FD seminar

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Two-term silting and τ-cluster morphism categories

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Plan:

§0: Setting and notation

§1: Two-term silting and τ-tilting

compatibility of reduction

§2: t-exactness and τ-cluster morphism categories

§3: New approaches

§0: Setting:

Let k be a field and let A be a non-positive proper dg k-algebra.

$$A = \cdots \xrightarrow{d} A^{-2} \xrightarrow{d} A^{-1} \xrightarrow{d} A^{0} \xrightarrow{d} A^{1} \xrightarrow{d} \cdots$$

w/ $d^2 = 0$ and an associative binary operation $\mu : A \otimes A \longrightarrow A$, $|\mu| = 0$,

s.t. i) $H^i A = 0 \quad \forall\, i > 0$,

ii) $\bigoplus_{i \in \mathbb{Z}} H^i A$ is finite-dimensional,

Rem. $H^0 A$ is a finite-dimensional k-algebra.

B denotes a finite-dimensional k-algebra throughout.

Let $D(A)$ be the derived category of A, where Σ denotes the suspension.

$$\mathrm{per}(A) \stackrel{\mathrm{def}}{=} \mathrm{thick}_{DA}(A) \subseteq D(A).$$

$$D_{fd}(A) \stackrel{\mathrm{def}}{=} \{ X \in D(A) \mid \dim_k \Big(\bigoplus_{i \in \mathbb{Z}} H^i X \Big) < \infty \} \subseteq D(A),$$

Since A is proper, we have that $\mathrm{per}(A) \subseteq D_{fd}(A)$.

§1:

Def. An object $P \in \mathrm{per}(A)$ is <u>two-term presilting</u> if

i) $P = \mathrm{cone}(P_1 \longrightarrow P_0)$, where $P_0, P_1 \in \mathrm{add}(A)$

ii) $\mathrm{Hom}_{\mathrm{per}(A)}(P, \Sigma^i P) = 0$ for all $i > 0$.

It is a <u>silting object</u> if, in addition:

iii) $\mathrm{thick}_{\mathrm{per}(A)}(P) = \mathrm{per}(A)$.

Notation: 2-presilt(A) and 2-silt(A), respectively.

$$\text{2-presilt}_P(A) = \{ X \in \text{2-presilt}(A) \mid X = P \oplus R \text{ for some } R \}.$$

Facts about two-term (pre)silting objects: (cf. [Iyama–Jørgensen–Yang])

i) Two-term presilting are precisely the direct summands of two-term silting objects.

ii) For $P \in 2\text{-presilt}(A)$, then $P \in 2\text{-silt}(A)$ iff $|P| = |H^0 A|$

iii) Indeed, one defines the Bongartz completion $T_P^+ \in 2\text{-silt}_P(A)$ of $P \in 2\text{-presilt}(A)$ by $T_P^+ = P \oplus Q^+$ where Q^+ is defined by the triangle

$$A \longrightarrow Q^+ \longrightarrow P' \xrightarrow{\beta} \Sigma A$$

and β is an $\mathrm{add}(P)$-approximation of ΣA.

Def. [Adachi–Iyama–Reiten]: Let B be a finite-dimensional $\Bbbk$-algebra.

A pair $(M,N) \in \mathrm{mod}(B) \times \mathrm{proj}(B)$ is <u>τ-rigid</u> if

i) $\mathrm{Hom}_B(M, \tau M) = 0$

ii) $\mathrm{Hom}_B(N, M) = 0$

It is <u>support τ-tilting</u> if, in addition

iii) $|M| + |N| = |B|$

Thm. [AIR]:

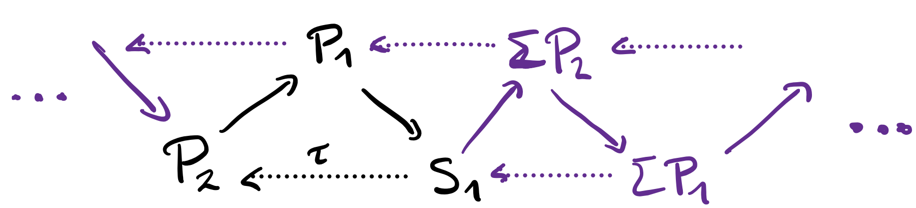
$$\begin{array}{ccc} H_A : 2\text{-presilt}(A) & \longrightarrow & \tau\text{-rigid pair}(A) \\ \cup\!\mid & & \cup\!\mid \\ 2\text{-silt}(A) & \longrightarrow & s\tau\text{-tilt}(A) \end{array}$$

where $H_A(N \oplus P_1' \xrightarrow{(0\ d)} P_0) \overset{\mathrm{def}}{=} (\mathrm{cok}\, d, N)$

