j.w. Kontsevich–Vlassopoulos, Rivera–Wang

1. Some string topology

1.1 Let M be closed manifold, dim d, oriented

Consider loop spaces $\underset{\text{based}}{\Omega M}$, $\underset{\text{free}}{LM}$

Work over field $\mathbb{k} \supseteq \mathbb{Q}$

Intersection product $C_p M \otimes C_q M \to C_{p+q-d} M$ lifts to two operations on homology of loop spaces

Loop product (Chas–Sullivan product)

$$H_p(\underline{LM}) \otimes H_q(LM) \to H_{p+q-d}(LM) \quad \left(\frac{\text{coh. deg}}{\underline{d}}\right)$$

Loop coproduct (Goresky–Hingston coproduct) (+ Sullivan, Wahl, ...)

$M \overset{\text{constant}}{\hookrightarrow} LM$

$$H_p(LM, M) \longrightarrow H_{p-\underline{d+1}}(LM \times LM, \ LM\times M \cup M \times LM)$$

i.e. defined modulo constant loops

More recently, shown to factor through

$H_*(LM, \chi(M).pt)$ (Naef–Hingston, Kaufmann)

Q1 How to describe algebraically?

Q2 What are invariance properties?

1.2 Algebraic models

(Jones) M simply connected $\Rightarrow$ $HH_*(\underset{\text{alg. of cochains}}{C^*(M)}) \cong \underline{H^*(LM)}$

alg. of chains + concatenation

(Goodwillie) Any $M \Rightarrow HH_*(C_*(\Omega M)) \cong \underline{H_*(LM)}$

A1. (Simply-connected) Rivera + Wang:

Loop prod + coprod can be recovered from $C^*(M)$ using "Tate-Hochschild homology"

Use model $A \cong C^*(M)$,

A dg symmetric Frobenius algebra

Def (Rivera-Wang) $\underline{\text{Tate-Hochschild}}$ complex

$$D^*(A,A) = \cdots \to C_1(A,A^\vee) \to A^\vee \xrightarrow[\chi]{} A \to C^1(A) \to \cdots$$

When $A = C^*(M)$, χ = $\underline{\text{Euler}}$ char.

$D^*(A,A)$ related to singularity category of A

A2 Loop product is homotopy invariant

(can be proven in $\pi_1 M = 0$ by above)

but coproduct $\underline{\text{isn't}}$

Ex (Naef) Lens spaces $L_{1,7}$ & $L_{2,7}$ (3-manifolds)

have different loop coproducts

even though htpy-equivalent

Not simple-homotopy equivalent

(j.w. Kontsevich – Vlassopoulos)

3. Calabi-Yau & Pre-Calabi-Yau structures

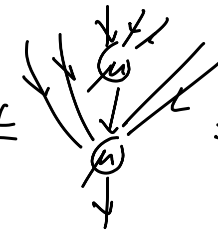
3.1 Proper & smooth CY structures

for simplicity, let A be A_∞ algebra

i.e. $\mathbb{Z}$-graded v.s., with $\{\mu^n\}_{n\geq 1}$

$\mu^n : A^{\otimes n} \to A[2-n]$, satisfying

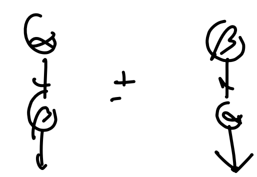
$\underline{\sum \pm \mu(\cdots \mu(\cdots)\cdots)} = 0$

$\sum \pm$  $= 0$

$\underline{\mu = \sum \mu^n}$ is Hochschild cochain $\in C^2(A,A)$

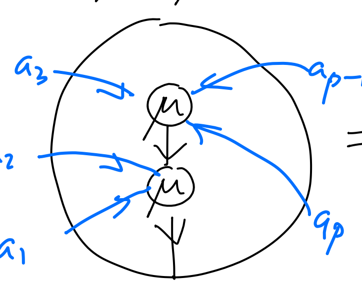
$C^*(A,A)[1]$ is dg Lie algebra,

$[\ ,\ ] =$ Gerstenhaber bracket



(another) Def A_∞-structure μ is element of deg. 1 in $C^*(A,A)[1]$ satisfying $[\mu,\mu]=0$

Graphically,



$\sum \pm$ (diagram) $= 0 \in C^*(A,A)$

Bimodule A has 2 duals:

