On $\mathcal{J}$-representation types with examples from the representation theory of valued quivers

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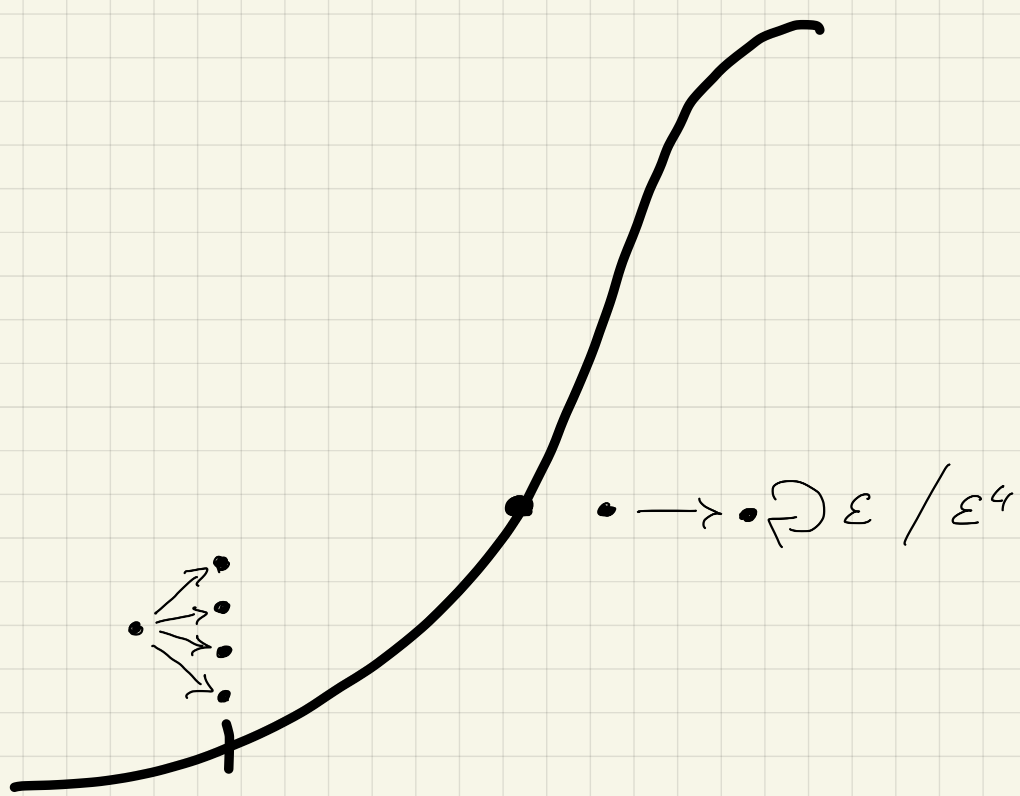
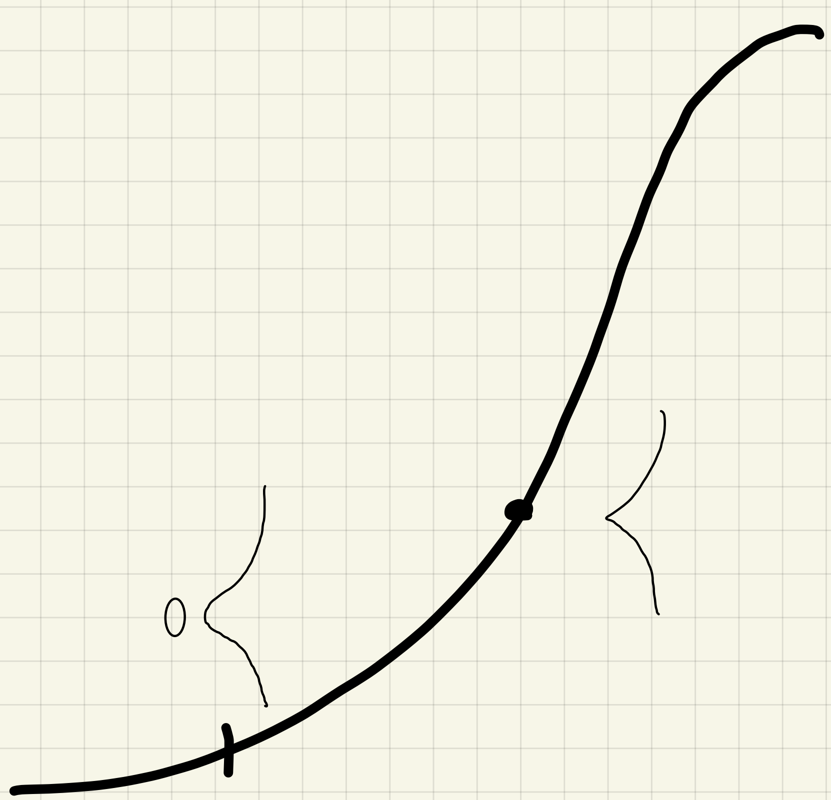
FD Seminar

