Algebras from surfaces: deformation, duality, quotients

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Online seminar on finite dimensional algebras and related topics

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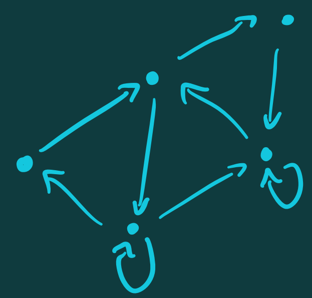


Outline

- Mixed-angulations on surfaces
- Gentle algebras
- Relative, graded Brauer graph algebras
- Ginzburg-type algebras
- Perverse schober perspective, stability conditions

Based on joint work with Merlin Christ and Yu Qiu:
“Perverse schobers, stability conditions, and quadratic differentials”
arXiv:2303.18249

Weighted marked surfaces

- S — compact, oriented surface
- $M \subset S$ — marked points (vertices)
- $\Delta \subset S$ — singular points (centers of polygons)
- $d : \Delta \to \mathbb{Z}_{\geq 1} \cup \{\infty\}$ — degree
- $\nu \in \Gamma(\mathbb{P}TS; S \setminus (M \cup \Delta))$ — grading structure

① $d(x) = \infty \iff x \in \partial S$

② $M \cap \Delta = \emptyset$

③ $|M \cap \mathrm{int}(S)| < \infty$

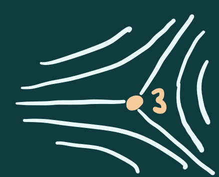
④ $B \subset \partial S$ component $\Rightarrow B \cap M \neq \emptyset$

⑤ $M \subset S \setminus \Delta$ discrete

$B \subset \partial S \setminus \Delta$ open component $\Rightarrow |B \cap M| = \infty$

⑥ $x \in \Delta \cap \mathrm{int}(S) \Rightarrow \mathrm{ind}_\nu(x) = d(x)$

e.g.



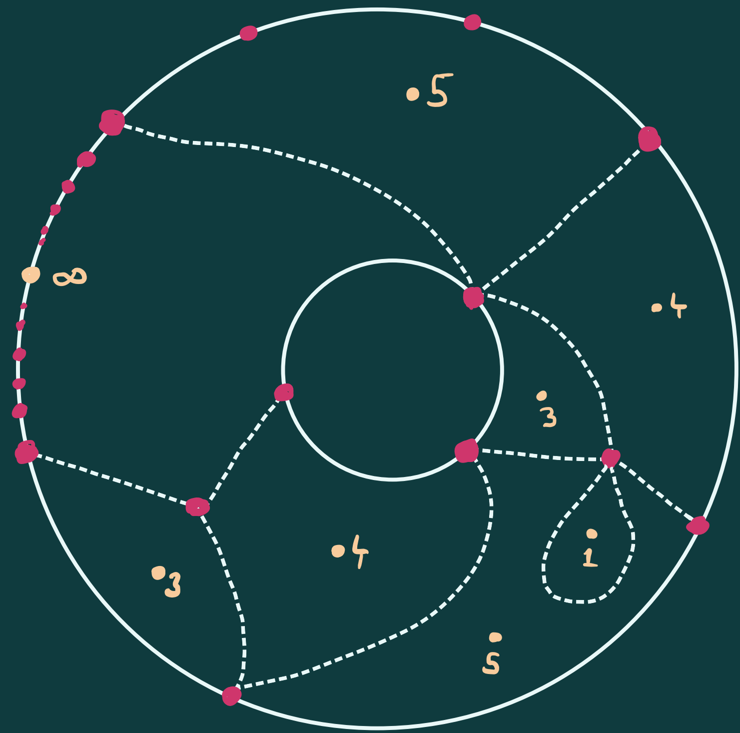
Mixed-angulation of weighted marked surface

- Collection of arcs in $S \setminus \Delta$

(1) endpoints in M

(2) arcs intersect only in endpoints

(3) arcs cut S into polygons with vertices in M, each n-gon contains exactly one point x of Δ and $d(x) = n$



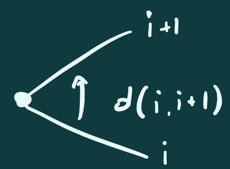
Dual graph of mixed-angulation (S-graph)