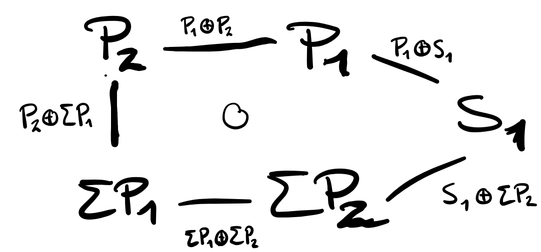
↳ maximal

A small example:

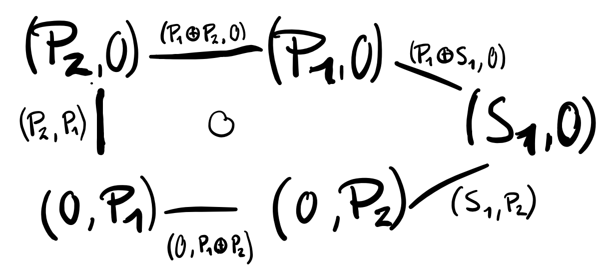
- Take $A = k[1 \leftarrow 2]$. The AR-quiver of $\mathrm{mod}(A)$ (resp. $\mathrm{per}(A)$) is



Then 2-presilt(A) =



and τ-rigid pair(A) =



Let $P \in 2\text{-presilt}(A)$. As $P \in \mathrm{per}(A)$, we have a recollement

$$\begin{tikzcd}
P^{\perp_{\mathbb{Z}}} \arrow[r, hook] \arrow[r, phantom, shift left=4, "\perp"] \arrow[r, phantom, shift right=4, "\perp"] & D(A) \arrow[l, bend right=40, "\pi_P"'] \arrow[l, bend left=40, leftarrow] \arrow[r] \arrow[r, phantom, shift left=4, "\perp"] \arrow[r, phantom, shift right=4, "\perp"] & \mathrm{Loc}(P) \arrow[l, bend right=40] \arrow[l, bend left=40]
\end{tikzcd},$$

where $P^{\perp_{\mathbb{Z}}} = \{ X \in D(A) \mid \mathrm{Hom}_{D(A)}(P, \Sigma^i X) = 0 \quad \forall i \in \mathbb{Z} \}$.

Facts:
- We can find a non-positive dg algebra C_P s.t. $P^{\perp_{\mathbb{Z}}} \cong D(C_P)$.
- The functor π_P induces a triangle equivalence $\mathrm{per}(A)/\mathrm{thick}(P) \longrightarrow \mathrm{per}(C_P)$.

Let T_P be the Bongartz completion of P. Then $H^0 C_P \simeq \mathrm{End}_{\mathrm{per}(A)}(T_P)/[\mathrm{add}(P)]$

Two-term silting reduction and support τ-tilting reduction

Thm. [Iyama–Yang]: The triangle equivalence $\pi_P : \mathrm{per}(A)/\mathrm{thick}(P) \longrightarrow \mathrm{per}(C_P)$ induces bijections

$$\begin{array}{ccc} \pi_P : 2\text{-}\mathrm{presilt}_P(A) & \longrightarrow & 2\text{-}\mathrm{presilt}(C_P) \\ \cup\!\shortmid & & \cup\!\shortmid \\ 2\text{-}\mathrm{silt}_P(A) & \longrightarrow & 2\text{-}\mathrm{silt}(C_P) \end{array}$$

Rem. (*) If $Q \in 2\text{-}\mathrm{presilt}(C_P)$, then $\pi_{\pi_P^{-1}(Q)} = \pi_Q \circ \pi_P$, i.e. this commutes

$$\begin{tikzcd} 2\text{-}\mathrm{presilt}_{\pi_P^{-1}(Q)}(A) \arrow[rr, "\pi_{\pi_P^{-1}(Q)}"] \arrow[dr, "\pi_P"'] & & 2\text{-}\mathrm{presilt}(C_{\pi_P^{-1}(Q)}) \\ & 2\text{-}\mathrm{presilt}_Q(C_P) \arrow[ur, "\pi_Q"'] & \end{tikzcd}$$

Thm. [Buan–Marsh]:

For any $(M,N) \in \tau\text{-rigid pair}(B)$, we have bijections

$$\begin{array}{ccc}
\mathcal{E}_{(M,N)} : \tau\text{-rigid pair}_{(M,N)}(B) & \longrightarrow & \tau\text{-rigid pair}(H^0 C_P) \\
\cup & & \cup \\
s\tau\text{-tilt pair}_{(M,N)}(B) & \longrightarrow & s\tau\text{-tilt pair}(H^0 C_P)
\end{array}$$

It is defined by six cases.

