# Indecomposables in the monomorphism category

(Joint with Nan Gao, Julian Külshammer and Chrysostomos Psaroudakis.)

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# Plan of the talk

1. Definition and previous work.
2. An epivalence.
3. Artinian uniserial rings of Loewy length 3.
4. Nilpotent endofunctors on abelian categories.
5. Representations of prospecies.

# Definition and previous work

$\mathcal{B}$ abelian category.
$Q = (Q_0, Q_1)$ a finite acyclic quiver.
For an arrow $\alpha\colon \mathtt{i} \to \mathtt{j}$ write $s(\alpha) = \mathtt{i}$ and $t(\alpha) = \mathtt{j}$.
$\operatorname{Rep}(Q, \mathcal{B})$ - category of representations of $Q$ in $\mathcal{B}$. Objects are tuples $(B_{\mathtt{i}}, B_\alpha)_{\mathtt{i}\in Q_0, \alpha\in Q_1}$ where $B_{\mathtt{i}}$ is an object in $\mathcal{B}$ for all vertices $\mathtt{i} \in Q_0$ and $B_\alpha\colon B_{s(\alpha)} \to B_{t(\alpha)}$ is a morphism in $\mathcal{B}$ for all arrows $\alpha \in Q_1$.
Morphism $\varphi\colon (B_{\mathtt{i}}, B_\alpha)_{\mathtt{i}\in Q_0, \alpha\in Q_1} \to (B'_{\mathtt{i}}, B'_\alpha)_{\mathtt{i}\in Q_0, \alpha\in Q_1}$ in $\operatorname{Rep}(Q, \mathcal{B})$ consists of morphisms $\varphi_{\mathtt{i}}\colon B_{\mathtt{i}} \to B'_{\mathtt{i}}$ in $\mathcal{B}$ for all $\mathtt{i} \in Q_0$ making the following diagram

$$\begin{array}{ccc} B_{s(\alpha)} & \xrightarrow{B_\alpha} & B_{t(\alpha)} \\ \downarrow{\scriptstyle \varphi_{s(\alpha)}} & & \downarrow{\scriptstyle \varphi_{t(\alpha)}} \\ B'_{s(\alpha)} & \xrightarrow{B'_\alpha} & B'_{t(\alpha)} \end{array}$$

commutative for all arrows $\alpha$.
$\operatorname{Rep}(Q, \mathcal{B})$ is an abelian category.

# Definition and previous work

## Definition

The *(separated) monomorphism category* $\mathrm{Mono}(Q, \mathcal{B})$ is the full subcategory of $\mathrm{Rep}(Q, \mathcal{B})$ consisting of all representations $(B_{\mathtt{i}}, B_\alpha)_{\mathtt{i} \in Q_0, \alpha \in Q_1}$ such that for any vertex $\mathtt{i}$ in $Q$ the morphism

$$\bigoplus_{\alpha \in Q_1, t(\alpha) = \mathtt{i}} B_{s(\alpha)} \xrightarrow{(B_\alpha)_\alpha} B_{\mathtt{i}}$$

is a monomorphism.

$\mathrm{Mod}\, R$ - category of right $R$-modules.
$\mathrm{Mono}(1 \to 2, \mathrm{Mod}\, R)$ consists of pairs $(N, M)$ where $M$ and $N$ are right $R$-modules and $N$ is a submodule of $M$.
$\mathrm{Mono}(1 \to 2 \to \cdots \to \mathtt{n}, \mathrm{Mod}\, R)$ consists of flags $M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots M_{n-1} \subseteq M_n$ of right $R$-modules.
$\mathrm{Mono}(1 \to 2 \leftarrow 3, \mathrm{Mod}\, R)$ consists of tuples $(K, M, L)$ where $K, L, M$ are right $R$-modules, $K, L$ are submodules of $M$, and $K \cap L = (0)$.

# Definition and previous work

mod $R$ - category of finitely generated right $R$-modules.