[P'23]: arXiv:2308.09576 & arXiv:2308.09587





Setting

K : algebraically closed field of characteristic 0.

$A = KQ/I$ basic finite-dimensional associative and unital K-algebra.

$\nearrow$ quiver $\nwarrow$ admissible ideal

$\mathrm{mod}(A)$:= category of finite-dimensional left A-modules.

$\cup$

$\mathrm{proj}(A)$:= full subcategory of projective A-modules.

τ_A : Auslander-Reiten translation for A.

$K_0(A) := K_0(\mathrm{mod}\,A) = \mathbb{Z}Q_0$ Grothendieck group of $\mathrm{mod}(A)$.

$\cup$

$K_0(A)^+ = \mathbb{N}Q_0$ submonoid of dimension vectors $\underline{\dim}(V)$ for $V \in \mathrm{mod}(A)$.

$K_0(\mathrm{proj}\,A)_{\mathbb{R}} := K_0(\mathrm{proj}\,A) \underset{\mathbb{Z}}{\otimes} \mathbb{R}$ $\qquad$ $K_0(A)^*_{\mathbb{R}} := \mathrm{Hom}_{\mathbb{Z}}(K_0(A), \mathbb{R})$

Euler pairing: $\langle -,- \rangle_A : K_0(\mathrm{proj}\,A) \times K_0(A) \to \mathbb{Z}$, $\langle P, V \rangle := \dim_K \mathrm{Hom}_A(P, V)$.

τ-tilting theory

Def: [AIR] $(V,P) \in \mathrm{mod}(A) \times \mathrm{proj}(A)$ is **τ-rigid** if

* $\mathrm{Hom}_A(P,V) = 0$
* $\mathrm{Hom}_A(V, \tau_A V) = 0$

Stability

Def: [King] $\theta \in K_0(A)^*_{\mathbb{R}}$. $V \in \mathrm{mod}(A)$ is **θ-semistable** if

* $\theta(V) = 0$
* $\theta(U) \leq 0 \;\forall\, 0 \neq U \subsetneq V$.

it is **θ-stable** if strict inequality holds.

Interplay of τ-tilting theory and stability: [DIJ], [BST], [Asai], [AI], ...

representation finite
finitely many indecomposable modules in $\mathrm{mod}(A)$

$\Swarrow$ $\qquad\qquad$ $\Searrow$

τ-tilting finite $\xRightarrow{[DIJ]}$ **stability finite**

τ-tilting finite: finitely many indecomposable τ-rigid modules in $\mathrm{mod}(A)$.

stability finite: finitely many stable modules in $\mathrm{mod}(A)$.