Thm. 3.5. We have a commutative square of bijections

$$\begin{tikzcd}
2\text{-presilt}_P(A) \arrow[r, "H_A"] \arrow[d, "\pi_P"'] \arrow[dr, phantom, "///"] & \tau\text{-rigid pair}_{H(P)}(H^0 A) \arrow[d, "\mathcal{E}_{H(P)}"] \\
2\text{-presilt}(C_P) \arrow[r, "H_{C_P}"] & \tau\text{-rigid pair}(H^0 C_P)
\end{tikzcd}$$

§2

Fact: [Alonso Tarrío – Jeremías López – Saúto Salorio, …] $D_{fd}(A)$ admits a bounded t-structure $(D_{fd}^{\leq 0}(A), D_{fd}^{>0}(A))$, where

$$D_{fd}^{\leq 0}(A) \overset{\text{def}}{=} \{ X \in D_{fd}(A) \mid H^i X = 0 \ \ \forall i > 0 \} \qquad D_{fd}^{>0}(A) \overset{\text{def}}{=} \{ X \in D_{fd}(A) \mid H^i X = 0 \ \ \forall i \leq 0 \}$$

i.e. [Beilinson – Bernstein – Deligne + Gabber, Keller – Vossieck]:

$D_{fd}^{\leq 0}(A)$ is closed under Σ and $D_{fd}^{\leq 0}(A) \hookrightarrow D_{fd}(A)$ admits a right adjoint

$\sigma^{\leq 0} : D_{fd}(A) \longrightarrow D_{fd}^{\leq 0}(A)$ called the <u>truncation functor</u>.

Boundedness: $D_{fd}(A) = \bigcup_{i \leq 0} \Sigma^i D_{fd}^{\leq 0}(A) = \bigcup_{i \geq 0} \Sigma^i D_{fd}^{>0}(A)$

$\iff D_{fd}(A) = \operatorname{thick}(D_{fd}^{\leq 0}(A) \cap \Sigma D_{fd}^{>0}(A))$

Def. [LLV+Y]: A thick subcategory $\mathcal{J} \subseteq D_{fd}(A)$ is <u>t-exact</u> if $\sigma^{\leq 0} X \in \mathcal{J} \quad \forall X \in \mathcal{J}$.

Cor 2.12 If P is two-term presilting, then $P^{\perp_{\mathbb{Z}}} \subseteq D_{fd}(A)$ is t-exact.

see also [Chen-Xi]

Fact: We have a cohomological functor $H^0 = \mathrm{Hom}_{D_{fd}(A)}(A,-) : D_{fd}(A) \to \mathrm{mod}(H^0 A)$.

Fact: If $\mathcal{J} \subseteq D_{fd}(A)$ is t-exact, then $H^0 \mathcal{J} \subseteq \mathrm{mod}(H^0 A)$ is wide,

i.e. closed under kernels, cokernels, and extensions

Thm. [Jasso, Demonet–Iyama–Reading–Reiten–Thomas; Brüstle–Smith–Treffinger]

For $(M,N) \in \tau$-rigid pair(B) then $J(M,N) \overset{\text{def}}{=} M^{\perp} \cap {}^{\perp}(\tau M) \cap N^{\perp} \subseteq \mathrm{mod}(B)$ is wide.

Thm. [Zhang–Cai, ...]. There are mutually inverse isomorphisms

$$\text{t-exact}(D_{fd}(A)) \underset{\mathrm{thick}_{D_{fd}(A)}(-)}{\overset{H^0}{\rightleftarrows}} \text{wide}(\mathrm{mod}\, H^0 A)$$

Prop. 2.10: For $P \in 2$-presilt(A), then $H^0(P^{\perp_{\mathbb{Z}}}) = J(H(P))$.