- $\mathrm{Mono}(1 \to 2, \mathrm{mod}\, \mathbb{Z}/p^m\mathbb{Z})$ for $p$ prime.
  - (Miller 1904, Hilton 1907, Birkhoff 1935) Subgroups of abelian groups.
  - (Plahotnik 76) Representation finite if and only if $m \leq 5$.
  - (Richman–Walker 99) Determined indecomposables when $m \leq 5$.
  - (Ringel–Schmidmeier 06) Proved controlled wildness when $m \geq 7$.
  - AR-quiver for $m = 6$ not known. Open question, called the Birkhoff problem.
- Analogous results for $\mathrm{Mono}(1 \to 2, \mathrm{mod}\, k[x]/(x^m))$.
  - (Plahotnik 76) Representation finite if and only if $m \leq 5$.
  - (Simson 02) Wild if $m \geq 7$ and $k$ algebraically closed.
  - (Ringel–Schmidmeier 08) Determined AR-quiver for $m \leq 6$.
    Challenging question: Is the AR quiver of $\mathrm{Mono}(1 \to 2, \mathrm{mod}\, k[x]/(x^6))$ similar to the AR-quiver of $\mathrm{Mono}(1 \to 2, \mathrm{mod}\, \mathbb{Z}/p^6\mathbb{Z})$?
    $k$ - finite field with $p$ elements.

# Definition and previous work

$\Lambda$ Artin algebra.

- Representation theory of $\text{Mono}(\mathtt{1} \to \mathtt{2} \to \cdots \to \mathtt{n}, \text{mod}\,\Lambda)$ studied by Bauer–Botnan–Oppermann–Steen, Moore, Plahotnik, Schmidmeier, Simson, $\cdots$.
- (Ringel–Schmidmeier 08, Luo–Zhang 13) $\text{Mono}(Q, \text{mod}\,\Lambda)$ has AR sequences for any finite acyclic quiver $Q$.
- (Ringel–Zhang 17) Described AR-quiver of $\text{Mono}(Q, \text{mod}\,k[x]/(x^2))$ in terms of AR-quiver of $\text{Rep}(Q, \text{mod}\,k)$.

# An epivalence

$\mathcal{B}$ - abelian category with enough injectives.
$\mathcal{I}$ - subcategory of injectives in $\mathcal{B}$.
$\mathrm{Mono}(Q,\mathcal{B})$ extension closed subcategory of $\mathrm{Rep}(Q,\mathcal{B})$, i.e. if

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

exact in $\mathrm{Rep}(Q,\mathcal{B})$, then $M_1 \in \mathrm{Mono}(Q,\mathcal{B})$ and $M_3 \in \mathrm{Mono}(Q,\mathcal{B})$ implies $M_2 \in \mathrm{Mono}(Q,\mathcal{B})$.
$\implies$ $\mathrm{Mono}(Q,\mathcal{B})$ inherits an exact structure making it into a Quillen exact category.
As an exact category $\mathrm{Mono}(Q,\mathcal{B})$ has enough injectives.
$\mathcal{J}$ - subcategory of injectives in $\mathrm{Mono}(Q,\mathcal{B})$.

$$\overline{\mathrm{Mono}}(Q,\mathcal{B}) = \frac{\mathrm{Mono}(Q,\mathcal{B})}{\mathcal{J}} \quad \text{and} \quad \overline{\mathcal{B}} = \frac{\mathcal{B}}{\mathcal{I}}$$

injectively stable categories of $\mathrm{Mono}(Q,\mathcal{B})$ and $\mathcal{B}$.

# An epivalence

## Definition

An *epivalence* is a full and dense functor, which reflects isomorphisms ($F(f)$ isomorphism $\implies$ $f$ isomorphism).

## Theorem (GKKP 23)

*$\mathcal{B}$ abelian category with enough injectives. The canonical functor*

$$\mathrm{Rep}(Q, \mathcal{B}) \to \mathrm{Rep}(Q, \overline{\mathcal{B}})$$

*vanishes on the injective objects in* $\mathrm{Mono}(Q, \mathcal{B})$*, and induces an epivalence*

$$\Phi \colon \overline{\mathrm{Mono}}(Q, \mathcal{B}) \to \mathrm{Rep}(Q, \overline{\mathcal{B}}).$$

*Furthermore, if $Q$ has at least one arrow, then $\Phi$ is an equivalence if and only if $\mathcal{B}$ is hereditary.*

