On Krull-Gabriel dimension of cluster repetitive categories and cluster-tilted algebras

Alicja Jaworska-Pastuszak
Nicolaus Copernicus University in Toruń

FD Seminar, 23.02.2023

1. Basic definitions and notation.
2. The motivation and some results.
3. Short reminder on Galois coverings.
4. Galois coverings preserving Krull-Gabriel dimension.
5. Krull-Gabriel dimension of locally support-finite repetitive categories.
6. Krull-Gabriel dimension of cluster repetitive categories.
7. Some generalization and its applications.

- $K = \overline{K}$
- R is a **locally bounded K-category**, that is, R is isomorphic with a bound quiver K-category of some locally finite quiver.
- Finite locally bounded K-categories are identified with bound quiver K-algebras.
- $\mathrm{MOD}(R)$ is the category of **right R-modules**, that is, K-linear contravariant functors $M : R \to \mathrm{MOD}(K)$.
- $\mathrm{Mod}(R)$ is the category of **locally finite dimensional R-modules**, that is, $M \in \mathrm{Mod}(R)$ iff

$$\forall_{x \in \mathrm{ob}(R)} \dim_K M(x) < \infty.$$

- $\mathrm{mod}(R)$ is the full subcategory of **finite dimensional R-modules**, that is, $M \in \mathrm{mod}(R)$ iff

$$\dim M = \sum_{x \in \mathrm{ob}(R)} \dim_K M(x) < \infty.$$

- $\mathcal{G}(R)$ is the category of contravariant K-linear functors $\mathrm{mod}(R) \to \mathrm{mod}(K)$.
- $\mathcal{F}(R)$ is the full subcategory of $\mathcal{G}(R)$ formed by **finitely presented functors**, that is, functors $T \in \mathcal{G}(R)$ such that there is an exact sequence of functors

$$_R(-, M) \xrightarrow{_R(-,f)} {}_R(-, N) \to T \to 0,$$

for some $M, N, f : M \to N \in \mathrm{mod}(R)$. Then $T \cong \mathrm{Coker}_R(-, f)$.

- Categories $\mathcal{G}(R), \mathcal{F}(R)$ are abelian.

Assume $\mathcal{C}$ is a skeletally small abelian category.

- The **Krull-Gabriel filtration** $(\mathcal{C}_\alpha)_\alpha$ of $\mathcal{C}$ is a filtration of $\mathcal{C}$ by Serre subcategories defined recursively as follows:
  (1) $\mathcal{C}_{-1} = 0$ and $\mathcal{C}_{\alpha+1}$ is the Serre subcategory of $\mathcal{C}$ formed by all objects of $\mathcal{C}$ having finite length in the Serre quotient category $\mathcal{C}/\mathcal{C}_\alpha$, for any ordinal number α,
  (2) $\mathcal{C}_\beta = \bigcup_{\alpha<\beta} \mathcal{C}_\alpha$, for any limit ordinal β.
- The **Krull-Gabriel dimension** $\mathrm{KG}(\mathcal{C})$ of $\mathcal{C}$ is the smallest ordinal number α such that $\mathcal{C}_\alpha = \mathcal{C}$, if it exists. Otherwise, set $\mathrm{KG}(\mathcal{C}) = \infty$.
- If $\mathrm{KG}(\mathcal{C}) = n \in \mathbb{N}$, then the Krull-Gabriel dimension of $\mathcal{C}$ is **finite**. If $\mathrm{KG}(\mathcal{C}) = \infty$, then the Krull-Gabriel dimension of $\mathcal{C}$ is **undefined**.
- We set $\mathrm{KG}(R) := \mathrm{KG}(\mathcal{F}(R))$.

Motivation: The conjecture of Prest.

An algebra A is of domestic representation type
if and only if $\mathrm{KG}(A)$ is finite.