For example, set $A = k[1 \leftarrow 2]$:

Then 2-presilt(A) =

$$\begin{tikzcd}
P_2 \arrow[r, dash, "P_1 \oplus P_2"] \arrow[d, dash, "P_2 \oplus \Sigma P_1"'] & P_1 \arrow[r, dash, "P_1 \oplus S_1"] \arrow[d, phantom, "\circlearrowleft"] & S_1 \\
\Sigma P_1 \arrow[r, dash, "\Sigma P_1 \oplus \Sigma P_2"'] & \Sigma P_2 \arrow[ur, dash, "S_1 \oplus \Sigma P_2"'] &
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
& \mathcal{D}_{fd}(A) \arrow[loop above, "\circlearrowleft"] \arrow[dl, shift right, "P_1"'] \arrow[dl, shift left, "\Sigma P_1"] \arrow[d, shift right, "P_2"'] \arrow[d, shift left, "\Sigma P_2"] \arrow[dr, "S_1"] & \\
P_1^{\perp_2} \simeq \mathcal{D}_{fd}(k) \arrow[dr, shift right, "P_2"'] \arrow[dr, shift left, "\Sigma P_2"] & P_2^{\perp_2} \simeq \mathcal{D}_{fd}(k) \arrow[d, shift right, "S_1"'] \arrow[d, shift left, "\Sigma S_1"] & S_1^{\perp_2} \simeq \mathcal{D}_{fd}(k) \arrow[dl, shift right, "P_1"'] \arrow[dl, shift left, "\Sigma P_1"] \\
& 0 &
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
& \mathrm{mod}(A) \arrow[loop above, "\circlearrowleft"] \arrow[dl, shift right, "(P_1,0)"'] \arrow[dl, shift left, "(0,P_1)"] \arrow[d, shift right, "(P_2,0)"'] \arrow[d, shift left, "(0,P_2)"] \arrow[dr, "(S_1,0)"] & \\
\mathcal{H}(P_1,0) \simeq \mathrm{mod}(k) \arrow[dr, shift right, "(P_2,0)"'] \arrow[dr, shift left, "(0,P_2)"] & \mathcal{H}(P_2,0) \simeq \mathrm{mod}(k) \arrow[d, shift right, "(S_1,0)"'] \arrow[d, shift left, "(0,S_1)"] & \mathcal{H}(S_1,0) \simeq \mathrm{mod}(k) \arrow[dl, shift right, "(P_1,0)"'] \arrow[dl, shift left, "(0,P_1)"] \\
& 0 &
\end{tikzcd}$$

+ five arrows from $\mathcal{D}_{fd}(A)$ to 0.

+ five arrows from $\mathrm{mod}(A)$ to 0.

Def. The <u>τ-cluster morphism category</u> of A is defined as follows:

- $\mathrm{ob}\,\mathfrak{M}_A = \{ \mathcal{J} \in t\text{-exact}(A) \mid \mathcal{J} = P^{\perp_{\mathbb{Z}}}$ for some $P \in 2\text{-presilt}(A) \}$

$\xrightarrow[\sim]{H^0} \{ W \in \mathrm{wide}(A) \mid W = \mathcal{J}(M,N)$ for some $(M,N) \in \tau\text{-rigid pair}(A) \} =: \tau\text{-perp wide}(A)$

- Add a map $P^{\perp_{\mathbb{Z}}} \xrightarrow{\pi_P P'} (P \oplus P')^{\perp_{\mathbb{Z}}}$ if $P \oplus P' \in 2\text{-presilt}_P(A)$

Think of "$\pi_P P' \in 2\text{-presilt}(\mathcal{C}_P)$".

- Composition: Given $\mathcal{J}_1 \xrightarrow{P} \mathcal{J}_2 \xrightarrow{Q} \mathcal{J}_3$

set $Q \circ P \overset{\text{def}}{=} P \oplus \pi_P^{-1}(Q) \in 2\text{-presilt}(\mathcal{C}_P)$

- Identity maps: $P^{\perp_{\mathbb{Z}}} \xrightarrow{0} P^{\perp_{\mathbb{Z}}}$

Thm. 4.3. The composition rule in $\mathfrak{M}_A$ is associative.

Proof: Given $\mathcal{S}_1 \xrightarrow{P} \mathcal{S}_2 \xrightarrow{Q} \mathcal{S}_3 \xrightarrow{R} \mathcal{S}_4$, we have that

$$
\begin{aligned}
(R \circ Q) \circ P &= \pi_Q^{-1}(R) \circ P \\
&= \pi_P^{-1}\left(\pi_Q^{-1}(R)\right) \qquad (*) \\
&= \pi_{\pi_P^{-1}(Q)}^{-1}(R) \\
&= \pi_{Q \circ P}^{-1}(R) \\
&= R \circ (Q \circ P).
\end{aligned}
$$

Q.E.D.

Thm. 4.4. There is an equivalence of categories $\mathfrak{M}_A \longrightarrow \mathfrak{M}^{bm}_{H^\circ A}$, where $\mathfrak{M}^{bm}_{H^\circ A}$ is defined by Guan–Marsh and Guan–Hanson.