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$\Phi$ induces a bijection

$$\left\{\begin{matrix}\text{Indecomposables} \\ \text{in } \overline{\mathsf{Mono}}(Q,\mathcal{B})\end{matrix}\right\} \xrightarrow{\cong} \left\{\begin{matrix}\text{Indecomposables} \\ \text{in } \mathsf{Rep}(Q,\overline{\mathcal{B}})\end{matrix}\right\}$$

Here we consider *isomorphism classes* of objects.

$$\left\{\begin{matrix}\text{Indecomposables} \\ \text{in } \overline{\text{Mono}}(Q,\mathcal{B})\end{matrix}\right\} \xrightarrow{\cong} \left\{\begin{matrix}\text{Indecomposables} \\ \text{in } \text{Rep}(Q,\overline{\mathcal{B}})\end{matrix}\right\}$$

Assume $\mathcal{B} = \operatorname{mod} \Lambda$ or $\mathcal{B} = \operatorname{Mod} \Lambda$,

$$\left\{\begin{matrix}\text{Indecomposables} \\ \text{in } \overline{\text{Mono}}(Q,\mathcal{B})\end{matrix}\right\} \xrightarrow{\cong} \left\{\begin{matrix}\text{Indecomposables} \\ \text{in } \text{Rep}(Q,\overline{\mathcal{B}})\end{matrix}\right\}$$

Assume $\mathcal{B} = \operatorname{mod}\Lambda$ or $\mathcal{B} = \operatorname{Mod}\Lambda$, where $\Lambda$ is an Artin algebra.

$$\begin{Bmatrix} \text{Indecomposables} \\ \text{in } \overline{\text{Mono}}(Q,\mathcal{B}) \end{Bmatrix} \xrightarrow{\cong} \begin{Bmatrix} \text{Indecomposables} \\ \text{in } \text{Rep}(Q,\overline{\mathcal{B}}) \end{Bmatrix}$$

Assume $\mathcal{B} = \operatorname{mod}\Lambda$ or $\mathcal{B} = \operatorname{Mod}\Lambda$, where $\Lambda$ is an Artin algebra.

$$\left\{\begin{matrix}\text{Indecomposable} \\ \text{non-injectives} \\ \text{in } \operatorname{Mono}(Q,\mathcal{B})\end{matrix}\right\} \xrightarrow{\cong} \left\{\begin{matrix}\text{Indecomposables} \\ \text{in } \overline{\operatorname{Mono}}(Q,\mathcal{B})\end{matrix}\right\} \xrightarrow{\cong} \left\{\begin{matrix}\text{Indecomposables} \\ \text{in } \operatorname{Rep}(Q,\overline{\mathcal{B}})\end{matrix}\right\}$$

- Useful if $\overline{\mathcal{B}}$ is an abelian category e.g. $\mathcal{B} = \operatorname{mod}\Lambda$ for $\Lambda$ radical-square-zero Nakayama algebra.
  For example, if $\Lambda = k[x]/(x^2)$, then $\overline{\operatorname{mod}}\, k[x]/(x^2) \cong \operatorname{mod} k$, and RHS is just representations of $Q$ over $k$. Recover results by Ringel and Zhang.
- Can construct an explicit inverse. Uses the Mimo-construction first introduced by Ringel and Schmidmeier for $Q = 1 \to 2$, and extended to arbitrary finite acyclic quivers $Q$ by Luo and Zhang.

# Artinian uniserial rings of Loewy length 3

$\Lambda$ artinian uniserial ring of Loewy length $n$ (e.g. $\Lambda = k[x]/(x^n)$ for $k$ a field, or $\Lambda = \mathbb{Z}/p^n\mathbb{Z}$ for $p$ a prime).
Isomorphism classes of finitely generated $\Lambda$-modules are in bijection with sequences $\overline{\alpha} = (\alpha_1 \geq \alpha_2 \geq \cdots \alpha_m > 0)$ of integers with $\alpha_1 \leq n$ (sometimes called partitions). Explicitly

$$\overline{\alpha} \mapsto M(\overline{\alpha}) := \bigoplus_{i=1}^{m} M(\alpha_i)$$

where $M(\alpha_i)$ is the unique (up to isomorphism) indecomposable with Loewy length $\alpha_i$ (e.g. $k[x]/(x^{\alpha_i})$ or $\mathbb{Z}/p^{\alpha_i}\mathbb{Z}$).
An object $(M_{\mathtt{i}}, M_\alpha)_{\mathtt{i}\in Q_0, \alpha\in Q_1} \in \operatorname{Rep}(Q, \operatorname{mod}\Lambda)$ has an associated *partition vector* $(\overline{\alpha}^{\mathtt{i}})_{\mathtt{i}\in Q_0}$ uniquely determined by

$$M(\overline{\alpha}^{\mathtt{i}}) \cong M_{\mathtt{i}} \text{ for all } \mathtt{i} \in Q_0.$$

