Relative cluster categories and Higgs categories with infinite-dimensional morphism spaces

joint work with Chris Fraser and Bernhard Keller (arXiv:2307.12279).

1. Cluster algebras

2. Ice quivers with potentials and Ginzburg functors

3. Relative cluster categories and Higgs categories

1. Cluster algebras

Q: finite quiver without loops nor 2-cycles

Example:

(1) $Q: \; 1 \longrightarrow 2$

(2) $Q:$
$$\begin{tikzcd} & 2 \arrow[dr] & \\ 1 \arrow[ur] & & 3 \arrow[ll] \end{tikzcd}$$

$\mathcal{F} = \mathbb{Q}(x_1, \ldots x_n)$, $n = \# |Q_0|$. $X = \{x_1, \ldots, x_n\}$

We call the pair $(Q, \underline{X})$ a seed.

Cluster

For each vertex $i \in Q_0$, we have a mutation operation

$$\mu_i : (Q, X) \longleftrightarrow (Q', X')$$

Example.

$$(1 \longrightarrow 2 , \{x_1, x_2\}) \overset{\mu_1}{\longleftrightarrow} (1 \longleftarrow 2 , \{\tfrac{1+x_2}{x_1}, x_2\})$$

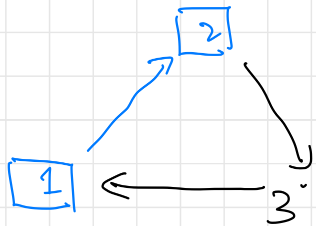
Def (Fomin-Zelevinsky 2002)

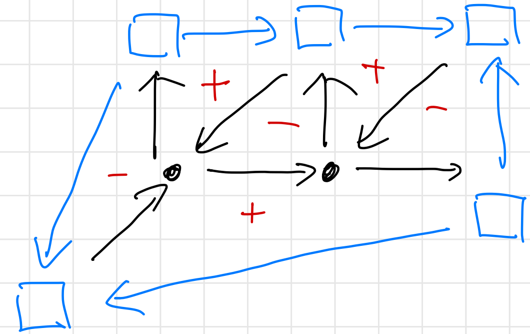
The cluster alg $\mathcal{A}_Q$ is the subalg of $\mathcal{F} = \mathbb{Q}(x_1, \ldots x_n)$ generated by all cluster variables.

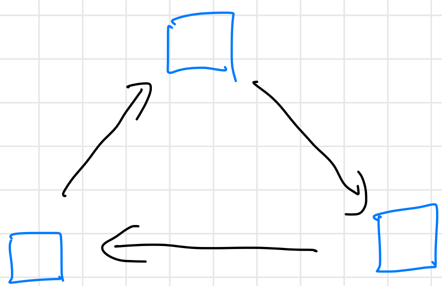
Ex. $Q: 1 \to 2$. Cluster Variables $= \left\{ x_1, x_2, \frac{1+x_2}{x_1}, \frac{1+x_1}{x_2}, \frac{1+x_1+x_2}{x_1 x_2} \right\}$

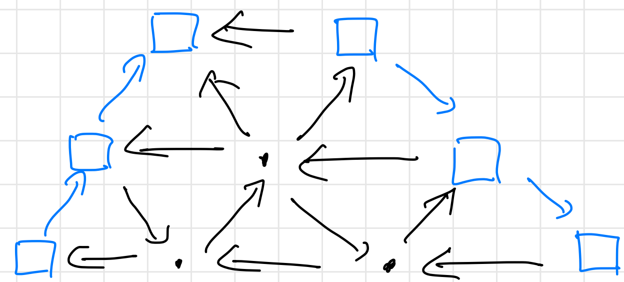
More generally, a cluster alg with coefficients is associated with an ice quiver $(\tilde{Q}, F)$.

Ex.









Additive Categorification

$\mathcal{C}$: triangulated Category

$$CC : \mathrm{obj}(\mathcal{C})/\sim \longrightarrow \mathcal{A}_Q.$$

such that

(1) $\left\{ \begin{array}{l} \text{(reachable) indecomposable} \\ \text{rigid objects} \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} \text{Cluster} \\ \text{variables} \end{array} \right\}$

(2) $\left\{ \begin{array}{l} \text{(reachable) Cluster-tilting} \\ \text{objs } T = \bigoplus_{i=1}^{n} T_i \end{array} \right\} \longmapsto \{\text{Clusters}\}$

$$T \longmapsto \{ CC(T_i) \}.$$
$$\{x_1, \cdots x_n\}$$

Constructions:

(1) Q: acyclic quiver

$\mathcal{C}_Q$: Buan-Marsh-Reineke-Reiten-Todorov.

