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CORRESPONDENCES FROM TILTING THEORY IN HIGHER HOMOLOGICAL ALGEBRA

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Motivation

Theorem ([AIR ’14])

Let A be a finite-dimension algebra over an algebraically closed field k. We have bijections between:

- *The set of functorially finite torsion classes in $\operatorname{mod} A$;*
- *The set of basic two-term silting complexes for A;*
- *The set of maximal τ-rigid pairs in $\operatorname{mod} A$.*

Overview

Higher homological algebra

(higher) Torsion classes

(higher) τ-tilting

Silting

Correspondence

d-cluster-tilting subcategories

Let $\mathcal{A}$ be an essentially small, finite length abelian category, satisfying the Krull–Remak–Schmidt property.
Fix some integer $d \geq 1$
Let $\mathcal{M} \subseteq \mathcal{A}$ be generating-cogenerating and assume

$$\begin{aligned}\mathcal{M} &= \{X \in \mathcal{A} \mid \operatorname{Ext}^i_{\mathcal{A}}(X, M) = 0 \text{ for } M \in \mathcal{M} \text{ and } i = 1, \dots, d-1\} \\ &= \{Y \in \mathcal{A} \mid \operatorname{Ext}^i_{\mathcal{A}}(M, Y) = 0 \text{ for } M \in \mathcal{M} \text{ and } i = 1, \dots, d-1\}.\end{aligned}$$

Then $\mathcal{M}$ is a d-cluster-tilting subcategory of $\mathcal{A}$ [Iyama '07].

d-abelian categories

The category $\mathcal{M}$ is *d-abelian*. [Jasso '16]
Amongst other things it has

d-exact sequences $0 \to X \to E_1 \to \cdots \to E_d \to Y \to 0$

d-kernels $0 \to K_1 \to \cdots \to K_d \to X \xrightarrow{f} Y$

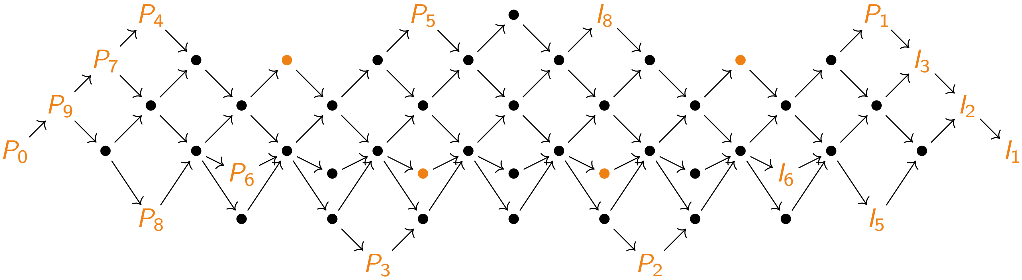
d-cokernels $X \xrightarrow{f} Y \to C_1 \to \cdots \to C_d \to 0$

Higher Auslander-Reiten translation $\tau_d X = \tau \Omega^{d-1} X$

Any d-abelian category can be obtained as a d-cluster-tilting subcategory of an abelian category [Kvamme '22, EN-I '22].
We will consider $\mathcal{M} \subseteq \mathcal{A} = \operatorname{mod} A$, where A is a finite-dimensional algebra over a field k.

Running example: A_4^2

$$\begin{tikzcd}
 & & & 4 \arrow[dr] & & & \\
 & & 3 \arrow[ur] \arrow[dr] & & 7 \arrow[dr] & & \\
 & 2 \arrow[ur] \arrow[dr] & & 6 \arrow[ur] \arrow[dr] & & 9 \arrow[dr] & \\
1 \arrow[ur] & & 5 \arrow[ur] & & 8 \arrow[ur] & & 0
\end{tikzcd}$$