# Artinian uniserial rings of Loewy length 3

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$$M(\overline{\alpha}^{\mathtt{i}}) \cong M_{\mathtt{i}} \text{ for all } \mathtt{i} \in Q_0.$$

Assume $k$ is a finite field with $p$ elements, i.e. $k = \mathbb{Z}/p\mathbb{Z}$. Then

$$\overline{\mathrm{mod}}\, k[x]/(x^n) \cong \overline{\mathrm{mod}}\, \mathbb{Z}/p^n\mathbb{Z} \text{ for } n \leq 3$$

$\implies \mathrm{Rep}(Q, \overline{\mathrm{mod}}\, k[x]/(x^n)) \cong \mathrm{Rep}(Q, \overline{\mathrm{mod}}\, \mathbb{Z}/p^n\mathbb{Z})$ for $n \leq 3$.
$\implies$ bijection between non-injective indecomposables in $\mathrm{Mono}(Q, \mathrm{mod}\, k[x]/(x^n))$ and $\mathrm{Mono}(Q, \mathrm{mod}\, \mathbb{Z}/p^n\mathbb{Z})$ for $n \leq 3$.

## Theorem (GKKP 23)

*Let $Q$ be a finite acyclic quiver, $n \leq 3$ a positive integer, and $k = \mathbb{Z}/p\mathbb{Z}$. There exists a bijection*

$$\left\{ \begin{matrix} \textit{indecomposable objects in} \\ \mathrm{Mono}(Q, \mathrm{mod}\, k[x]/(x^n)) \end{matrix} \right\} \xrightarrow{\cong} \left\{ \begin{matrix} \textit{indecomposable objects in} \\ \mathrm{Mono}(Q, \mathrm{mod}\, \mathbb{Z}/p^n\mathbb{Z}) \end{matrix} \right\}$$

*which preserves the underlying partition vectors.*

If one could extend this to $n \leq 6$, then this could give a solution to the Birkhoff problem.

# Nilpotent endofunctors on abelian categories

To avoid the combinatorics arising from quiver representations we prove the results using a different language.
Fix a finite acyclic quiver $Q$ and an abelian category $\mathcal{B}$.
Set $\mathcal{C} = \prod_{\mathtt{i} \in Q_0} \mathcal{B}$ - abelian category.
The arrows of the quiver $Q$ defines an endofunctor $X\colon \mathcal{C} \to \mathcal{C}$ by

$$X((B_{\mathtt{i}})_{\mathtt{i}\in Q_0}) = (\bigoplus_{\substack{\alpha \in Q_1 \\ t(\alpha)=\mathtt{i}}} B_{s(\alpha)})_{\mathtt{i}\in Q_0}$$

For example:

- $Q = 1 \to 2 \to \cdots \to \mathtt{n}$ , then $X(B_1, B_2, \cdots, B_{\mathtt{n}}) = (0, B_1, \cdots, B_{\mathtt{n}-1})$
- $Q = 1 \to 2 \leftarrow 3$ then $X(B_1, B_2, B_3) = (0, B_1 \oplus B_3, 0)$.

Note that a morphism $X((B_{\mathtt{i}})_{\mathtt{i}\in Q_0}) \to (B_{\mathtt{i}})_{\mathtt{i}\in Q_0}$ is the same as a representation of $Q$ in $\mathcal{B}$.