$A^e = A^{op} \otimes A$

Linear dual $\quad A^\vee = \mathrm{Hom}_{\Bbbk}(A, \Bbbk)$

$\bar{A} = A/\Bbbk$

Bimodule dual $\quad A^! = $ "$\mathrm{Hom}_{A^e}(A, A^e)$"

$= C^*(A, A^e)$

resolve by bar complex $B(\bar{A}[1])$

Def. A is *proper* if $\dim H^*(A,\mu^1) < \infty$ $\quad A^{\vee\vee} \simeq A$

A is *smooth* if A is compact bimodule $\quad A^{!!} \simeq A$

(i.e. has finite-length resolution)

Ex. M = manifold (or fin. CW complex)

$C^*(M)$ is proper, $C_*(\Omega M)$ is smooth
(Abbaspour)

(A,μ) has Hochschild chain complex $(C_*(A,A), b_\mu)$

Fact If A is proper, $(C_*(A,A))^\vee \cong \mathrm{Hom}_{A^e}(A, \underline{A^\vee})$ lin

If A is smooth, $C_*(A,A) \cong \mathrm{Hom}_{A^e}(A^!, A)$ $\underline{\text{bimodule}}$

Def $C_*(A,A) \to \Bbbk[-d]$ is (weak) proper CY structure of dim d if induces isomorphism $A \cong A^\vee[-d]$

$\Bbbk[d] \to C_*(A,A)$ is (weak) smooth CY structure of dim d if induces isomorphism $A^! \cong A[-d]$

(non-weak CY = factors thru cyclic/negative cyclic homology)

Ex $A = C^*(M)$, M orientable, $\underline{[M]}$ is proper CY

$C_*(M) \to C_*(LM)$

$A = C_*(\Omega M)$, " , $\exists$ smooth CY coming from $\underline{[M]}$

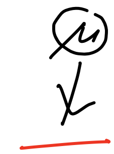
e.g. $M = S^1$, $A = \Bbbk[t^{\pm 1}]$

$A \otimes A[1]$

Hochschild chain $\omega = t^{-1}[t]$ is smooth CY.

3.2 Pre-Calabi-Yau structures (Kontsevich-T.-Vlassopoulos)

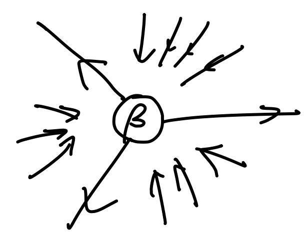
"A_∞-structure = vertex with 1 output"



Can extend this to more outputs

Def k-th higher Hochschild cochains

$$C^*_{(k)}(A) = \prod_{n_i \geq 0} \mathrm{Hom}\left(A[1]^{\otimes n_1} \otimes \cdots \otimes A[1]^{\otimes n_k}, \underline{A^{\otimes k}} \right)$$



Note $C^*_{(1)}(A) = C^*(A,A) \ni \mu$

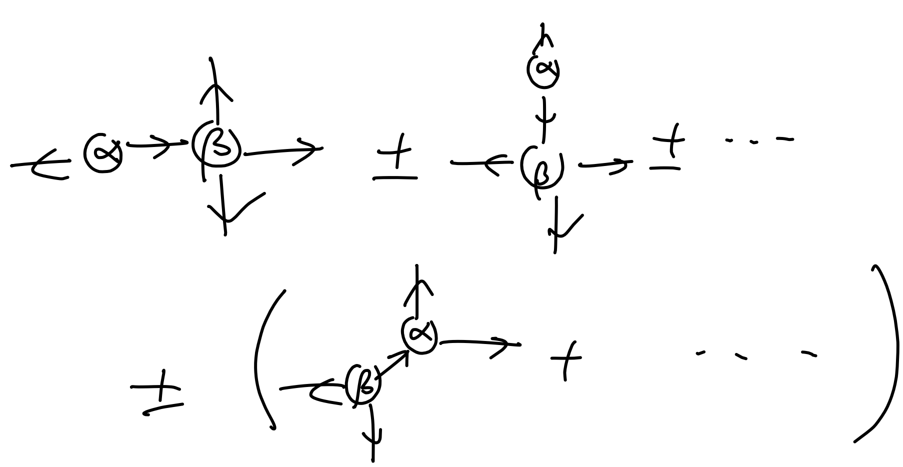
Def (KTV) $m = \mu + m_{(2)} + m_{(3)} + \cdots$

where $m_{(k)} \in C^*_{(k)}(A)$, cyclically (anti) symmetric

is *pre-CY structure* of $\overset{\text{deg}}{}$ $\underline{dk - d - 2k + 4}$ dim d

if $\underline{[m, m] = 0}$ (necklace bracket)