§3: A novel approach [Schroll–Tattar–Treffinger–Williams]:

Consider τ-rigid pair(B) as a poset $\mathcal{T}(B)$

$$(M_1, N_1) \leq (M_1 \oplus M_2, N_1 \oplus N_2)$$

We have an order reversing epimorphism:

$$\begin{aligned} \mathcal{J} : \mathcal{T}(B) &\longrightarrow \tau\text{-perp wide}(B) \\ \rotatebox{90}{$\in$} \quad & \qquad\qquad \rotatebox{90}{$\in$} \\ (M,N) &\longmapsto \mathcal{J}(M,N) = M^{\perp} \cap {}^{\perp}(\tau N) \cap N^{\perp} \end{aligned}$$

and an equivalence relation on $\mathcal{T}(B)$

$$(M_1, N_1) \sim (M_2, N_2) \quad \text{if} \quad \mathcal{J}(M_1, N_1) = \mathcal{J}(M_2, N_2).$$

Define a category $\mathcal{G}_B$ as follows

$\mathrm{ob}\,\mathcal{G}_B$ = equivalence classes of $\sim$

$$\mathrm{Hom}_{\mathcal{G}_B}\left([(M_1,N_1)],[(M_2,N_2)]\right) = \bigsqcup_{\substack{(M_1',N_1')\in[(M_1,N_1)] \\ (M_2',N_2')\in[(M_2,N_2)]}} \mathrm{Hom}_{\tau(B)}\left((M_1,N_1),(M_2,N_2)\right)\Big/ R$$

with $\begin{matrix}(M\oplus\hat{M},N\oplus\hat{N}) \\ | \\ (M,N)\end{matrix} \; R \; \begin{matrix}(M'\oplus\hat{M}',N'\oplus\hat{N}') \\ | \\ (M',N')\end{matrix}$ if $\mathcal{E}_{(M,N)}(M\oplus\hat{M},N\oplus\hat{N}) = \mathcal{E}_{(M',N')}(M'\oplus\hat{M}',N'\oplus\hat{N}')$

$\in \tau$-rigid pair $(H^0 C_p)$

Composition: Induced from $\tau(B)$

Thm. [SSTM]: There is an equivalence of categories $\mathcal{G}_B \longrightarrow \mathfrak{M}_B$

Why τ-cluster morphism categories?

- One can define the picture space of A by $|\mathfrak{M}_A|$. and the picture group by $\pi_0|\mathfrak{M}_A|$.

- Can study scattering diagrams.

§4 Footnotes

If P is three-term presilting, then $P^{\perp_{\mathbb{Z}}}$ need not be t-exact.

Let $A = k[1 \xleftarrow{\alpha} 2 \xleftarrow{\beta} 3]/(\alpha\beta)$ consider $P = P_3 \longrightarrow P_2 \longrightarrow P_1$

deg $\quad -2 \quad -1 \quad 0$

Then $P^{\perp_{\mathbb{Z}}} = \mathrm{thick}(P_3 \to P_2,\ P_2 \to P_1)$

but $\sigma^{\leq 0}(\underset{-1}{0} \longrightarrow \underset{0}{P_2} \to \underset{1}{P_1}) = 3$

and $\mathrm{Hom}_{D_{fd}(A)}(P, \Sigma^2 P_3) \neq 0$.

Let $\mathcal{S} \subseteq D_{fd}(A)$ be t-exact.

Then $H^0\mathcal{S}$ is closed under kernels, cokernels, and extensions.

Also, we have $\mathrm{thick}(H^0\mathcal{S}) = \mathcal{S}$ by boundness of the the t-structure on $\mathcal{S}$.

Can check that $H^0\,\mathrm{thick}(W) = W \quad \forall\, W \in \mathrm{wide}(A)$.

C_P need not be an algebra, even when A is:

Let $A = k[1 \xleftarrow{\alpha} 2 \xleftarrow{\beta} 3]/(\alpha\beta)$ consider $P = P_2$

Then $P^{\perp x} = \text{thick}\left(\cdots \to 0 \to P_2 \xrightarrow{\begin{pmatrix} f \\ 1 \\ 0 \end{pmatrix}} P_1 \oplus P_2 \oplus P_3 \to 0 \to \cdots\right)$

so

$$\begin{array}{ccccccccccc} C_P = & \cdots & \longrightarrow & 0 & \longrightarrow & k & \xrightarrow{\beta} & k & \longrightarrow & 0 & \longrightarrow \cdots \\ \text{deg} & & & -2 & & -1 & & 0 & & 1 & \end{array}$$