- Vertices = $\triangle$
- Edges : dual to internal edges of mixed-angulation

S-graph: Graph S with

- linear or cyclic order on halfedges meeting any vertex

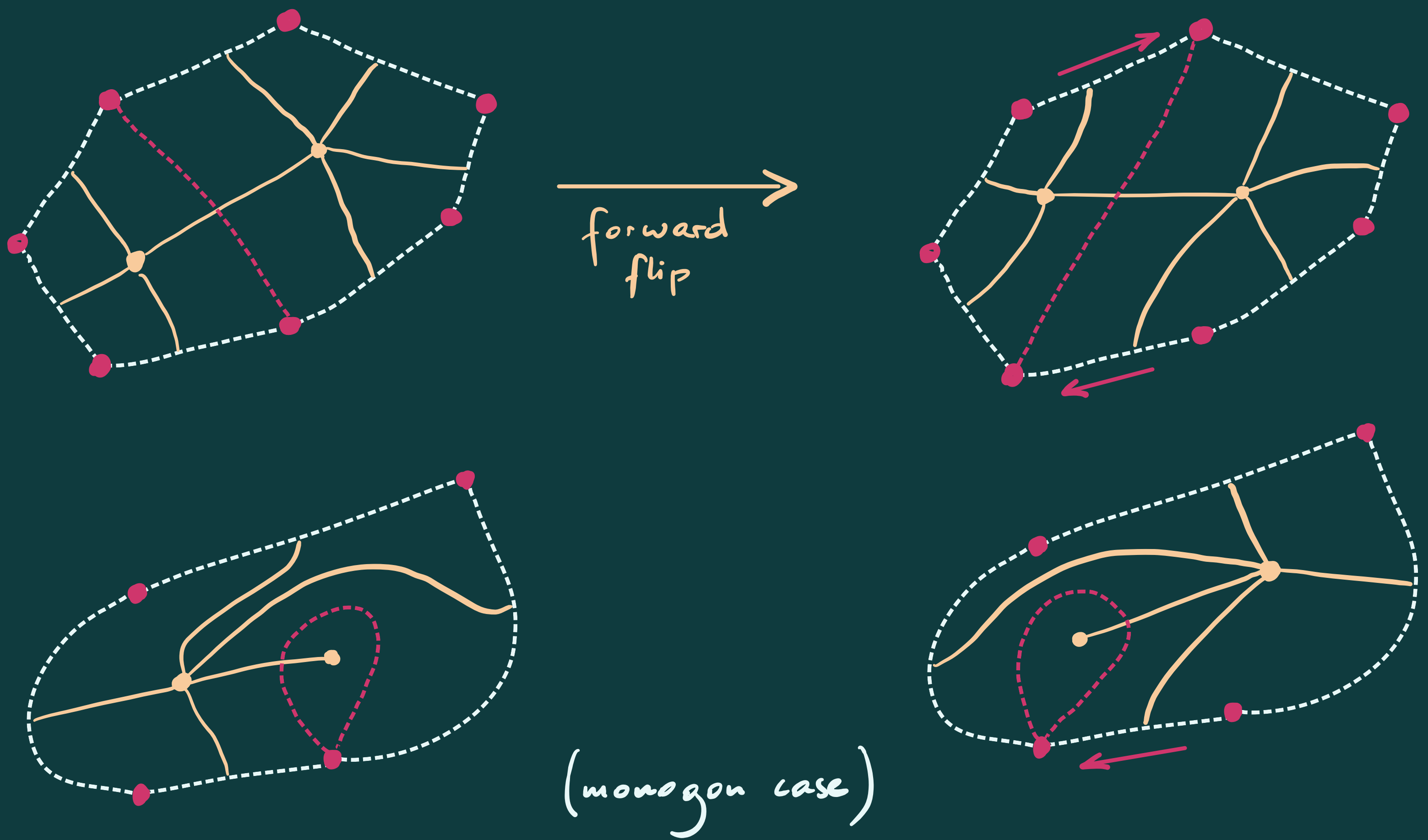
- $d(i,i+1) \in \mathbb{Z}_{\geq 1}$ for any pair $(i,i+1)$ of consecutive halfedges