CC: Caldero-Chapoton.

(2) (Q, W): Quiver with Potential

$\mathcal{C}(Q, W)$: Generalized Cluster Category (Amiot) (Hom-finite)

CC: Derksen-Weyman-Zelevinsky.
Palu.

(3) (Q, W): any Quiver with Potential

$\mathcal{D}(Q,W)$: Plamondon. $\mathcal{D}(Q,W) \hookrightarrow \mathcal{C}(Q,W)$

CC : he also constructed the CC map.

(1), (2), (3) : Categorification of cluster alg without coeffs.

(4) If Q comes from Lie theory, the category $\mathcal{C}$ would be a Frobenius category.

c.f. the series of works by Geiß-Leclerc-Schröer

Aim : Under some technical conditions, we generalize (2), (3), (4) to ice quivers with potential. (Jacobi-infinite).

2. Ice quiver with potentials and Ginzburg functors.

(Q, F).

Let W be a potential on Q, i.e. an element in

$$HH_0(\widehat{kQ}) = \frac{\widehat{kQ}}{\overline{[\widehat{kQ}, \widehat{kQ}]}}.$$

formal sum of cycles.

Ex.

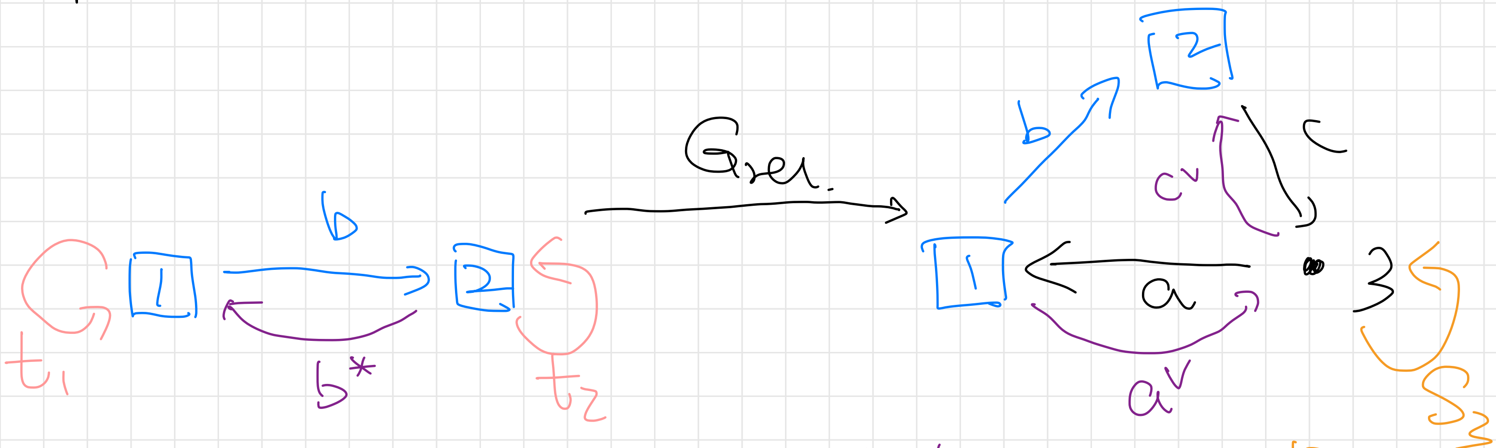
$$\begin{tikzcd} & \boxed{2} \arrow[dr, "c"] & \\ \boxed{1} \arrow[ur, "b"] & & 3 \arrow[ll, "a"] \end{tikzcd}$$

$W = cba.$

$(Q, F, W) \rightsquigarrow$ Ginzburg functor

$$G_{rel} : \underbrace{\Pi_2(kF)}_{\text{2-Calabi-Yau completion of } kF} \longrightarrow \underbrace{\Gamma_{rel}(Q, F, W)}_{\text{relative Ginzburg dg alg.}}$$

Example:



$|t_1| = |t_2| = -1$ $\quad |b^*| = 0$

$|c^\vee| = |a^\vee| = -1$. $\quad |s_3| = -2$

$d(t_1) = b\, b^*$.