τ-tilting finite $\overset{[DIJ]}{\Updownarrow}$ g-finite

stability finite $\Downarrow$ wall finite

g-finite $\xLeftarrow{[BST]}$ **wall finite**

g-finite: finitely many cones in the g-vector fan $\mathrm{Fan}(A) \subseteq K_0(\mathrm{proj}\,A)_{\mathbb{R}}$

wall finite: finitely many walls in $K_0(A)^*_{\mathbb{R}}$
$\mathcal{W}(V) := \{\theta \mid V \text{ is } \theta\text{-semistable}\}$

g-finite $\overset{[DIJ]}{\Downarrow}$ rationally g-complete $\qquad$ g-finite $\overset{?}{\Uparrow}$ rationally g-complete

rationally g-complete

$K_0(\mathrm{proj}\,A)_{\mathbb{Z}} \subseteq \mathrm{Fan}(A)$

Demonet's conjecture:
Suppose A is τ-tilting infinite.
Then, there exists $\gamma \in K_0(\mathrm{proj}\,A)$ such that $\gamma \notin \mathrm{Fan}(A)$.

Brauer-Thrall I

Thm: [Roiter] Suppose A is representation infinite.
For every $d \geq 0$ exists an indecomposable
$V \in \mathrm{mod}(A)$ with $\dim_K(V) \geq d$.

τ-rigid BT I

Prop: Suppose A is τ-tilting infinite.
For every $d \geq 0$ exists an indecomposable
τ-rigid $V \in \mathrm{mod}(A)$ with $\dim_K(V) \geq d$.

Pf: Let $d \in K_0(A)^+$.

$\mathrm{Rep}(A,d)$:= affine variety of representations of A with dimension vector d.

$\circlearrowleft$

$GL(K,d) := \prod_{i \in Q_0} GL(K,d_i)$

$V \in \mathrm{Rep}(A,d) \rightsquigarrow \mathcal{O}(V) := GL(K,d).V$

V τ-rigid $\overset{\text{Voigt}}{\Longrightarrow}$ $\mathcal{O}(V) \subseteq \mathrm{Rep}(A,d)$ open

$\Longrightarrow$ $\overline{\mathcal{O}(V)} \subseteq \mathrm{Rep}(A,d)$ irreducible component.

$\Rightarrow$ $\exists < \infty$ τ-rigid modules with dimension vector d. $\square$

stable BT I

Thm: [ST] Suppose A is τ-tilting infinite.
For every $d \geq 0$ exists a stable $V \in \mathrm{mod}(A)$
with $\dim_K(V) \geq d$.

Pf: [ST] using g- & c-vectors
works over arbitrary fields.
[MP] using similar strategy as for
τ-rigid BT I over $K = \overline{K}$.

Notation: $\mathrm{Irr}(A) := \bigcup_{d \in K_0(A)^+} \{Z \mid Z$ is an irreducible component of $\mathrm{Rep}(A,d)\}$

Brauer-Thrall II

Thm: [B] Suppose A is representation infinite.
There exist $d \in K_0(A)^+$ and ∞-many indecomposable
$V \in \operatorname{mod}(A)$ with $\underline{\dim}(V) = d$.

Def: [GLS'12] $\mathcal{Z} \in \operatorname{Irr}(A)$ is τ-reduced if

generic number of parameters $c_A(\mathcal{Z}) = \hom_A^\tau(\mathcal{Z})$ $:= \min_{V \in \mathcal{Z}} \dim_K \operatorname{Hom}_A(V, \tau_A V)$

Rk: "$\leq$" holds by a Lemma of Voigt

$$\begin{array}{ccc} V & & \{V \in \operatorname{mod}(A)\ \tau\text{-rigid}\}/\cong \\ \downarrow & & \updownarrow 1:1 \\ \overline{\mathcal{O}(V)} & & \{\mathcal{Z} \in \operatorname{Irr}(A)\ \tau\text{-reduced with } c_A(\mathcal{Z}) = 0\}. \end{array}$$

Thm: [King] Let $\theta \in K_0(A)^*$, $d \in K_0(A)$.
There exists a quasi-projective variety
$\mathcal{M}od(A, d, \theta)^{st}$
parametrizing isomorphism classes
of θ-stable A-modules $V \in \operatorname{Rep}(A, d)$.