Running example: $\mathcal{M} \subset \operatorname{mod} A_4^2$

$$\begin{tikzcd}
 & & & P_4 \arrow[rr] & & P_3 \arrow[r] \arrow[dr] & P_2 \arrow[rr] \arrow[dr] & & P_1 \arrow[dr] & & & \\
 & & P_7 \arrow[ur] \arrow[rr] & & P_6 \arrow[ur] \arrow[r] \arrow[dr] & P_5 \arrow[ur] \arrow[dr] & I_8 \arrow[r] & I_6 \arrow[rr] \arrow[dr] & & I_3 \arrow[dr] & & \\
 & P_9 \arrow[ur] \arrow[rr] & & P_8 \arrow[ur] \arrow[dr] & & \bullet \arrow[ur] \arrow[r] & \bullet \arrow[ur] \arrow[dr] & & I_5 \arrow[rr] & & I_2 \arrow[dr] & \\
P_0 \arrow[ur] & & & & \bullet \arrow[ur] & & & \bullet \arrow[ur] & & & & I_1
\end{tikzcd}$$

Torsion Classes

A pair $(\mathcal{T}, \mathcal{F})$ of subcategories of $\mathcal{A}$ is a torsion pair if the following conditions are satisfied:

1. For every $X \in \mathcal{A}$, there exists a short exact sequence
$$0 \to tX \to X \to fX \to 0$$
where $tX \in \mathcal{T}$ and $fX \in \mathcal{F}$.
2. $\mathrm{Hom}_{\mathcal{A}}(X, Y) = 0$ for all $X \in \mathcal{T}$ and $Y \in \mathcal{F}$.

We say that $\mathcal{T}$ is a torsion class and $\mathcal{F}$ a torsion free class.

Theorem ([Dickson '66])

A subcategory $\mathcal{T}$ of $\mathcal{A}$ is a torsion class if and only if $\mathcal{T}$ is closed under extensions and quotients.

Higher Torsion Classes [Jørgensen '16]

Let $\mathcal{M}$ be a d-abelian category. A subcategory $\mathcal{U}$ of $\mathcal{M}$ is a d-torsion class if for every M in $\mathcal{M}$, there exists a d-exact sequence

$$0 \to U_M \to M \to V_1 \to \cdots \to V_d \to 0$$

such that the following conditions are satisfied:

1. The object U_M is in $\mathcal{U}$.
2. The sequence $0 \to \mathrm{Hom}_{\mathcal{M}}(U, V_1) \to \cdots \to \mathrm{Hom}_{\mathcal{M}}(U, V_d) \to 0$ is exact for every U in $\mathcal{U}$.

Characterisation of higher torsion classes

Theorem ([AJST '22])

Let $\mathcal{U} \subseteq \mathcal{M} \subseteq \operatorname{mod} A$ be a d-torsion class in the d-cluster tilting subcategory $\mathcal{M}$ of $\operatorname{mod} A$. Then the minimal torsion class of $\operatorname{mod} A$ containing $\mathcal{U}$ is the unique torsion class $\mathcal{T}$ satisfying:

1. $\forall M \in \mathcal{M}$, $tM \in \mathcal{U}$;
2. $\mathcal{T}$ *is the minimal torsion class containing all* tM *for* $M \in \mathcal{M}$;
3. $\forall M, N \in \mathcal{M}$, $\operatorname{Ext}_{\mathcal{A}}^{d-1}(tM, fN) = 0$.

Moreover, in this case we have $\mathcal{U} = \mathcal{M} \cap \mathcal{T}$ *and* $tM \cong U_M$ *for all* $M \in \mathcal{M}$.

Characterization of higher torsion classes

Theorem ([AHJKPT ’23])

Let $\mathcal{M}$ be a d-cluster tilting subcategory of an abelian length category $\mathcal{A}$. A subcategory $\mathcal{U} \subseteq \mathcal{M}$ is a d-torsion class if and only if it is closed under d-extensions and d-quotients.