$d(t_2) = -\, b^* b$

$d(s_3) = cc^\vee - a^\vee a$.

$d(a^\vee) = \partial_a W = cb$

$d(c^\vee) = \partial_c W$

Remark:

(1) $\mathcal{G}_{rel}$ is the relative deformed 3-CY completion of $kF \hookrightarrow kQ$ w.r.t to the potential W. (Yeung)

(2) $\mathcal{G}_{rel}$ has a canonical relative left 3-CY structure (Brav-Dyckerhoff)

For a dg alg A, we introduce the following notations:

- $D(A) = \mathcal{C}(A)[q, s^{-1}]$

- Perfect derived Category

$$\mathrm{Per}(A) = \mathrm{thick}(A_A) \subseteq D(A)$$

- Perfectly valued derived Category

$$\mathrm{pvd}(A) = \{ M \in D(A) \mid M|_k \in \mathrm{per}\, k \}.$$

We say A is <u>Smooth</u> if $A \in \mathrm{per}(A^e)$

<u>Ex</u>. $\Pi_2(kF)$, $\Gamma_{rel}(Q, F, W)$ are smooth.

$\Longrightarrow$ $\mathrm{pvd}(A) \subseteq \mathrm{per}(A)$ (A smooth).

Def. The relative cluster category $\mathcal{C}(Q,F,W)$ is defined as the idempotent completion of

$$\frac{\operatorname{per}\Gamma_{rel}}{\operatorname{pvd}_F(\Gamma_{rel})}$$

where $\operatorname{pvd}_F(\Gamma_{rel}) = \{ M \in \operatorname{pvd}\Gamma_{rel} \mid M|_{e_F} = 0 \}$

$= \operatorname{thick}\langle S_i \mid i \in F_0 \rangle$.

In this talk, $H^0(\Gamma_{rel})$ is infinite-dimensional.
(Jacobian-alg)

- $$\mathrm{pr}^F \mathcal{T}_{rel} = \left\{ \mathrm{Cone}(X_1 \xrightarrow{f} X_0) \;\middle|\; \begin{array}{l} X_i \in \mathrm{add}\, \mathcal{T}_{rel}, \\ \begin{tikzcd} X_1 \arrow[r, "f"] \arrow[d, "\forall \alpha"'] & X_0 \arrow[dl, dashed, "\exists"] \\ I & \end{tikzcd} \quad I \in \mathrm{add}(e_F \mathcal{T}_{rel}). \end{array} \right\}$$

$$\subseteq \mathrm{per}\, \mathcal{T}_{rel}$$

- $$\mathrm{copr}^F \mathcal{T}_{rel} = \left\{ \Sigma^{-1} \mathrm{Cone}(X^0 \xrightarrow{f} X^1) \;\middle|\; \begin{array}{l} X^i \in \mathrm{add}(\mathcal{T}_{rel}). \\ \begin{tikzcd} & P \arrow[d, "\forall \alpha"] \arrow[dl, dashed, "\exists"'] \\ X^0 \arrow[r, "f"'] & X^1 \end{tikzcd} \\ P \in \mathrm{add}(e_F \mathcal{T}_{rel}) \end{array} \right\}$$

$$\begin{tikzcd}
\mathrm{Per}\,\mathcal{T}_{rel} \arrow[r, "\pi_{rel}"] & \mathcal{C}(Q,F,W) = \dfrac{\mathrm{Per}\,\mathcal{T}_{rel}}{\mathrm{prd}_F(\mathcal{T}_{rel})} \\
\cup & \\
\mathrm{Pr}^F\mathcal{T}_{rel} \arrow[uur, hook] & \\
\mathrm{CoPr}^F\mathcal{T}_{rel} \arrow[uuur, hook] \arrow[uuu, bend left] &
\end{tikzcd}$$

Def : We define <u>Higgs Category</u> $\mathcal{H}(Q,F,W)$ as

$$\mathcal{H}(Q,F,W) = \left\{ M \in \mathcal{T}_{rel}(\mathrm{Pr}^F\mathcal{T}_{rel} \cap \mathrm{CoPr}^F\mathcal{T}_{rel}) \;\middle|\; \mathrm{Ext}^1_{\mathcal{C}(Q,F,W)}(\mathcal{T}_{rel}, M) \text{ is finite-dimensional} \right\}$$

$\subseteq \mathcal{C}(Q, F, W)$

<u>Theorem</u>. The Higgs Category $\mathcal{H}(Q, F, W)$ is an extensional Closed subCategory of $\mathcal{C}(Q, F, W)$.