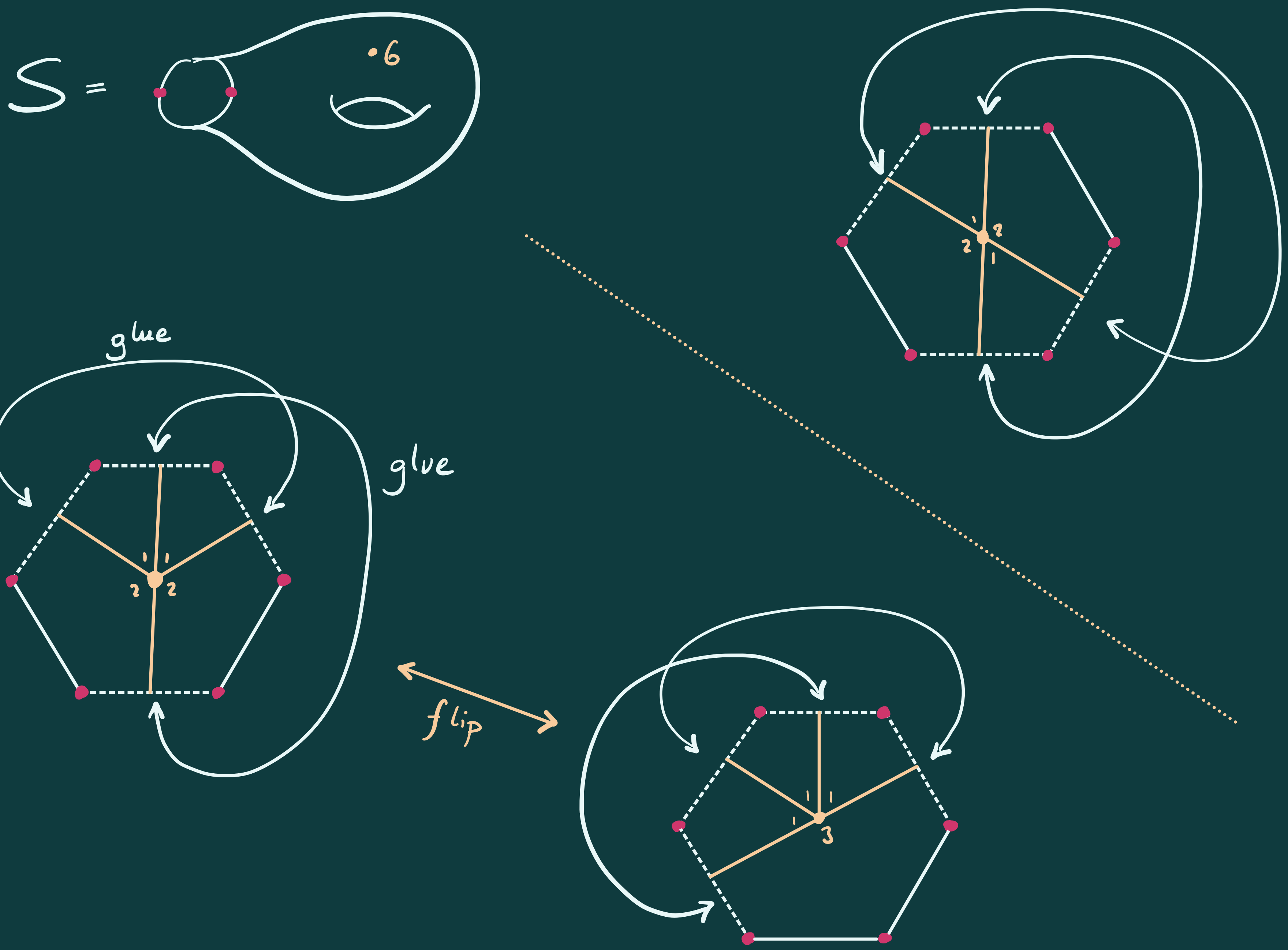
H-Katzarkov-Kontsevich: *“Flat surfaces and stability structures”*

Flips

For any internal (not boundary) edge of a mixed-angulation there is a forward/backward flip giving a new mixed-angulation.