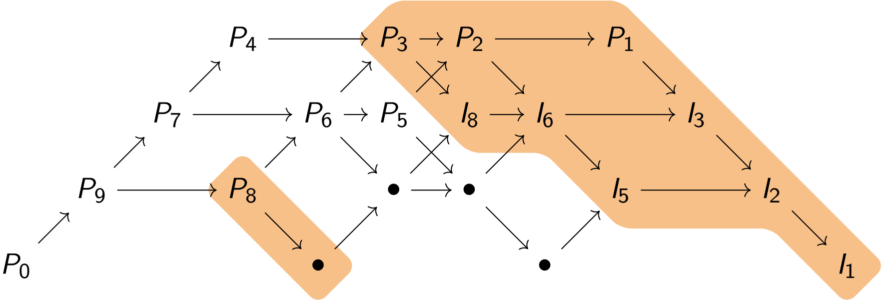
Closure under d-quotients:

$$M \xrightarrow{f} U \to E_1 \to E_2 \to \cdots \to E_d \to 0$$

Closure under d-extensions:

$$0 \to X \to E_1 \to \cdots \to E_d \to Y \to 0$$

Example



Maximal τ-rigid pairs

Definition ([AIR '14])

Consider a pair (M, P) with $M \in \operatorname{mod} A$ and $P \in \operatorname{proj} A$.

- M is called τ-rigid if $\operatorname{Hom}_A(M, \tau M) = 0$.
- (M, P) is called a τ-rigid pair in $\mathcal{M}$ if M is τ-rigid and $\operatorname{Hom}_A(P, M) = 0$.
- (M, P) is called a maximal τ-rigid pairs if either of the following equivalent conditions are satisfied:
    - $|M| + |P| = |A|$ (also known as a support τ-tilting pair)
    - If $\operatorname{Hom}(M, \tau X) = 0$, $\operatorname{Hom}(X, \tau M) = 0$ and $\operatorname{Hom}(P, X) = 0$ then $X \in \operatorname{add} M$.

Maximal τ_d-rigid pairs

Definition ([JJ '20, ZZ '23])

Let $\mathcal{M}$ be a d-cluster tilting subcategory of $\operatorname{mod} A$ and consider a pair (M, P) with $M \in \mathcal{M}$ and $P \in \operatorname{proj} A$.

- M is called τ_d-rigid if $\operatorname{Hom}_A(M, \tau_d M) = 0$.
- (M, P) is called a τ_d-rigid pair in $\mathcal{M}$ if M is τ_d-rigid and $\operatorname{Hom}_A(P, M) = 0$.
- (M, P) is called a maximal τ_d-rigid pair in $\mathcal{M}$ if it satisfies:
  - If N is in $\mathcal{M}$, then

$$N \in \operatorname{add}(M) \iff \begin{cases} \operatorname{Hom}_A(M, \tau_d N) = 0, \\ \operatorname{Hom}_A(N, \tau_d M) = 0, \\ \operatorname{Hom}_A(P, N) = 0. \end{cases}$$

  - If Q is in $\operatorname{proj} A$, then

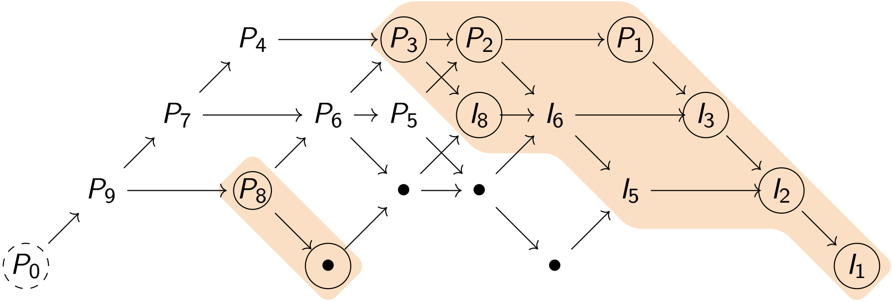
$$Q \in \operatorname{add}(P) \iff \operatorname{Hom}_A(Q, M) = 0.$$

From torsion classes to τ_d-rigid pairs [AHJKPT (wip)]

- Start with a functorially finite d-torsion class $\mathcal{U} \subseteq \mathcal{M} \subseteq \operatorname{mod} A$.
- Let $M_{\mathcal{U}}$ be a basic additive generator of Ext^d-projectives in $\mathcal{U}$.
- let $P_{\mathcal{U}}$ be the maximal basic projective A-module such that $\operatorname{Hom}_A(P_{\mathcal{U}}, \mathcal{U}) = 0$
- Then $(M_{\mathcal{U}}, P_{\mathcal{U}})$ is a basic τ_d-rigid pair in $\mathcal{M}$ with $|M_{\mathcal{U}}| + |P_{\mathcal{U}}| = |A|$.