All known results support the conjecture of Prest. In particular:

- A is of finite representation type if and only if $\mathrm{KG}(A) = 0$ (Auslander'82).
- $\mathrm{KG}(A) \neq 1$ (Krause'98).
- If A is hereditary of Euclidean type, then $\mathrm{KG}(A) = 2$ (Geigle'86).
- $\mathrm{KG}(A) = \infty$ for the following classes of algebras: wild (Prest'88), tubular (Geigle'86), string of non-domestic type (Schröer'00), pg-critical (Kasjan-Pastuszak'14).
- If A is a string algebra of domestic type, then $\mathrm{KG}(A)$ is finite (Laking-Prest-Puninski'16).
- A is strongly simply connected: A is of domestic type if and only if $\mathrm{KG}(A)$ is finite (Wenderlich'96).
- A is generalized multicoil algebra: A is of domestic type if and only if $\mathrm{KG}(A)$ is finite (Malicki'15).
- A is a cycle-finite algebra of infinite representation type: A is domestic if and only if $\mathrm{KG}(A)$ is finite (Skowroński'16).

A locally bounded K-category R is cycle-finite, if for any cycle

$$M_0 \xrightarrow{f_1} M_1 \to \ldots \to M_{r-1} \xrightarrow{f_r} M_r = M_0$$

of non-zero non-isomorphisms in $\mathrm{ind}(R)$, we have $f_1, \ldots, f_r \notin \mathrm{rad}_R^{\infty}$.

Question (Skowroński'16).

Is it possible to apply the result for cycle-finite algebras in the study of Krull-Gabriel dimension of standard selfinjective algebras of infinite representation type?
These algebras have "nice" Galois coverings (by a cycle-finite categories). Do they preserve KG dimension?

Theorem (Pastuszak'19).

Assume R is a locally support-finite locally bounded K-category, G is a torsion-free admissible group of K-linear automorphisms of R. Assume that $A = R/G$ is the orbit category and $F : R \to A$ the associated Galois covering. Then $\mathrm{KG}(R) = \mathrm{KG}(A)$.

Let R, A be locally bounded K-categories, G a group of K-linear automorphisms of R acting freely on $\mathrm{ob}(R)$ (that is $gx = x$ if and only if $g = 1$, for any $g \in G$, $x \in \mathrm{ob}(R)$). Then a K-linear functor $F : R \to A$ is a **Galois covering**, if:

- $F : R \to A$ induces isomorpisms

$$\bigoplus_{g\in G} R(gx, y) \cong A(F(x), F(y)) \cong \bigoplus_{g\in G} R(x, gy)$$

  of vector spaces, for any $x, y \in \mathrm{ob}(R)$,
- $F : R \to A$ induces a surjective function $\mathrm{ob}(R) \to \mathrm{ob}(A)$,
- $Fg = F$, for any $g \in G$,
- for any $x, y \in \mathrm{ob}(R)$ such that $F(x) = F(y)$ there is $g \in G$ with $gx = y$.

In the above case, the functor F induces an isomorphism $A \cong R/G$ where R/G is the **orbit category**.

Assume R is a locally bounded K-category, G is a group of K-linear automorphisms of R acting freely on $\mathrm{ob}(R)$ and $F : R \to A \cong R/G$ the associated Galois covering. Then:

- The **pull-up functor** $F_\bullet : \mathrm{MOD}(A) \to \mathrm{MOD}(R)$ is the exact functor $(-) \circ F^{op}$.
- $F_\bullet$ has the left adjoint $F_\lambda : \mathrm{MOD}(R) \to \mathrm{MOD}(A)$ and the right adjoint $F_\rho : \mathrm{MOD}(R) \to \mathrm{MOD}(A)$ which are the **push-down functors**.
- Assume $M \in \mathrm{MOD}(R)$, $a \in \mathrm{ob}(A)$ and $a = F(x)$, for $x \in \mathrm{ob}(R)$. Then $F_\lambda(M)(a) = \bigoplus_{g \in G} M(gx)$ and $F_\rho(M)(a) = \prod_{g \in G} M(gx)$. Note that $F_\lambda(\mathrm{mod}(R)) \subseteq \mathrm{mod}(A)$ and $F_\lambda|_{\mathrm{mod}(R)} = F_\rho|_{\mathrm{mod}(R)}$.
- The group G acts on $\mathrm{mod}(R)$ as ${}^g M := M \circ g^{-1}$ and on homomorphisms in a natural way.
- If G is torsion-free, then it acts freely on $\mathrm{ind}(R)$, that is, ${}^g N \cong N$ yields $g = 1$, for any $N \in \mathrm{ind}(R)$.