Example of non flip-equivalent mixed-angulations



Mixed-angulations and quadratic differentials

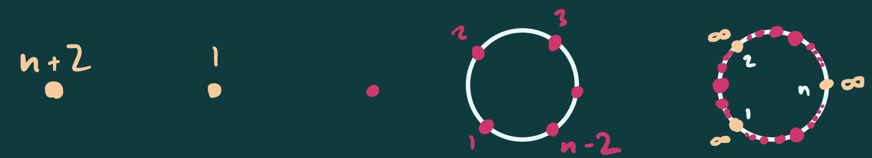
There is a correspondence:

$$\left\{ \begin{array}{l} \text{surfaces with mixed-angulation} \\ \text{and numbers } z_e \in \mathbb{C},\ \mathrm{Im}(z_e) > 0, \text{ for} \\ \text{every internal edge } e \end{array} \right\}$$

$$\Updownarrow$$

$$\left\{ \begin{array}{l} \text{compact Riemann surfaces} \\ \text{with generic quadratic differential} \\ \text{with 1) zeros, 2) poles} \\ \text{3) exponential singularities} \\ \qquad e^{z^{-n}} f(z)\, dz^2 \end{array} \right\}$$

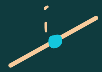
no finite length horizontal trajectories



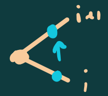
$$\underbrace{\begin{array}{c} z^n dz^2 \\ n \geq 1 \end{array}}_{1)} \qquad \underbrace{\frac{dz^2}{z} \qquad \frac{dz^2}{z^2} \qquad \begin{array}{c} \frac{dz^2}{z^n} \\ n \geq 3 \end{array}}_{2)} \qquad \underbrace{\begin{array}{c} e^{z^{-n}} dz^2 \\ n \geq 1 \end{array}}_{3)}$$

Graded gentle algebra of an S-graph

S-graph $\mathbb{S}$ $\rightsquigarrow$ graded algebra $G(\mathbb{S})$ given by graded quiver with relations

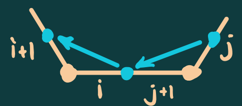
edge 

$\longleftrightarrow$ vertex i

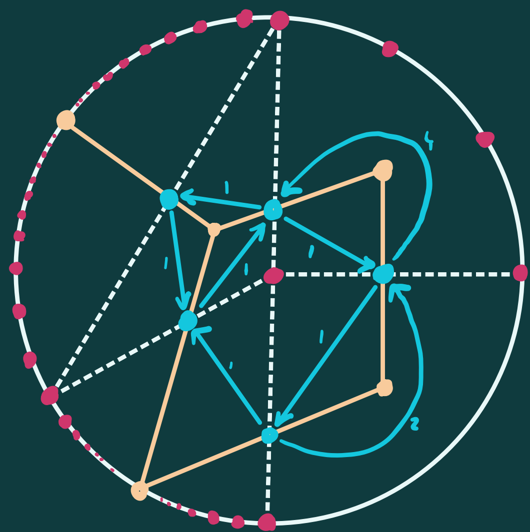
corner 

$\longleftrightarrow$ arrow $a_i : i \to i+1$

$d(i,i+1) \in \mathbb{Z}_{\geq 1}$ $\longleftrightarrow$ $|a_i| \in \mathbb{Z}_{\geq 1}$



pair of corners on same side of edge $\longleftrightarrow$ $a_i a_j = 0$

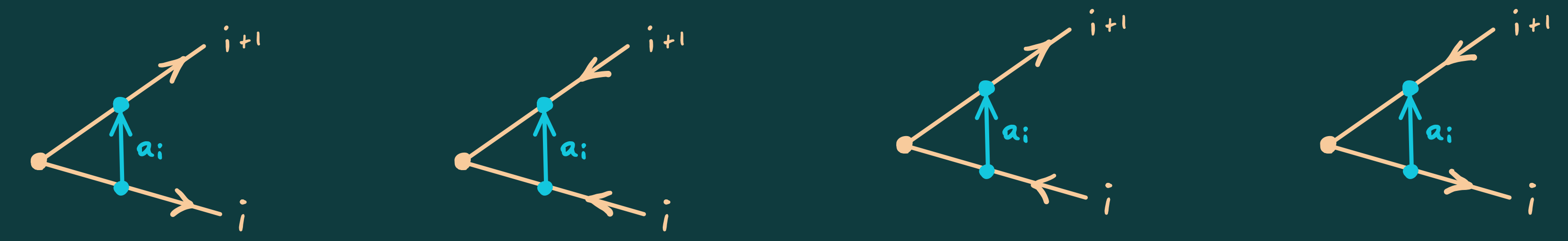


$\mathrm{Perf}(G(\mathbb{S}))$ is in general a full subcategory of a partially wrapped Fukaya category

