension-2 leaf
- In fact, Γ is a Kleinian group, and locally $V /\!\!/ G \cong \mathbb{C}^2/\Gamma$
- The normalizer $N_G(\Gamma)$ acts on $\mathrm{Irr}(G)$ by conjugation
- Theorem (Bellamy): This is the Dynkin automorphism associated to the leaf

Thanks for coming!