Let R be locally bounded K-category.

- For $M \in \mathrm{MOD}(R)$, the **support** $\mathrm{supp}(M)$ of M is the full subcategory of R formed by all objects x in R such that $M(x) \neq 0$.
- The category R is **locally support-finite**, if for any object x of R the union of the sets $\mathrm{supp}(N)$, where $N \in \mathrm{ind}(R)$ and $N(x) \neq 0$, is finite.

Theorem.

Assume R is a locally support-finite K-category, G an admissible torsion-free group of K-linear automorphisms of R and $F : R \to A$ the Galois covering. Then the functor $F_\lambda : \mathrm{mod}(R) \to \mathrm{mod}(A)$ is a Galois covering of module categories, that is,

$$\mathrm{mod}(R)/G \cong \mathrm{mod}(A).$$

In particular: F_λ is dense, preserves indecomposable modules and Auslander-Reiten sequences.

The proof that $\mathrm{KG}(R) = \mathrm{KG}(A)$ is based on the general facts:

Fact 1.

Assume $\mathcal{C}, \mathcal{D}$ are abelian categories and $F : \mathcal{C} \to \mathcal{D}$ is an exact functor.

(1) If F is full and dense, then $\mathrm{KG}(\mathcal{D}) \leq \mathrm{KG}(\mathcal{C})$.

(2) If F is faithful, then $\mathrm{KG}(\mathcal{C}) \leq \mathrm{KG}(\mathcal{D})$.

Fact 2.

Assume R is locally support-finite locally bounded K-category and G is an admissible group of K-linear automorphisms of R. There is a finite convex subcategory B_R of R, the **fundamental domain** of R, such that for any $M \in \mathrm{ind}(R)$ there is $g \in G$ with $\mathrm{supp}({}^{g}M) \subseteq B_R$.

The sketch of the proof of (Pastuszak'19).

Recall that $F : R \to A$ is a Galois covering with R-lsf, G- torsion-free. We define two exact functors

$$\Phi : \mathcal{F}(R) \to \mathcal{F}(A) \text{ and } \Lambda : \mathcal{F}(A) \to \mathcal{F}(B_R),$$

and use Fact 1.

Assume $T \in \mathcal{F}(R)$, then $T = \mathrm{Coker}_R(-, f)$, for $f : M \to N$ in $\mathrm{mod}(R)$.

R is lsf: $\mathrm{Ind}(R) = \mathrm{ind}(R) \Rightarrow$
$F_\bullet(\mathrm{mod}(A)) \subseteq \mathrm{Add}(\mathrm{mod}(R))$

$$\begin{array}{ccc} \mathrm{mod}(R) & \xrightarrow{T} & \mathrm{MOD}(K) \\ & & \nearrow \\ \mathrm{mod}(A) & & \end{array}$$

$T \longmapsto \hat{T} : \mathrm{Add}(\mathrm{mod}(R)) \to \mathrm{MOD}(K)$ - additive closure of T
$(\hat{T}(\oplus M_i) = \oplus T(M_i))$
Define $\Phi : \mathcal{F}(R) \to \mathcal{G}(A)$ as $\Phi(T) = \hat{T} \circ F_\bullet$. It can be shown that

$$\Phi(T) = \hat{T} \circ F_\bullet = \mathrm{Coker}_A(F_\bullet(-), f) \cong \mathrm{Coker}_A(-, F_\lambda f) \in \mathcal{F}(A)$$

since $(F_\bullet, F_\rho)$ is an adjoint pair and $F_\lambda = F_\rho$ on $\mathrm{mod}(R)$.