Extra mod 2 grading

S – S-graph with chosen <u>orientation</u> of all edges

$\rightsquigarrow$ extra $\mathbb{Z}/2$-grading on $G(S)$:



$|a_i| + \pi(a_i)$ even $\qquad\qquad$ $|a_i| + \pi(a_i)$ odd

$\mathbb{Z} \times \mathbb{Z}/2$ - grading on $G(S)$

$\ni$

$(|a_i|, \pi(a_i))$

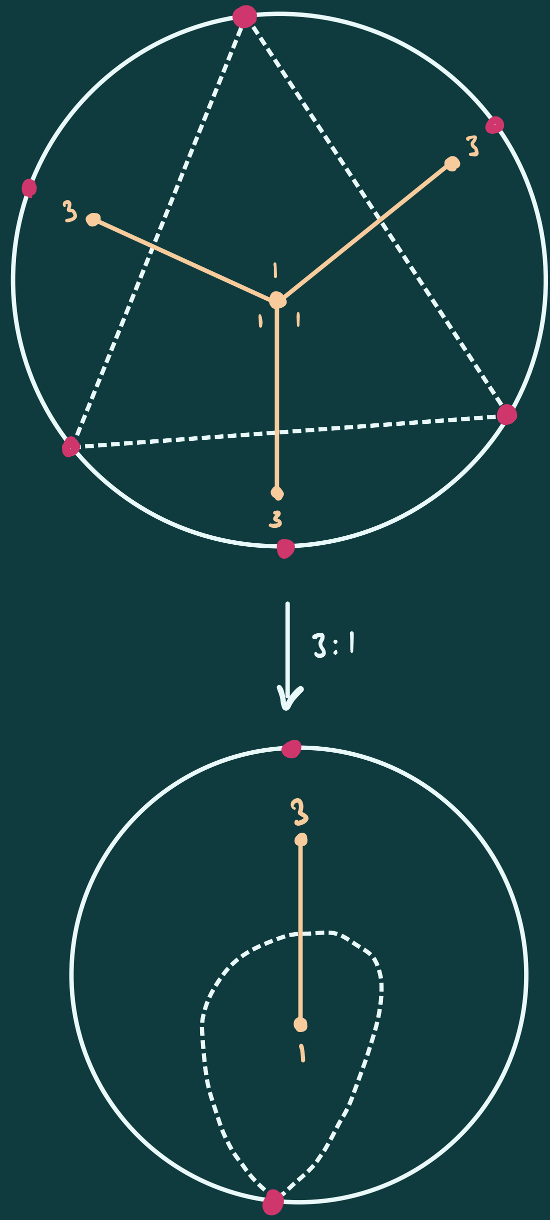
Mixed-angulated as quotient of n-angulated

mixed-angulated surface with only ∞-gons and n-gons for some fixed $n \geq 1$

$\big\downarrow$ finite-sheeted cover

mixed-angulated surface

(such covering exists for any mixed-angulated surface)



Deforming the gentle algebra

Fix $n \geq 1$ which is common multiple of all degrees of vertices of $\mathbb{S}$.

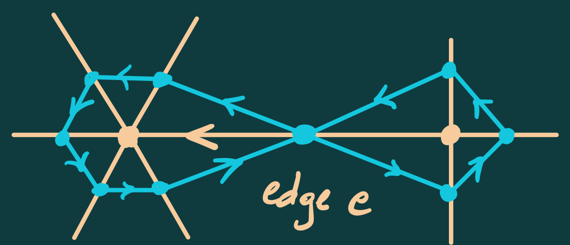
$\mathbb{Z} \times \mathbb{Z}/2$-graded algebra

$\mathbb{Z} \times \mathbb{Z}/2$-graded DG-algebra

$$G(\mathbb{S}) \rightsquigarrow \left(G(\mathbb{S})[\tau],\ d\right)$$

$$|\tau| = n-1, \quad \pi(\tau) = n \bmod 2 \qquad (\Rightarrow \tau^2 = 0)$$

$$d(\tau) = \sum_{w \xleftarrow{e} v} c_{e,+}^{\,n/d(w)} + (-1)^{n}\, c_{e,-}^{\,n/d(v)}$$



cycle $c_{e,+}$ cycle $c_{e,-}$

$(c_{e,\pm} = 0$ if vertex with $d(v) = \infty)$

Graded Brauer graph algebras

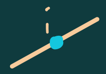
1) $\mathbb{S}$-graph $\mathbb{S}$

assume $d(v) < \infty \ \forall v$

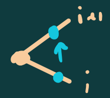
2) multiple of degrees: $n \geq 1$

$\rightsquigarrow$ finite-dimensional graded algebra $A(\mathbb{S}, n)$ (quasi-isomorphic to $G(\mathbb{S})[\tau]$)