τ-reduced BT II

Conj: Suppose A is τ-tilting infinite.
There exist $d \in K_0(A)^+$ and $\mathcal{Z} \in \operatorname{Irr}(A)$
such that $\mathcal{Z}$ is τ-reduced with $c_A(\mathcal{Z}) \geq 1$.

stable BT II

Conj: Suppose A is τ-tilting infinite.
There exist $d \in K_0(A)^+$ and $\theta \in K_0(A)^*$
such that $\dim \mathcal{M}od(A, d, \theta)^{st} \geq 1$.

Remarks on the history:

- BT I & II < 1950 first published in [Jans]
- brick BT II first introduced in [Mo2] stated with this name in [STV].
- Conj: [Mo2] Dense orbit property $\Rightarrow$ τ-tilting finite.

Verified for:

- hereditary algebras. ✓
- minimal rep.-infinite special biserial and
  " " " non-distributive [Mo182]
- special biserial algebras [STV]
- biserial algebras [MP]

Brauer-Thrall II

Thm: [B] Suppose A is representation infinite.
There exist $d \in K_0(A)^+$ and ∞-many indecomposable $V \in \mathrm{mod}(A)$ with $\underline{\dim}(V) = d$.

τ-reduced BT II

$\Longleftarrow ? \Longrightarrow$

stable BT II

Conj: Suppose A is τ-tilting infinite.
There exist $d \in K_0(A)^+$ and $Z \in \mathrm{Irr}(A)$ such that Z is τ-reduced with $c_A(Z) \geq 1$.

Conj: Suppose A is τ-tilting infinite.
There exist $d \in K_0(A)^+$ and $\theta \in K_0(A)^*$ such that $\dim \mathcal{M}od(A, d, \theta)^{st} \geq 1$.

[Pla]

[Jasso, BST]

Demonet's conjecture

Conj: Suppose A is τ-tilting infinite.
Then, there exists $g \in K_0(\mathrm{proj}\, A)$ such that $g \notin \mathrm{Fan}(A)$.

Representation tame

Indecomposables of any fixed dimension are parametrized by finitely many points and rational curves.

$\Downarrow$ [CC]

generically τ-reduced tame

For all generically indecomposable and τ-reduced $\mathcal{Z} \in \operatorname{Irr}(A)$ is

$$c_A(\mathcal{Z}) \leq 1.$$

$\Downarrow$ [GLFS]

E-tame [DF, AI]

For all τ-reduced $\mathcal{Z} \in \operatorname{Irr}(A)$ is

$$\hom_A^{\tau}(\mathcal{Z}, \mathcal{Z}) = 0$$

$$\min_{V,W \in \mathcal{Z}} \dim_K \operatorname{Hom}_A(V, \tau_A W)$$

$\Downarrow$ [PY] (Representation tame $\Rightarrow$ g-tame)

$\Downarrow$ [CC] (Representation tame $\Rightarrow$ stability tame)

stability tame

For all $\theta \in K_0(A)^*$, $d \in K_0(A)^+$ is

$$\dim \operatorname{Mod}(A, d, \theta)^{st} \leq 1$$

g-tame [AY]

$$K_0(\operatorname{proj} A)_{\mathbb{Z}} \subseteq \overline{\operatorname{Fan}(A)}$$

$\Leftarrow$ [Asai, PY]

wall tame [BST]

$$\overline{\bigcup_{V \neq 0} \mathcal{W}(V)} \subseteq K_0(A)^*_{\mathbb{R}}$$

has finite measure.

τ-reduced BT II $\Longleftarrow ? \Longrightarrow$ stable BT II

Conj: Suppose A is τ-tilting infinite.
There exist $d \in K_0(A)^+$ and $Z \in \operatorname{Irr}(A)$
such that Z is τ-reduced with $c_A(Z) \geq 1$.

Conj: Suppose A is τ-tilting infinite.
There exist $d \in K_0(A)^+$ and $\Theta \in K_0(A)^*$
such that $\dim \operatorname{Mod}(A, d, \Theta)^{st} \geq 1$.

[Pla] [Jasso, BST]

Demonet's conjecture

Conj: Suppose A is τ-tilting infinite.
Then, there exists $\gamma \in K_0(\operatorname{proj} A)$
such that $\gamma \notin \operatorname{Fan}(A)$.