- Φ is well-defined (does not depend on the presentation of T) and exact (as a composition).
- Φ is faithful (F_λ is dense and $F_\bullet(F_\lambda(M)) \cong \bigoplus_{g \in G} {}^g M$).
- **Hence we obtain** $\mathrm{KG}(R) \leq \mathrm{KG}(A)$.

Assume $U \in \mathcal{F}(A)$ and $\mathcal{E} : \mathrm{mod}(B_R) \to \mathrm{mod}(R)$ is the extension by zeros.

Define $\Lambda : \mathcal{F}(A) \to \mathcal{F}(B_R)$ as $\Lambda(U) = U \circ F_\lambda \circ \mathcal{E}$.

$$\begin{array}{ccc} \mathrm{mod}(A) & \xrightarrow{U} & \mathrm{mod}(K) \\ \uparrow F_\lambda & & \\ \mathrm{mod}(R) & & \\ \uparrow \mathcal{E} & & \\ \mathrm{mod}(B_R) & & \end{array}$$

- $\mathcal{E}$ is exact, full and faithful, hence $\mathrm{KG}(B_R) \leq \mathrm{KG}(R)$.
- It can be shown that Λ is well-defined ($U \circ F_\lambda \circ \mathcal{E} \in \mathcal{F}(B_R)$).
- $F_\lambda \circ \mathcal{E}$ is dense (since B_R is a fundamental domain),
- Λ is exact and faithful (as a composition with a dense functor).
- **Hence we obtain** $\mathrm{KG}(A) \leq \mathrm{KG}(B_R) \leq \mathrm{KG}(R)$.

Conclusion.

$\mathrm{KG}(R) = \mathrm{KG}(B_R) = \mathrm{KG}(A)$

A a finite dimensional K-algebra, $D(A) = \mathrm{Hom}_K(A, K)$ - A-A-bimodule

$$\widehat{A} = \begin{bmatrix} \ddots & & & & 0 \\ \ddots & A_{-1} & & & \\ & D(A)_0 & A_0 & & \\ & & D(A)_1 & A_1 & \\ 0 & & & \ddots & \ddots \end{bmatrix} \text{ – repetitive category of } A$$

- It is locally fin. dim. K-algebra (locally bounded K-category), $A_i = A$ and $D(A)_i = D(A)$, and there are only finitely many non-zero entries.
- Identity maps $A_i \to A_{i-1}$, $D(A)_i \to D(A)_{i-1}$ induce an automorphism $\nu : \widehat{A} \to \widehat{A}$.
- There is a Galois covering $G : \widehat{A} \to \widehat{A}/\langle \nu \rangle = T(A)$, where $T(A) \cong A \ltimes D(A)$ is a trivial extension algebra.

Theorem. (Assem-Skowroński'93)

The repetitive category $\widehat{A}$ of algebra A is lsf and tame if and only if $\widehat{A} \cong \widehat{B}$ where B is tilted algebra of Dynkin or Euclidean type, or tubular algebra.

Corollary 1. (Pastuszak'19)

Let A be an algebra such that $\widehat{A}$ is lsf. Then $\mathrm{KG}(\widehat{A}) \in \{0, 2, \infty\}$ and:

(a) $\mathrm{KG}(\widehat{A}) = 0$ if and only if $\widehat{A} \cong \widehat{B}$ for B tilted of Dynkin type;

(b) $\mathrm{KG}(\widehat{A}) = 2$ if and only if $\widehat{A} \cong \widehat{B}$ for B tilted of Euclidean type;

(c) $\mathrm{KG}(\widehat{A}) = \infty$ if and only if $\widehat{A}$ is wild or $\widehat{A} \cong \widehat{B}$ for B tubular.