This gives an injection ϕ from the set of functorially finite d-torsion classes to the set of τ_d-rigid pairs.

Example



Let $M = \bigoplus \odot$ and $P = P_0$.
(M, P) is a τ_d-rigid pair.

Silting complexes

Definition

The complex $S \in K^b(\text{proj } A)$ is a presilting complex if $\text{Hom}_{K^b(\text{proj } A)}(S, S[i]) = 0$ for all $i > 0$.

We say that S is silting if moreover **thick**$(S) = K^b(\text{proj } A)$, i.e., the smallest triangulated full subcategory containing S and closed under direct summands in $K^b(\text{proj } A)$.

A (pre)-silting complex $S \in K^b(\text{proj } A)$ is a $(d+1)$-term (pre)-silting complex if it is concentrated in homological degrees $0, 1, \ldots, d$, and has homology concentrated in degrees 0 and d.

Connection to τ_d-rigid pairs

Let $P_\bullet^U$ be the complex given by the minimal projective resolution of U, with the projective cover in degree zero.

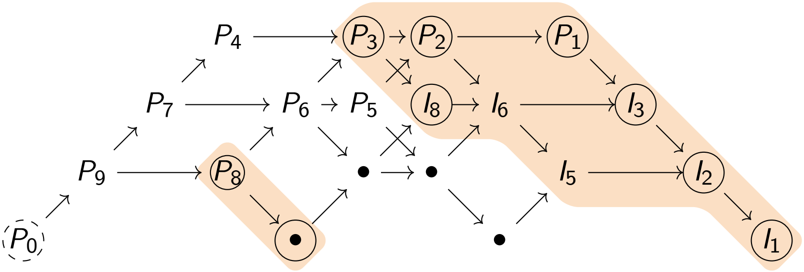
Proposition (Consequence of [MM])

Let (U, P_U) be a τ_d-rigid pair in the d-cluster tilting subcategory $\mathcal{M}$. Then $P_\bullet^{(U,P_U)} := P_\bullet^U \oplus P_U[d]$ is a $(d+1)$-term presilting object in $K^b(\text{proj } A)$.

Theorem

Let A be a finite-dimensional algebra and let $\mathcal{U}$ be a functorially finite d-torsion class in a d-cluster tilting subcategory $\mathcal{M} \subset \text{mod } A$. If $(M_\mathcal{U}, P_\mathcal{U})$ is the basic τ_d-rigid pair induced by $\mathcal{U}$, then $P_\bullet^{(M_\mathcal{U},P_\mathcal{U})}$ is a silting object in $K^b(\text{proj } A)$.

Example



$$\begin{tikzcd}
0 \arrow[d] \\
P_0^3 \oplus P_5 \oplus P_6 \oplus P_7 \oplus P_8 \arrow[d] \\
P_2 \oplus P_3^2 \oplus P_4^2 \oplus P_9 \arrow[d] \\
P_1^4 \oplus P_2^2 \oplus P_3^2 \oplus P_8^2 \arrow[d] \\
0
\end{tikzcd}$$

Consequence for ϕ

Proposition

Let A be a finite-dimensional algebra and let $\mathcal{U}$ be a functorially finite d-torsion class in a d-cluster tilting subcategory $\mathcal{M}$ in $\operatorname{mod} A$.
Then the τ_d-rigid pair $(M_{\mathcal{U}}, P_{\mathcal{U}})$ is maximal.