# Nilpotent endofunctors on abelian categories

The arrows of the quiver $Q$ defines an endofunctor $X\colon \mathcal{C} \to \mathcal{C}$ by

$$X((B_{\mathtt{i}})_{\mathtt{i}\in Q_0}) = (\bigoplus_{\substack{\alpha\in Q_1\\ t(\alpha)=\mathtt{i}}} B_{s(\alpha)})_{\mathtt{i}\in Q_0}$$

We get equivalences (write $C = (B_{\mathtt{i}})_{\mathtt{i}\in Q_0}$)
$\operatorname{Rep}(Q, \mathcal{B}) \cong \mathcal{C}^{T(X)} = \{\text{morphisms } X(C) \to C \text{ in } \mathcal{C}\}$.
$\operatorname{Mono}(Q, \mathcal{B}) \cong \operatorname{Mono}(X) = \{\text{monomorphisms } X(C) \to C \text{ in } \mathcal{C}\}$.
$\operatorname{Rep}(Q, \overline{\mathcal{B}}) \cong \overline{\mathcal{C}}^{T(X)} = \{\text{morphisms } X(C) \to C \text{ in } \overline{\mathcal{C}}\}$.
The right hand sides are categories, where objects are morphisms $X(C) \to C$ in $\mathcal{C}$ or $\overline{\mathcal{C}}$, respectively, and where a morphism $(X(C) \to C) \to (X(C') \to C')$ is given by commutative square

$$\begin{array}{ccc} X(C) & \xrightarrow{X(g)} & X(C') \\ \downarrow & & \downarrow \\ C & \xrightarrow{g} & C'. \end{array}$$

$\mathrm{Rep}(Q, \mathcal{B}) \cong \mathcal{C}^{T(X)} = \{\text{morphisms } X(C) \to C \text{ in } \mathcal{C}\}$.
$\mathrm{Mono}(Q, \mathcal{B}) \cong \mathrm{Mono}(X) = \{\text{monomorphisms } X(C) \to C \text{ in } \mathcal{C}\}$.
$\mathrm{Rep}(Q, \overline{\mathcal{B}}) \cong \overline{\mathcal{C}}^{T(X)} = \{\text{morphisms } X(C) \to C \text{ in } \overline{\mathcal{C}}\}$.
$X$ is nilpotent, exact, and preserves injectives. This follows from $Q$ being finite and acyclic, and the formula

$$X((B_{\mathtt{i}})_{\mathtt{i} \in Q_0}) = \Big( \bigoplus_{\substack{\alpha \in Q_1 \\ t(\alpha) = \mathtt{i}}} B_{s(\alpha)} \Big)_{\mathtt{i} \in Q_0}$$

**Theorem (GKKP 23)**

*Assume $\mathcal{C}$ is an abelian category with enough injectives, and $X \colon \mathcal{C} \to \mathcal{C}$ is an exact, nilpotent, endofunctor preserving injectives. Then $\mathrm{Mono}(X)$ is an exact category with enough injectives, and the canonical functor*

$$\overline{\mathrm{Mono}}(X) \to \overline{\mathcal{C}}^{T(X)}$$

*is an epivalence. Furthermore, it is an equivalence if and only if any object in the image of $X$ has injective dimension $1$.*

# Representations of prospecies

The setting also applies to prospecies, introduced by Gabriel and further studied by Dlab and Ringel.
$Q$ - finite acyclic quiver.
For each $\mathtt{i} \in Q_0$, let $\Lambda_{\mathtt{i}}$ be a finite-dimensional selfinjective algebra.
For each arrow $\alpha \colon \mathtt{i} \to \mathtt{j}$ let $\Lambda_\alpha$ be a finite-dimensional $\Lambda_{\mathtt{j}}$-$\Lambda_{\mathtt{i}}$-bimodule, which is projective as a left $\Lambda_{\mathtt{j}}$-module, and as a right $\Lambda_{\mathtt{i}}$-module. We call such a tuple $(\Lambda_{\mathtt{i}}, \Lambda_\alpha)$ a *prospecies* over $Q$. Note that

$$- \otimes_{\Lambda_{\mathtt{j}}} \Lambda_\alpha \colon \operatorname{mod} \Lambda_{\mathtt{j}} \to \operatorname{mod} \Lambda_{\mathtt{i}}$$

is exact and preserves injectives. Define $\mathcal{C}$ and $X \colon \mathcal{C} \to \mathcal{C}$ by

$$\mathcal{C} = \prod_{\mathtt{i} \in Q_0} \operatorname{mod} \Lambda_i \quad \text{and} \quad X((M_{\mathtt{i}})_{\mathtt{i} \in Q_0}) = (\bigoplus_{\alpha \in Q_1, t(\alpha) = \mathtt{i}} M_{s(\alpha)} \otimes_{\Lambda_{s(\alpha)}} \Lambda_\alpha)_{\mathtt{i} \in Q_0}.$$

Then $X$ is exact, nilpotent, and preserves injectives.