Thm: [P'23] Suppose A is E-tame. Then, the following implication holds:

$$A \text{ satisfies } \tau\text{-reduced BT II} \Longrightarrow A \text{ satisfies the stable BT II}$$

Pf Idea: Given τ-reduced $Z \in \operatorname{Irr}(A)$ with $c_A(Z) \geq 1$

generic g-vector [Pla]

Consider $\Theta := \langle g(Z), - \rangle_A \in K_0(A)^*$

Construct Θ-stable modules as factors of generic elements of Z. $\square$

Rk: It is enough to prove Demonet's conjecture for g-tame algebras.

An example of affine type $\mathbb{B}\mathbb{C}_1$

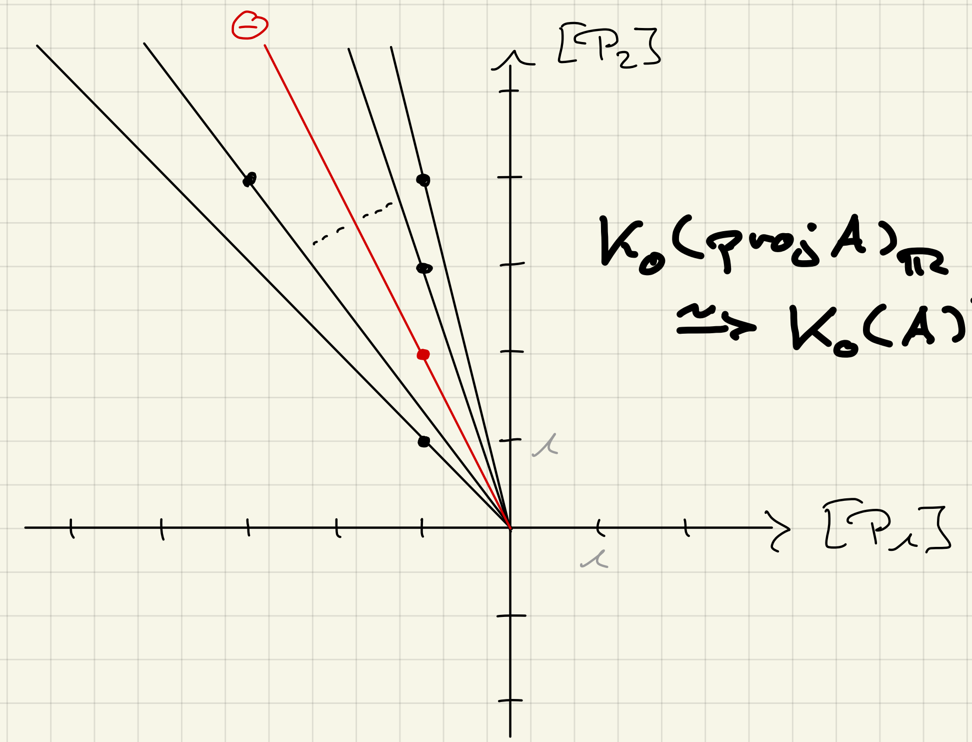
$A := KQ/I$, $\;Q: 2 \longrightarrow 1 \circlearrowleft \varepsilon$, $\;I := \langle \varepsilon^4 \rangle$. is representation wild

τ-tilting exchange quiver

$$\begin{tikzcd}[column sep=small]
 & (1,0)(0,1) \arrow[dl] \arrow[dr] & \\
(-1,4)(0,1) \arrow[d] & & \\
(-1,4)(-1,3) \arrow[d] & & (1,0)(0,-1) \arrow[ddddl] \\
\vdots \arrow[d] & & \\
(-3,4)(-1,1) \arrow[d] & & \\
(-1,0)(-1,1) \arrow[dr] & & \\
 & (-1,0)(0,-1) &
\end{tikzcd}$$

wall & chamber structure

$\Theta := (-1,2) \in K_0(A)^*$



$K_0(\operatorname{proj} A)_{\mathbb{R}} \xrightarrow{\sim} K_0(A)^*_{\mathbb{R}}$