Hence it becomes an extriangulated Category in the sense of Nakaoka-palu.

<u>Assumption</u>: ① $\mathcal{P} = \mathrm{add}(e_F \Gamma_{\mathrm{rel}})$ is functorially finite in $\mathrm{add}\,\Gamma_{\mathrm{rel}}$

Theorem: Under this assumption,

(1) The additive quotient $\dfrac{\mathcal{H}(Q,F,W)}{[\mathcal{P}]}$ is equivalent to

$$\mathcal{D}(\underline{\bar{Q},\bar{W}})$$ (plamondon).

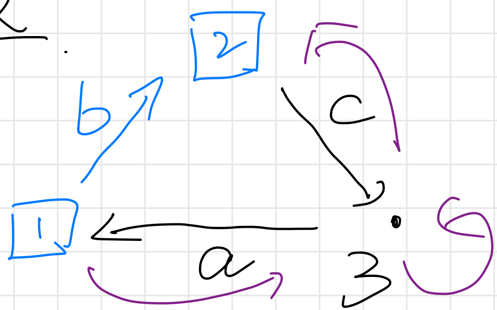
obtained from (Q,F,W) by deleting the frozen part.

(2) If moreover, $(\bar{Q},\bar{W})$ is Jacobi-finite, then $\mathcal{H}(Q,F,W)$ is Frobenius extriangulated category with

proj-inj objects $\mathcal{P} = \mathrm{add}(e_F T_{rel})$. And T_{rel} is a canonical cluster-tilting obj of $\mathcal{H}$ with endomorphism alg $\mathrm{End}_{\mathcal{H}}(T_{rel}) = H^0(T_{rel})$.

Example

(1)



$W = cba$.

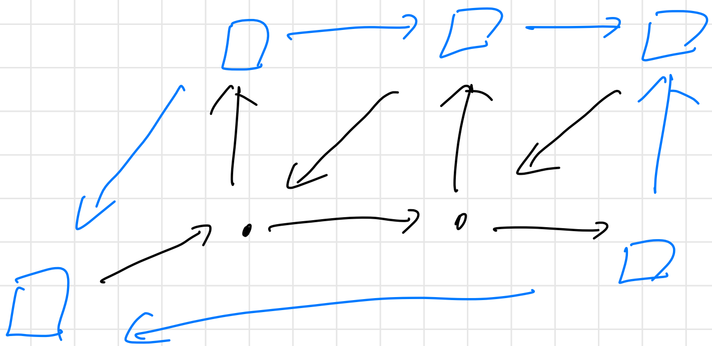
① $T_{rel} \xrightarrow{\sim} H^0(T_{rel})$. finite dimensional.

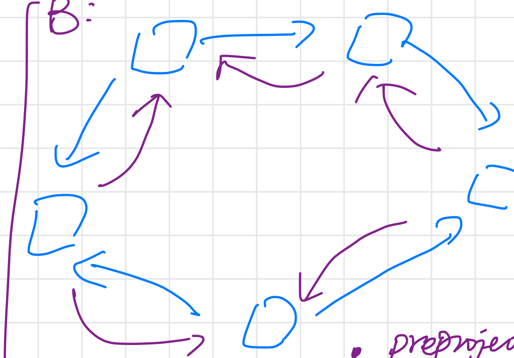
② The Higgs Category is equivalent to

$\mathrm{mod}\, B$. $B = e_F T_{rel}\, e_F$

$=$ preprojective alg of type A_2.

(2)





• preprojective relation

• ....

• $T_{rel} \xrightarrow{\sim} H^P(T_{rel})$.   Infinite-dimensional

• This Higgs Category $\mathcal{H} \xrightarrow{\sim} CM(B)$.

(3)

$$\begin{tikzcd} 1 \arrow[r] \arrow[d] & 2 \arrow[d] \\ \boxed{1'} & \boxed{2'} \end{tikzcd}$$

principle case.

- Tree not concentrated in degree 0.