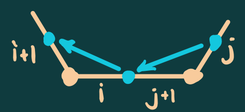
same as for $G(\mathbb{S})$

edge 

$\longleftrightarrow$ vertex i

corner 

$\longleftrightarrow$ arrow $a_i : i \to i+1$, $|a_i| = d(i, i+1)$



$\longleftrightarrow$ $a_i a_j = 0$

$\longleftrightarrow$ $c_{e,+}^{n/d(w)} = (-1)^{n-1} c_{e,-}^{n/d(v)}$

In ungraded case $(n=0)$, is different from usual sign $(+1)$ of Donovan–Freislich

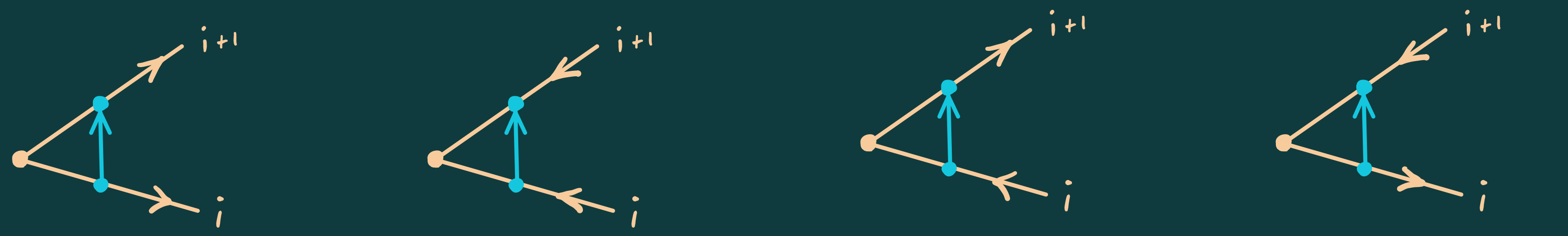
n-Calabi-Yau structure

For a graded algebra A over k, an nCY structure is a functional $tr: A^n \to k$ such that

1) $(a,b) \longmapsto tr(ab)$ is non-degenerate pairing $A \otimes A \to k$
2) $tr(ab) = (-1)^{|a|\cdot|b|}\, tr(ba)$

sufficient conditions for existence of nCY structure on $A(S,n)$:

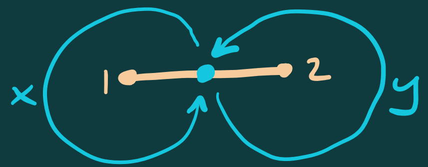
A) n odd
B) n even, $\exists$ orientation of edges of S such that $|a_i| \equiv \pi(a_i) \mod 2$, i.e.



$\underbrace{\qquad\qquad\qquad\qquad}_{\Rightarrow\ d(i,i+1)\ \text{even}}$ $\underbrace{\qquad\qquad\qquad\qquad}_{\Rightarrow\ d(i,i+1)\ \text{odd}}$

Non Calabi-Yau example

Sometimes, $A(S,n)$ is not Calabi-Yau for *any* choice of signs in the defining relations:



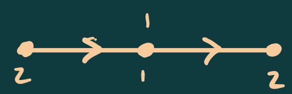
$$A = A(S,2) = k\langle x,y\rangle \big/ {}_{xy,\, yx,\, x^2 = \pm y} \quad , \quad |x|=1,\ |y|=2$$

degree	0	1	2
basis	1	x	x^2

$\dim(A^1) = 1 \Rightarrow \nexists$ skew symmetric bilinear form

$$A^1 \times A^1 \to k$$

but $(a,b) \mapsto tr(ab)$ would give such a form

Note: Example is $\mathbb{Z}/2$ - quotient of  which *does* have 2CY structure, but not an invariant one.