# Representations of prospecies

The category of representations of the prospecies $(\Lambda_{\mathtt{i}}, \Lambda_\alpha)$ is

$$\mathcal{C}^{T(X)} = \{\text{morphisms } X(C) \to C \text{ in } \mathcal{C}\}.$$

$\mathcal{C}^{T(X)}$ is equivalent to $H$-mod, the category of finite-dimensional left $H$-modules, where $H$ is the tensor algebra

$$H = \Lambda \oplus M \oplus (M \otimes_\Lambda M) \oplus \cdots = \bigoplus_{i \geq 0} M^{\otimes i}$$

and $\Lambda = \prod_{\mathtt{i} \in Q_0} \Lambda_{\mathtt{i}}$, and $M = \bigoplus_{\alpha \in Q_1} \Lambda_\alpha$.
In this case,

$$\text{Mono}(X) = \{\text{monomorphisms } X(C) \to C \text{ in } \mathcal{C}\}$$

is equivalent to the category $\text{Gproj}\, H = \{N \in H\text{-}\text{mod} \mid \text{Ext}^1_H(N, H) = 0\}$ of finite-dimensional *Gorenstein projective* left $H$-modules.

# Representations of prospecies

As a special case we can consider algebras defined by the quivers with relations associated to symmetrizable Cartan matrices introduced by Geiss, Leclerc, and Schröer, in order to extend their results relating cluster algebras and Lusztig's semicanonical basis.

The algebras $H = H(C, D, \Omega)$ they introduce are defined in terms of a symmetrizable Cartan matrix $C = (c_{i,j})_{i,j \in I}$, with symmetrizer $D = \mathrm{diag}(d_i \mid i \in I)$ and orientation $\Omega$ (of $C$).

They show that $H$-mod is equivalent to representations of a prospecies $(H_i, {}_jH_i)$ where $H_i = k[x]/(x^{d_i})$ and ${}_jH_i$ are certain explicitly defined bimodules.

Using our results we can investigate the Gorenstein projective $H$-modules, and characterize when $H$ is *Cohen-Macaulay finite*, i.e. has finitely many isomorphism classes of indecomposable finite-dimensional Gorenstein projective modules.

# Representations of prospecies

## Theorem (GKKP 23)

*Let $C = (c_{\mathtt{i},\mathtt{j}})_{\mathtt{i},\mathtt{j} \in I}$ be a symmetrizable Cartan matrix with symmetrizer $D = \operatorname{diag}(d_{\mathtt{i}} \mid \mathtt{i} \in I)$ and orientation $\Omega$. Assume $d_{\mathtt{i}} \leq 2$ for all $\mathtt{i} \in I$. Let $I' \subseteq I$ be the subset of all elements $\mathtt{i}$ for which $d_{\mathtt{i}} = 2$, and let $C|_{I' \times I'} = (c_{\mathtt{i},\mathtt{j}})_{\mathtt{i},\mathtt{j} \in I'}$ be the corresponding submatrix of $C$. Then $H = H(C, D, \Omega)$ is Cohen-Macaulay finite if and only if $C|_{I' \times I'}$ is Dynkin as a symmetric Cartan matrix. Furthermore, in this case there is a bijection between the positive roots of $C|_{I' \times I'}$ and the isomorphism classes of indecomposable objects in the singularity category of $H$.*

The singularity category of $H$ is equivalent to the stable category $\overline{\operatorname{Gproj}} \, H$ by a result of Buchweitz.

We show there exists an epivalence $\overline{\operatorname{Gproj}} \, H \to kQ$-mod where $Q$ is the quiver defined by $C|_{I' \times I'}$ with orientation $\Omega|_{I' \times I'}$. Gives a bijection between their indecomposables. No finiteness assumptions on $C|_{I' \times I'}$ necessary for this.