- B- Euclidean type $\Rightarrow$ $\widehat{B}$ - cycle finite of domestic type $\Rightarrow$ fund. domain C cycle finite of domestic type $\Rightarrow$ $\mathrm{KG}(\widehat{B}) = \mathrm{KG}(C) = 2$
- B - tubular $\Rightarrow$ $B \subset \widehat{B}$ - convex $\Rightarrow$ $\mathrm{KG}(B) \leq \mathrm{KG}(\widehat{B}) \Rightarrow \mathrm{KG}(\widehat{B}) = \infty$

Corollary 2. (Pastuszak'19)

A standard selfinjective algebra of infinite type

(a) if A domestic then $\mathrm{KG}(A) = 2$;

(b) if A nondomestic of polynomial growth then $\mathrm{KG}(A) = \infty$.

- $A \cong \widehat{B}/G$, $G \cong \mathbb{Z}$, $\widehat{B}$ - cycle-finite, tame and lsf:
  B - tilted Euclidean $\Rightarrow$ $\mathrm{KG}(A) = \mathrm{KG}(\widehat{B}) = 2$
  B - tubular $\Rightarrow$ $\mathrm{KG}(A) = \mathrm{KG}(\widehat{B}) = \infty$

C tilted algebra, $E = \mathrm{Ext}^2_C(DC, C)$ - C-C-bimodule

$$\check{C} = \begin{bmatrix} \ddots & & & & 0 \\ \ddots & C_{-1} & & & \\ & E_0 & C_0 & & \\ & & E_1 & C_1 & \\ 0 & & & \ddots & \ddots \end{bmatrix} \text{ – cluster repetitive category of } C$$

Identity maps $C_i \to C_{i-1}$, $E_i \to E_{i-1}$ induce an automorphism $\nu : \check{C} \to \check{C}$ and we have a Galois covering $G : \check{C} \to \check{C}/\langle\nu\rangle = \tilde{C}$

$$\tilde{C} \cong C \ltimes \mathrm{Ext}^2_C(DC, C) \text{ relation extension algebra}$$

$\mathrm{End}_{\mathcal{C}_Q}(T)$ - cluster tilted algebra of type Q

Theorem. (Assem-Brüstle-Schiffler'08)

A is a cluster tilted algebra of type Q if and only if $A = \tilde{C}$ for tilted algebra C of type Q.

Proposition.

There exists a fundamental domain B of $\check{C}$ and hence $\check{C}$ is lsf and $\mathrm{KG}(\check{C}) = \mathrm{KG}(\widetilde{C})$.

- Assem-Brüstle-Schiffler defined a fundamental domain for push-down functor $G_\lambda : \mathrm{mod}(\check{C}) \to \mathrm{mod}\,\tilde{C}$.
- The cluster duplicated algebra $\begin{bmatrix} C_0 & 0 \\ E & C_1 \end{bmatrix}$ is a fund. domain of $\check{C}$.

Theorem. (Assem-Brüstle-Schiffler'08)

There exists an additive K-linear functor $\phi : \mathrm{mod}(\widehat{C}) \to \mathrm{mod}(\check{C})$ which is full, dense (and exact) such that $\mathrm{Ker}(\phi)$ equals the class of all homomorphisms in $\mathrm{mod}(\widehat{C})$ which factorize through $\mathrm{add}(\mathcal{K}_C)$, where

$$\mathcal{K}_C = \{\widehat{P}_x, \tau^{1-i}\Omega^{-i}(C) \mid x \in (\widehat{C})_0, i \in \mathbb{Z}\} \subset \mathrm{mod}(\widehat{C}).$$

- $\widehat{P}_x$ is an indecomposable projective $\widehat{C}$-module at the vertex $x \in (\widehat{C})_0$
- $\tau = \tau_{\widehat{C}}$ is the Auslander-Reiten translation in $\mathrm{mod}(\widehat{C})$, Ω - syzygy functor

Aim: $\mathrm{KG}(\check{C}) \leq \mathrm{KG}(\widehat{C})$

Theorem 1. (– -Pastuszak'22)

(1) $\mathcal{K}_C$ is hom-support finite, that is for any $N \in mod(\widehat{C})$ there is only finitely many objects $X \in \mathcal{K}_C$ such that ${}_{\widehat{C}}(X, N) \neq 0$.