$\Longrightarrow$

- * minimal τ-tilting infinite [Mo, Wang]
- * wall & g-tame
- * stability tame
- * generically τ-reduced tame

Θ-stable modules: $\overline{V}_\infty := K \xrightarrow{\begin{bmatrix}1\\0\end{bmatrix}} K^2 \circlearrowleft \begin{bmatrix}0&0\\1&0\end{bmatrix}$, $\;V_\lambda := K^2 \xrightarrow{\begin{bmatrix}1&0\\0&1\\0&\lambda\\0&0\end{bmatrix}} K^4 \circlearrowleft \begin{bmatrix}0&0&0&0\\1&0&0&0\\0&1&0&0\\0&0&1&0\end{bmatrix}$ for $\lambda \in K$.

Thm: [P'23] These are pairwise Hom-orthogonal with $\tau_A(V_\lambda) \cong V_\lambda$ $\forall \lambda \in K$.
Moreover, every Θ-stable A-module is isomorphic to $\overline{V}_\infty$ or V_λ for some $\lambda \in K$.

Valued quivers & GLS algebras

A valued quiver Γ consists of

* Γ_0 : a finite set of vertices
* $\Gamma_1 \subseteq \Gamma_0 \times \Gamma_0$ a set of half edges s.t. $(i,j) \in \Gamma_1 \Rightarrow (j,i) \in \Gamma_1$ and $i \neq j$.
* $\Omega \subseteq \Gamma_1$ an orientation s.t. $(i,j) \in \Omega \Rightarrow (j,i) \notin \Omega$.
  ← assumed to be acyclic.
* $v : \Gamma_1 \to \mathbb{N}_{\geq 1}$ a valuation
* $c : \Gamma_0 \to \mathbb{N}$ a symmetrizer s.t. $c_i v_{ij} = c_j v_{ji}$ $\forall (j,i) \in \Gamma_1$.

$$\underset{c_i}{i} \xrightarrow{v_{ji} \mid v_{ij}} \underset{c_j}{j}$$

Rk: Equivalent to the datum (C, Ω, D) of a generalized Cartan matrix C with orientation Ω and symmetrizer D.

↦ [GLS'17] Quiver $Q = Q(\Gamma)$

* $Q_0 := \Gamma_0$,
* $Q_1 := \{ \alpha_{ji}^{(g)} : i \to j \mid (j,i) \in \Omega,\ 1 \leq g \leq g_{ji} := \gcd(v_{ji}, v_{ij}) \}$
  $\cup \{ \varepsilon_i : i \to i \mid i \in \Gamma_0 \}$.

$$\begin{tikzcd} i \arrow[loop left, "\varepsilon_i"] \arrow[r, shift left=2, "\alpha_{ji}^{(1)}"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2, "\alpha_{ji}^{(g_{ji})}"'] & j \arrow[loop right, "\varepsilon_j"] \end{tikzcd}$$

with ideal $I \trianglelefteq KQ$ generated by relations

* nilpotency: $\varepsilon_i^{c_i} = 0$ $\forall i \in \Gamma_0$,
* commutativity: $\varepsilon_j^{f_{ij}} \alpha_{ji}^{(g)} = \alpha_{ji}^{(g)} \varepsilon_i^{f_{ji}}$ $\forall (i,j) \in \Omega$.

↦ GLS algebra

$H(\Gamma) := KQ/I$.

Valued quivers & GLS algebras

$\rightsquigarrow$ [GLS'17] Quiver $Q = Q(\Gamma)$

* $Q_0 := \Gamma_0$,
* $Q_1 := \{\alpha_{ji}^{(g)} : i \to j \mid (j,i)\in\Omega,\ 1\le g\le g_{ji} := \gcd(v_{ji}, v_{ij})\}$
  $\cup \{\varepsilon_i : i \to i \mid i\in\Gamma_0\}$.

$$\varepsilon_i \circlearrowright i \underset{\alpha_{ji}^{(g_{ji})}}{\overset{\alpha_{ji}^{(1)}}{\underset{\vdots}{\rightrightarrows}}} j \circlearrowleft \varepsilon_j$$

with ideal $I \trianglelefteq KQ$ generated by relations $\rightsquigarrow$ GLS algebra $H(\Gamma) := KQ/I$.