Relative graded Brauer graph (RGB) algebras

1) $\mathbb{S}$-graph $\mathbb{S}$ (any)

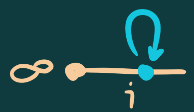
2) multiple of degrees: $n \geq 1$

$\rightsquigarrow$ finite-dimensional DG-algebra $A(\mathbb{S}, n)$

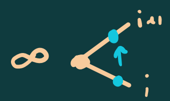
(quasi-isomorphic to $G(\mathbb{S})[\tau]$)

additionally:

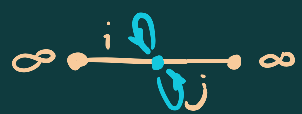
halfedge attached to vertex with $d = \infty$



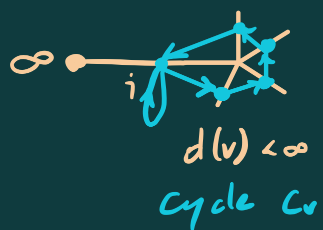
$\longleftrightarrow$ arrow $\tau_i : i \to i$, $|a_i| = d-1$



$\longleftrightarrow$ $a_i \tau_i = (-1)^{|a_i|} \tau_{i+1} a_i$



$\longleftrightarrow$ $\tau_i = (-1)^n \tau_j$



$\longleftrightarrow$ $d\tau_i = (-1)^n c_v^{n/d(v)}$

Koszul duality

(Keller, Lefèvre-Hasegawa, Van den Bergh)

k — field, $R = k^r$ (semisimple k-algebra)

A — DG algebra $/R$ with augmentation $A = R \oplus \bar{A}$

assume $\dim_k A < \infty$, all $x \in \bar{A}$ nilpotent

Koszul dual of A: $A^! = \mathrm{Cobar}(\bar{A}^\vee) = \bigoplus_{i \geq 0} (\bar{A}^\vee[1])^{\otimes i}$

$\mathrm{Hom}_R(\bar{A}, R)$
coalgebra

$$\boxed{\mathrm{End}_{A^!}(R) \underset{\text{QIS}}{\simeq} A}$$

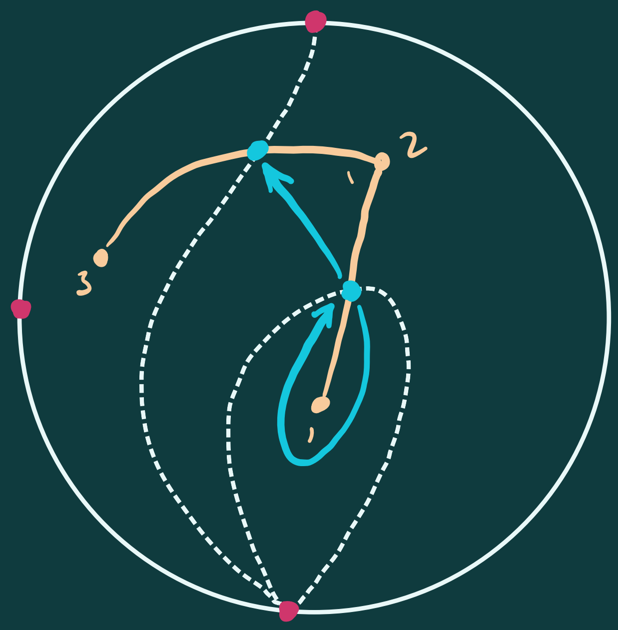
A	$A^!$
basis $e_1, \dots, e_n$	generators $e^1, \dots, e^n$, $\lvert e^i \rvert = 1 - \lvert e_i \rvert$
$de_i = \sum_j d_j^i e_j$	$de^k = -\sum_i d_k^i e^i + \sum_{i,j} (-1)^{\lvert e^i \rvert} m_k^{ji} e^i \otimes e^j$
$e_i \cdot e_j = \sum_k m_k^{ij} e_k$	

Koszul dual of RGB algebra

Generalizes:

1) Ginzburg algebra of triangulated surface
(without term in potential for interior marked points)
2) (Relative) Ginzburg algebra of n-angulated surface (M. Christ)

Example:



$A(5,3)^! =$ Ginzburg DG-algebra of quiver with potential

$$\alpha \circlearrowright \bullet \xrightarrow{\beta} \bullet \qquad W = \alpha^3$$

c.f. Labardini-Fragoso, Mou: *Gentle algebras arising from surfaces with orbifold points of order 3, Part I: scattering diagrams*

Perverse schober perspective

A perverse schober $\mathcal{E}$ (Kapranov - Schechtman) on a surface S is roughly: 1) A local system of stable ∞-categories on $S \setminus \Delta$, $\Delta \subset S$ discrete

2) A spherical adjunction for each $x \in \Delta$

$\rightsquigarrow$ stable ∞-category of global sections $\mathcal{F}(S;\mathcal{E})$

"Fukaya category of S with coefficients in $\mathcal{E}$"

Slogan: Derived categories attached to surfaces arise as $\mathcal{F}(S;\mathcal{E})$ for various choices of $\mathcal{E}$.