(2) $\mathrm{add}(\mathcal{K}_C)$ is contravariantly finite class in $\mathrm{mod}(\widehat{C})$, that is for any $N \in \mathrm{mod}(\widehat{C})$ there exists $M_N \in \mathrm{add}(\mathcal{K}_C)$ and $\alpha_N : M_N \to N$ such that

$$
{}_{\widehat{C}}(*, M_N) \xrightarrow{{}_{\widehat{C}}(*, \alpha_N)} {}_{\widehat{C}}(*, N) \to 0 \quad \text{is exact for } * \in \mathrm{add}(\mathcal{K}_C).
$$

(3) The functor $\Lambda_\phi : \mathcal{F}(\check{C}) \to \mathcal{G}(\widehat{C})$ defined as the composition $(-) \circ \phi$ satisfies the condition $\mathrm{Im}(\Lambda_\varphi) \subseteq \mathcal{F}(\widehat{C})$.

Sketch of the proof:

(1)

$$\mathcal{K}_C = \{\widehat{P}_x, \tau^{1-i}\Omega^{-i}(C) \mid x \in (\widehat{C})_0, i \in \mathbb{Z}\}$$

- C tilted of Euclidean or wild type Δ:

$$\Gamma_{\widehat{C}} = \bigvee_{q \in \mathbb{Z}} (\mathcal{X}_q \vee \mathcal{C}_q)$$

where $q \in \mathbb{Z}$, stable part $\mathcal{X}_q^s$ is of the form $\mathbb{Z}\Delta$ $\mathcal{C}_q^s$ is a union either of stable tubes or of components of the form $\mathbb{Z}\mathbb{A}_\infty$

(2) Take $M_N = \bigoplus_{X \in \mathcal{K}_C} ({}_{\widehat{C}}(X, N) \otimes_K X)$,
$\alpha_N : M_N \to N$, $\alpha_N(f \otimes x) = f(x)$, for any $f \in {}_A(X, N)$ and $x \in X$.

(3) Let $U \in \mathcal{F}(\check{C})$, hence

$$_{\check{C}}(-, X) \xrightarrow{\check{C}(-,f)} {}_{\check{C}}(-, Y) \to U \to 0$$

is exact. Then $U\phi \in \mathcal{G}(\widehat{C})$ and

$$_{\check{C}}(\phi(-), X) \xrightarrow{\check{C}(\phi(-),f)} {}_{\check{C}}(\phi(-), Y) \to U\phi \to 0$$

is exact. It's enough to show $_{\check{C}}(\phi(-), Z) \in \mathcal{F}(\widehat{C})$ since $\mathcal{F}(\widehat{C})$ is abelian.
Since ϕ is dense, $_{\check{C}}(\phi(-), Z) = {}_{\check{C}}(\phi(-), \phi(N))$ for some $N \in \mathrm{mod}(\widehat{C})$.
Applying (2) we can show that

$$_{\widehat{C}}(-, M_N) \xrightarrow{\widehat{C}(-,\alpha_N)} {}_{\widehat{C}}(-, N) \xrightarrow{\widetilde{\phi}} {}_{\check{C}}(\phi(-), \phi(N)) \to 0,$$

where $\tilde{\phi}$ is natural transformation of functors ($\tilde{\phi}_X(f) = \phi(f)$), is exact.

Corollary.

$\mathrm{KG}(\check{C}) \leq \mathrm{KG}(\widehat{C})$

- $\Lambda_\phi \colon \mathcal{F}(\check{C}) \to \mathcal{F}(\widehat{C})$ (composition $(-) \circ \phi$) is exact and faithful.

Theorem 2. (– -Pastuszak'22)

$\mathrm{KG}(\widetilde{C}) = \mathrm{KG}(\check{C}) = \mathrm{KG}(\widehat{C}) \in \{0, 2, \infty\}$, for any tilted algebra C, and the following assertions hold:

(1) C is tilted of Dynkin type if and only if $\mathrm{KG}(\widetilde{C}) = 0$.

(2) C is tilted of Euclidean type if and only if $\mathrm{KG}(\widetilde{C}) = 2$.