* nilpotency: $\varepsilon_i^{c_i} = 0 \ \forall i\in\Gamma_0$,
* commutativity: $\varepsilon_j^{f_{ij}}\alpha_{ji}^{(g)} = \alpha_{ji}^{(g)}\varepsilon_i^{f_{ji}} \ \forall (i,j)\in\Omega$. where $f_{ji} := v_{ji}/g_{ji}$.

Thm: Let Γ be a valued quiver and $H := H(\Gamma)$.

* [GLS'17] H is 1-Iwanaga-Gorenstein
* [Pla'13] $\xRightarrow{[P'23]}$ $Z\in\operatorname{Irr}(H)$ is τ-reduced iff Z is generically locally free

V_i is free $K[\varepsilon_i]/\langle\varepsilon_i^{c_i}\rangle$-module $\forall i\in Q_0$.

[GLS'18]: $\{Z\in\operatorname{Irr}(H) \text{ gen. loc. fr.}\} \xrightarrow{1:1} \mathbb{N}\Gamma_0$

$Z(r) \longmapsfrom r$

$Z \longmapsto \mathrm{rk}(Z)$ generic rank vector

rank $\mathrm{rk}(V) := \left(\mathrm{rk}_{K[\varepsilon_i]/\varepsilon_i^{c_i}}(V_i)\right)_{i\in Q_0}$

* [GLS'20] H is a (formal) degeneration of a species over $K((\varepsilon))$

$\rightsquigarrow$ H is τ-tilting finite if and only if Γ is of finite type (i.e. Tits form q_Γ is positive definite)

$\rightsquigarrow$ H is g-tame if and only if Γ is of affine type (i.e. q_Γ is pos. semidefinite)

Valued quivers & GLS algebras

Examples: * Γ simply laced (i.e. $c_i = 1\ \forall i \in \Gamma_0$) $\Longrightarrow$ H is ordinary path algebra.

* $\Gamma: \underset{1}{2} \xrightarrow{1|4} \underset{4}{1}$ affine type $\widetilde{\mathbb{BC}}_1$ $\rightsquigarrow H = K(2 \longrightarrow 1 \circlearrowleft \varepsilon)/\langle \varepsilon^4 \rangle \xrightarrow[\varepsilon^4 = te_1]{\text{defo.}}$ path alg. of
$$\begin{tikzcd} & 1 \\ & 1 \\ 2 \arrow[ur] \arrow[uur] \arrow[r] \arrow[dr] & 1 \\ & 1 \end{tikzcd}$$

* $\Gamma: \underset{2}{3} \xrightarrow{2|1} \underset{1}{2} \xrightarrow{1|2} \underset{2}{1}$ affine type $\widetilde{\mathbb{C}}_2$ $\rightsquigarrow H = K(\overset{\varepsilon_3}{\circlearrowleft}3 \longrightarrow 2 \longrightarrow \overset{\varepsilon_1}{\circlearrowleft}1)/\langle \varepsilon_3^2, \varepsilon_1^2 \rangle$ gentle

$\xrightarrow[\varepsilon_i^{c_i} = te_i]{\text{defo.}}$ path algebra of
$$\begin{tikzcd} 3 \arrow[dr] & & 1 \\ & 2 \arrow[ur] \arrow[dr] & \\ 3 \arrow[ur] & & 1 \end{tikzcd}$$

* $\Gamma: \underset{1}{4} \xrightarrow{1|2} \underset{2}{3} \longrightarrow \underset{2}{2} \xrightarrow{2|1} \underset{1}{1}$ affine type $\widetilde{\mathbb{B}}_3$ $\rightsquigarrow H = K(4 \longrightarrow \overset{\varepsilon_3}{\circlearrowleft}3 \xrightarrow{\alpha} \overset{\varepsilon_2}{\circlearrowleft}2 \longrightarrow 1)/\left\langle \begin{matrix} \varepsilon_3^2, \varepsilon_2^2, \\ \varepsilon_2\alpha = \alpha\varepsilon_3 \end{matrix} \right\rangle$