(e.g. derived categories of gentle algebras, Ginzburg-type algebras, RGB algebras)

Christ - H - Qiu

Stability conditions

S — weighted marked surface

$\mathcal{E}$ — perverse schober on S with "arc system kit"

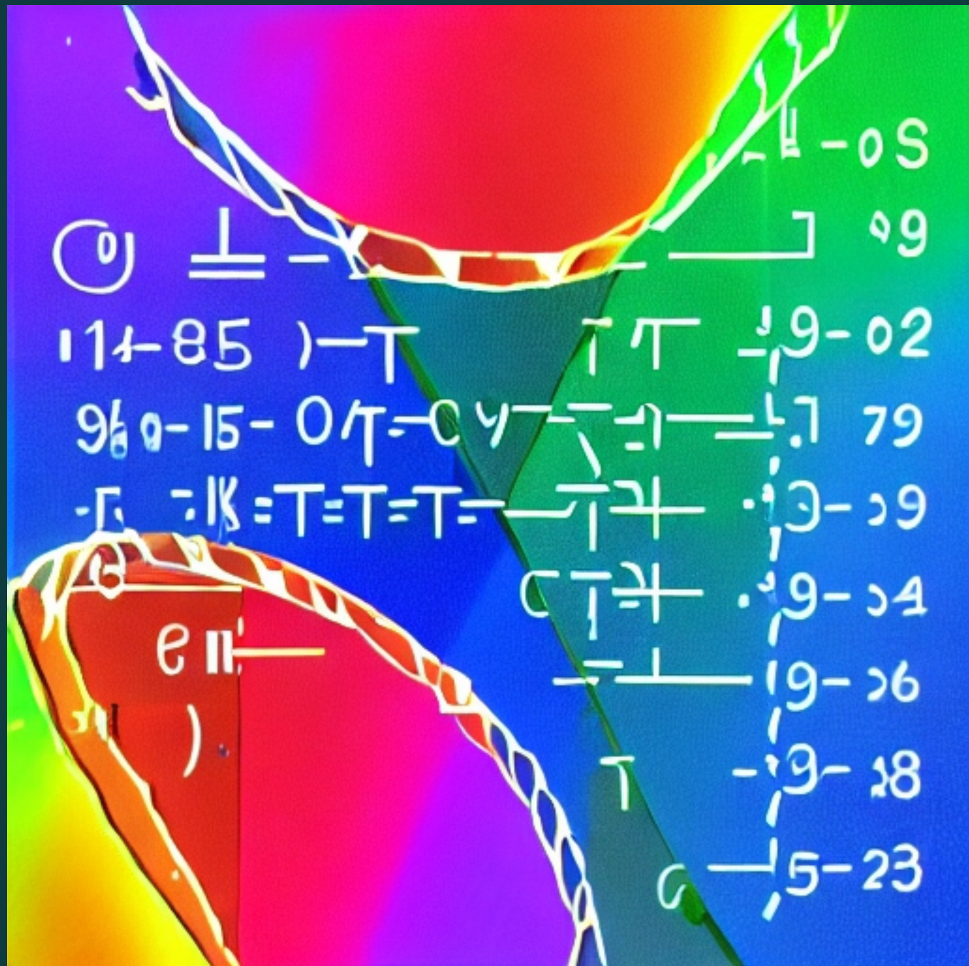
$\mathcal{C}(\mathbb{S};\mathcal{E}) \subset \mathcal{F}(S;\mathcal{E})$ full subcategory generated by edges of some S-graph $\mathbb{S}$ on S

Theorem (Christ – H – Qiu): There is a map, biholomorphic onto a union of connected components

$$\mathrm{FQuad}(S) \longrightarrow \mathrm{Stab}(\mathcal{C}(\mathbb{S};\mathcal{E}))$$

moduli space of framed quadratic differentials

space of Bridgeland stability conditions



RGB algebra