$\alpha, \beta \quad \alpha \in C^*_{(2)} \quad \beta \in C^*_{(3)}$



$[\alpha, \beta] \in C^*_{(4)}(A)$

$[\mu, \mu] = 0 \quad A_\infty$

$[\mu, m_{(2)}] = 0 \quad$ $m_{(2)}$ is μ-closed

$[\mu, m_{(3)}] = \frac{1}{2}[m_{(2)}, m_{(2)}]$

Ex X = Fano variety, $s \in \Gamma(X, \omega_X^{-1})$,

E = generator of $D^b\mathrm{Coh}\,X$, $A = \mathrm{End}(E)$

$\Rightarrow$ A has pre-CY structure $\dim X$

("$m_{(2)} = s$")

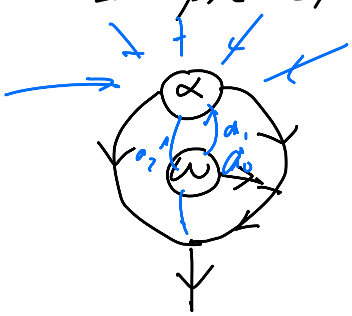
No need to be proper or smooth! But:

Thm (KTV) A has proper/smooth CY $\Rightarrow$ A has pre-CY.

Smooth case: $(A, \omega \in Cd(A,A))$ smooth CY,

$\exists$ pre-CY structure $m = \mu + m_{(2)} + \cdots$ s.t.

$m_{(2)}$ "$= \alpha$"



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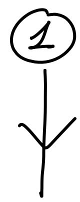
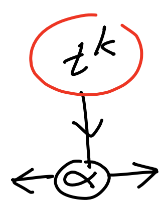


$+ [\mu, \beta]$ ("α & ω are inverses")

Ex $A = \Bbbk[t^{\pm 1}]$ $(\simeq C_*(\Omega S^1))$ $\deg t = 0$

$\omega = t^{-1}[t]$

$\alpha = m_{(2)} = m_{(2)}^{1,0}$

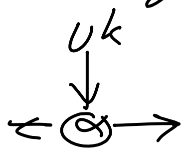


$= \boxed{\frac{1}{2}(1 \otimes t^k + t^k \otimes 1) + \sum_{1 \le i \le k-1} t^i \otimes t^{k-i}} \in A \otimes A$

$\tau = m_{(3)} = m_{(3)}^{0,0,0}$ $\qquad \tau = \boxed{\frac{1}{4}(1 \otimes 1 \otimes 1)}$

$m_{(\ge 4)} = 0$

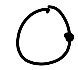
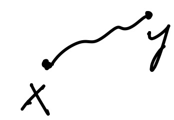
$A = \Bbbk[U]$, $\deg U = 2n \geq 2$ $\left(\simeq C_*(\Omega S^{2n+1})\right)$

$\alpha = m_{(2)}^{1,0}$  $= \sum_{0 \le i \le k-1} U^i \otimes U^{k-i-1}$ $\quad m_{(\geq 3)} = 0$

3.3 TFT structure

Cobordism "hypothesis": CY structure $\Leftrightarrow$ some type 2d TFT

Kontsevich,
Costello ('00s): proper CY structure on $\mathcal{C}$

$\Rightarrow$ "open-closed" TFT closed strings  $\mapsto HH_*(\mathcal{C})$
open strings  $\mapsto \mathcal{C}(x,y)$

Restricting to closed sector

$$HH_*(\mathcal{C})^{\otimes m} \otimes H_*(M_{g,m,n}) \longrightarrow HH_*(\mathcal{C})^{\otimes n} \qquad \begin{array}{l} m \geq 1 \\ n \geq 0 \\ g \geq 0 \end{array}$$

Others (Wahl-Westerland, Caldararu-Tu-Costello etc.) have extended to cyclic A_∞-algebras