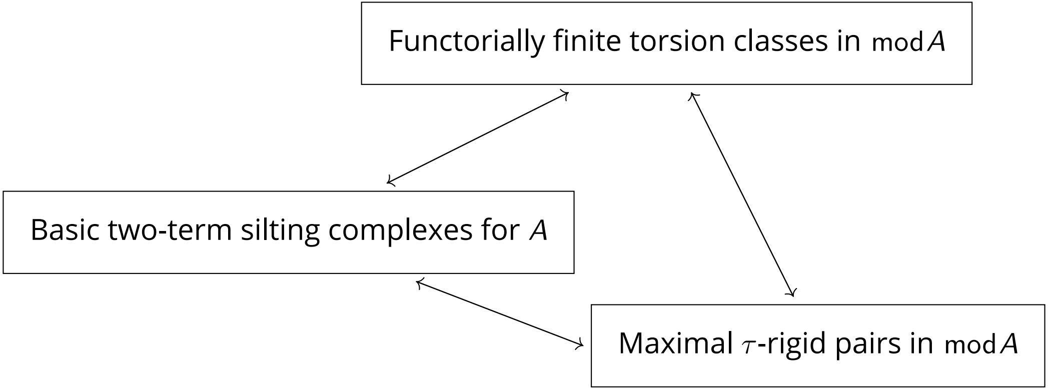
Proof.
$P_\bullet^{(M_{\mathcal{U}}, P_{\mathcal{U}})} = P_\bullet^{M_{\mathcal{U}}} \oplus P_{\mathcal{U}}[d]$ is silting.
Hence $K^b(\operatorname{proj} A) = \textbf{thick}(P_\bullet^{(M_{\mathcal{U}}, P_{\mathcal{U}})})$
Maximality is shown by lifting to $K^b(\operatorname{proj} A)$. □

In other words, we have an injection ϕ from the set of functorially finite d-torsion classes to the set of maximal τ_d-rigid pairs.

Main Result [AHJKPT (wip)]

$$\begin{tikzcd}
\left\{\begin{array}{c}\text{functorially finite torsion}\\ \text{classes } \mathcal{T} \subseteq \operatorname{mod} A \text{ such that}\\ \mathcal{T} \cap \mathcal{M} \text{ is a } d\text{-torsion class}\\ \text{and } t\mathcal{M} \subseteq \mathcal{M}\end{array}\right\} \arrow[r, "\phi_1"] \arrow[d, "- \cap \mathcal{M}"] & \left\{\begin{array}{c}\text{basic maximal}\\ \tau\text{-rigid pairs in } \operatorname{mod} A\end{array}\right\} \arrow[d, dashed] \\
\left\{\begin{array}{c}\text{functorially finite } d\text{-torsion}\\ \text{classes in } \mathcal{M}\end{array}\right\} \arrow[r, "\phi_d"] & \left\{\begin{array}{c}\text{basic maximal}\\ \tau_d\text{-rigid pairs in } \mathcal{M}\end{array}\right\}
\end{tikzcd}$$

Classical correspondence

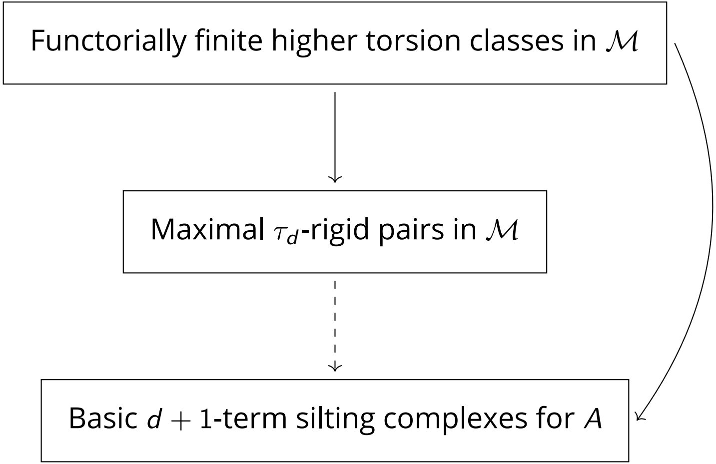


Functorially finite torsion classes in $\operatorname{mod} A$

Basic two-term silting complexes for A

Maximal τ-rigid pairs in $\operatorname{mod} A$

Main result [AHJKPT (wip)]



Functorially finite higher torsion classes in $\mathcal{M}$

Maximal τ_d-rigid pairs in $\mathcal{M}$

Basic $d+1$-term silting complexes for A

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Thanks for your attention!