$\xrightarrow[\varepsilon_i^{c_i} = te_i]{\text{defo.}}$ path algebra of
$$\begin{tikzcd} & 3 \arrow[r] & 2 \arrow[dr] & \\ 4 \arrow[ur] \arrow[dr] & & & 1 \\ & 3 \arrow[r] & 2 \arrow[ur] & \end{tikzcd}$$

* $\Gamma: \underset{1}{3} \xrightarrow{1|2} \underset{2}{2} \xrightarrow{1|2} \underset{4}{1}$ affine type $\widetilde{\mathbb{BC}}_2$ $\rightsquigarrow H = K(3 \longrightarrow \overset{\varepsilon_2}{\circlearrowleft}2 \xrightarrow{\alpha} \overset{\varepsilon_1}{\circlearrowleft}1)/\left\langle \begin{matrix} \varepsilon_2^2, \varepsilon_1^4, \\ \varepsilon_1^2\alpha = \alpha\varepsilon_2 \end{matrix} \right\rangle$

$\xrightarrow[\varepsilon_i^{c_i} = te_i]{\text{defo.}}$ path algebra of
$$\begin{tikzcd} & & 1 \\ & 2 \arrow[ur] \arrow[dr] & \\ & & 1 \\ 3 \arrow[uuur] \arrow[dr] & & \\ & & 1 \\ & 2 \arrow[ur] \arrow[dr] & \\ & & 1 \end{tikzcd}$$

Affine GLS algebras

<u>Thm</u>: [P'23] Suppose Γ is connected of affine type with minimal symmetrizer. (i.e. $\gcd(c_i) = 1$)

Then $H := H(\Gamma)$ is generically τ-reduced tame.

More precisely, every $v \in \mathbb{N}\Gamma_0$ can be decomposed as

$$v = m \cdot \eta + w, \quad (\text{"generalized Kac decomposition"})$$

for some $w \in \mathbb{N}\Gamma_0$, $m \in \mathbb{N}$ and $\eta \in \mathbb{N}\Gamma_0 \setminus \{0\}$ the primitive null root (i.e. minimal with $q_\Gamma(\eta) = 0$)

such that

$$Z(v) = \overline{Z(\eta)^m \oplus Z(w)},$$

$$Z(w) = \overline{\mathcal{O}(W)}, \qquad Z(\eta) = \overline{\bigcup_{\lambda \in \mathbb{P}^1} \mathcal{O}(V_\lambda)}$$

where $W \in \mathrm{mod}(A)$ is τ-rigid

and $V_\lambda \in \mathrm{mod}(A)$ is an explicit 1-parameter family of locally free modules

Moreover, for almost all $\lambda \in \mathbb{P}^1$ is

* $\tau_H(V_\lambda) \cong V_\lambda$
* V_λ is ∂-stable for the defect $\partial := \langle \eta, - \rangle_H : K_0(A)^*$.

<u>Question</u>: Are these together with the stable modules associated to τ-rigid modules all stable H-modules i.e. is H stability tame?

c.f. [BDG] on plane degenerations of elliptic curves.

Thank

You!

Abbreviation of names: (in order of appearance)

AIR : Adachi - Iyama - Reiten.

AI : Asai - Iyama

BST : Brüstle - Smith - Treffinger

DIJ : Demonet - Iyama - Jasso

ST : Schroll - Treffinger

MP : Mousavand - Paquette

B : Bautista

GLS : Geiß - Leclerc - Schröer

Mo : Mousavand

STV : Schroll - Treffinger - Valdivieso

Pla : Plamondon

CC : Carroll - Chindris

GLFS : Geiß - Labardini - Fragoso - Schröer

DF : Derksen - Fei

AY : Aoki - Yurikusa

PY : Plamondon - Yurikusa

BDG : Bodnarchuk - Drozd - Greuel