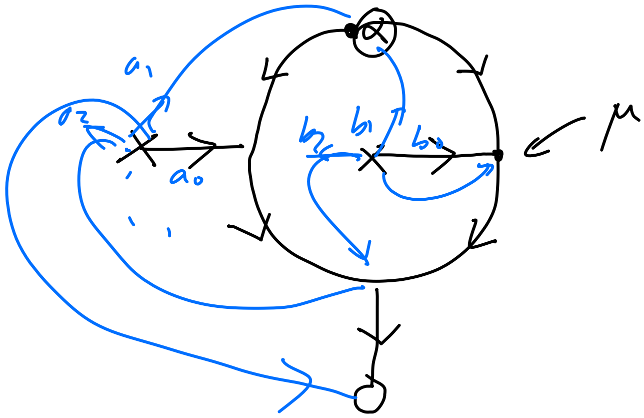
$\rightsquigarrow$ ribbon graph models

We extended this to pre-CY using ribbon quivers (oriented)

Thm (KTV) (A, m) pre-CY category, then have chain level

$$C_*(A,A)^{\otimes k} \otimes C_*(M_{g,k,\ell}) \longrightarrow C_*(A,A)^{\otimes \ell} \qquad \boxed{k, \ell \geq 1} \quad g \geq 0$$

In particular, have a product, associative & skew/commutative on $HH_*(A)$



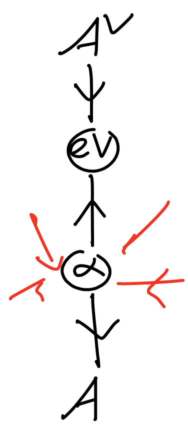
4. Pre-CY structures & products on cones (j.w. Rivera & Wang)

4.1 Let $A = A_\infty$-algebra

$m = \mu + \underset{m_{(2)}}{\alpha} + \tau + m_{(\geq 4)}$ pre-CY structure

α defines a map of bimodules $A^\vee[-d] \xrightarrow{f_\alpha} A$

(actually, $B\bar{A}[1] \otimes A^\vee[-d] \otimes B\bar{A}[1] \longrightarrow A$)



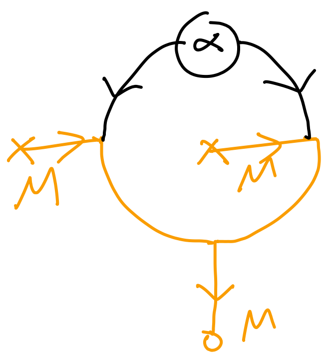
Def $M = \mathrm{Cone}(f_\alpha) =$ " $A^\vee[-d] \oplus A$ " $\circlearrowright$ $d_{A^\vee} + d_A + f_\alpha$

Prop From data of $(A, \overset{\dim d}{m})$, get structure of dim-d "pre-CY bimodule over A" on $M = \mathrm{Cone}(f_\alpha)$ such that $M \leftarrow m^M_{(2)} \rightarrow M = A \leftarrow \alpha \rightarrow A$

Cor. Given (A, m), have chain-level product of deg d

$$C_*(A,M) \otimes C_*(A,M) \xrightarrow{\pi} C_*(A,M)$$

intertwining differentials, extending product on $C_*(A,A)$



π has components $AA \xrightarrow{\pi_A} A$,
$A^\vee A \to A$, $A^\vee A \to A^\vee$, $AA^\vee \to A$, $AA^\vee \to A^\vee$
$A^\vee A^\vee \to A$, $A^\vee A^\vee \to A^\vee$

In general, π is associative on homology but not (skew) commutative.

4.2 Efimov's "categorical formal punctured neighborhood of ∞" (2016)

X smooth non-proper alg. variety $\rightsquigarrow$ $\hat{X}_\infty = \widehat{(\overline{X})}_{X_\infty}$

A smooth dg-category ("nc space") $\rightsquigarrow$ $\hat{A}_\infty$

Thm (Efimov) $\exists$ canonical dg functor $A \to \hat{A}_\infty$ such that, as bimodule,

$$C^*(A, A^\vee \underset{k}{\otimes} A) \longrightarrow C^*(A, \hom(A,A)) \longrightarrow \hat{A}_\infty$$
$$(\cong A^! \underset{A}{\otimes} A^\vee) \qquad\qquad (\cong A)$$

Recently, extended to A_∞-categories & related to "Rabinowitz Floer homology" by Ganatra - Gao - Venkatesh.

Taking $C_*(-)$

$$C_*(A, A^! \underset{A}{\otimes} A^\vee) \longrightarrow C_*(A, A) \longrightarrow C_*(A, \hat{A}_\infty)$$
$$\text{\rotatebox{90}{$\cong$}}$$
$$C^*(A, A^\vee) = (C_*(A,A))^\vee$$

(Efimov) map $(C_*(A,A))^\vee \to C_*(A,A)$ is given by "Shklyarov copairing" = Chern character of A

$$ch(A) \in HH_*(A,A) \otimes HH_*(A,A)$$

Now, if A is smooth d-CY, have quasi-iso $A^! \cong A[-d]$

Thm (RTW) (A, w) smooth d-CY A_∞-cat w/ compatible pre-CY structure $m = \mu + \alpha + (m_{(\geq 3)})$, $\exists$ homotopy

(1)
$$\begin{tikzcd} A^! \otimes_A A^\vee \arrow[r] \arrow[d] \arrow[dr, phantom, "\Swarrow"] & A \arrow[d, "id"] \\ A^\vee \arrow[r, "f_\alpha"] & A \end{tikzcd}$$

$\Rightarrow$ $\exists$ quasi-iso $\hat{A}_\infty \cong M = \mathrm{Cone}(f_\alpha)$

dg cat $\quad A_\infty$

& get associative product on $HH_*(A, \hat{A}_\infty)$ deg d

(2) This product is (skew) commutative.

Should be part of E_2 structure, already claimed by Efimov

pre-CY structure $\Rightarrow$ explicit formulas

5. Lift to coproduct

Recall $C_*(A, \hat{A}_\infty) \cong (C_*(A,A))^\vee[1] \oplus C_*(A,A)$ $\quad\pi$

Q1. Can we lift π to product on $(C_*(A,A))^\vee[1]$ and how?

Q2. When is this product associative / commutative ?

Q3. When is it dual to coproduct on $C_*(A,A)$?

A1. When $[E] = 0$, $E = ch(A) \in C_*(A) \otimes C_*(A)$

$$(C_*(A,A))^\vee \xrightarrow{E^\#} C_*(A,A) \longrightarrow C_*(A, \hat{A}_\infty) \xrightarrow{p} (C_*(A,A))^\vee[1]$$

$H^\#$ $\quad 1 + H^\#$

Pick homotopy $dH = E$, $dH^\# + H^\# d = E^\#$

$\pi_H = p \circ \pi((1+H^\#)-, (1+H^\#)-)$

This case described by Naef - Safronov (see talks) "nc volume forms"

If $A = \underline{C_*(\Omega X)}$, $ch(A) = \chi(X) \times ([pt] \otimes [pt])$

so this is case $\underline{\chi(X) = } 0$

We can extend this to case $\chi(X) \neq 0$.

Let A be non-positively graded (e.g. $C_*(\Omega X)$)

Thm (RTW) Given H such that $H^\#$ is homotopy

$$\begin{tikzcd}
(C_*(A,A))^\vee \arrow[rr, "E^\#"] \arrow[dr] & \arrow[d, phantom, "\Downarrow h"] & C_*(A,A) \\
& W \arrow[ur] &
\end{tikzcd}$$

W = $\underline{\text{vector space in deg. 0}}$

gives a lift of π to a product π_H on the space

$$\ker(C_*\underline{(A,A)^\vee \to} W)$$

Az (Skew)-commutative when H is (skew)-symmetric, always possible to choose if A is dCY

<u>NOT nec.</u> associative! But

Prop If A is non-positively graded, and $\underline{d-CY}$ with $\underline{d \geq 3}$, π_H is associative on homology

(compare to Cieliebak - Oancea)

(may be for choices of H if $\underline{d = 1, 2}$)

A^3 Always dualizes to coproduct on $HH_*(A,A)$ ($[E]=0$)
or $(HH_*(A,A)/W)$ (in general)
as long as homotopy $h = H^\#$ comes from H as above.

Conj For appropriate choice of H, under $HH_*(A,A) \cong H_*(LX)$ when $A = C_*(\Omega X)$, this is Goresky-Hingston coproduct (checked for spheres)

For $A = C_*(\Omega X)$, if $H_1(LX, \Bbbk) = 0$, have canonical choice of H (up to d-exact) $\Rightarrow$ canonical coproduct

Call this "algebraic GH coproduct"

Thm? (in progress) This algebraic GH coproduct is a simple homotopy invariant
RTW

Proof uses a dg category equivalent to $C_*(\Omega X)$, modeled on simplicial set for X.

[If A is proper + pre-CY structure $\exists$
$\overset{\text{q. iso}}{\wr\wr}$
A' s.t. $A' \oplus (A')^\vee[1-d]$ is cyclic A_∞]