(3) C is tilted of wild type if and only if $\mathrm{KG}(\widetilde{C}) = \infty$.

In particular, Prest conjecture is valid for cluster-tilted algebras.

- C of Euclidean type $\Rightarrow \mathrm{KG}(\widetilde{C}) = \mathrm{KG}(\check{C}) \leq \mathrm{KG}(\widehat{C}) = 2$, but $\mathrm{KG}(\widetilde{C}) \neq 0, 1$
- C either of Dynkin, or of Euclidean or of wild type - equivalences

Theorem. (Bobiński'22)

If $H = KQ$ is a hereditary algebra and $\widetilde{C}$ is a cluster-tilted algebra of type Q, then $\mathrm{KG}(\widetilde{C}) = \mathrm{KG}(H)$.

- (Geigle'86): $\mathcal{C}$ be a category such that $\mathcal{F}(\mathcal{C})$ is abelian and $\mathcal{B}$ be a full subcategory of $\mathcal{C}$ with only finitely many indecomposable objects up to isomorphism, $S_X \in \mathcal{F}(\mathcal{C})$ for each $X \in \mathcal{B}$:

$$\mathrm{KG}(\mathcal{C}) = \mathrm{KG}(\mathcal{C}/[\mathcal{B}])$$

- $\mathrm{KG}(\widetilde{C}) = \mathrm{KG}(\mathcal{C}_Q)$ and $\mathrm{KG}(H) = \mathrm{KG}(\mathcal{C}_Q)$

Assume $F : R \to A$ is a Galois covering.

(1) Recall that if R is lsf, then $F_\lambda : \mathrm{mod}(R) \to \mathrm{mod}(A)$ is dense and

$$F_\bullet(\mathrm{mod}(A)) \subseteq \mathrm{Add}(\mathrm{mod}(R)).$$

Hence we may define $\widehat{T} : \mathrm{Add}(\mathrm{mod}(R)) \to \mathrm{Mod}(K)$ (additive closure of T) and set $\Phi(T) := \widehat{T} \circ F_\bullet$.

(2) If R is arbitrary (and thus F_λ may not be dense), the construction of $\Phi : \mathcal{F}(R) \to \mathcal{F}(A)$ such that

$$\Phi(\mathrm{Coker}_R(-, f)) = \mathrm{Coker}_A(-, F_\lambda(f))$$

is as follows.

Assume C is locally bounded and let $\mathcal{H}(C)$ be the **morphism category** of C, that is:

- objects of $\mathcal{H}(C)$ are homomorphisms in $\mathrm{mod}(C)$.
- morphisms in $\mathcal{H}(C)$ are pairs $(a, b) : f \to f'$ of homomorphisms in $\mathrm{mod}(C)$ such that the following diagram commutes:

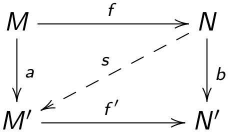
$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} \\ M' & \xrightarrow{f'} & N' \end{array}$$

Define the functor

$$\mathrm{Ck}_C : \mathcal{H}(C) \to \mathcal{F}(C)$$

as $f \mapsto \mathrm{Coker}_C(-, f)$ (on objects). Properties:

- Ck_C is full and dense.
- The kernel $K_C := \mathrm{Ker}(\mathrm{Ck}_C)$ is formed by **null-homotopic** morphisms, that is, morphisms $(a, b) : f \to f'$ for which there is $s : N \to M'$ with $b = f's$:



- We obtain $\mathcal{H}(C)/K_C \cong \mathcal{F}(C)$.

Define the functor

$$F_\lambda^{\mathcal{H}} : \mathcal{H}(R) \to \mathcal{H}(A)$$

as $f \mapsto F_\lambda(f)$ (on objects). Properties:

- easy to see that $F_\lambda^{\mathcal{H}}(K_R) \subseteq K_A$, so $F_\lambda^{\mathcal{H}}$ induces

$$\Phi : \mathcal{F}(R) \cong \mathcal{H}(R)/K_R \to \mathcal{H}(A)/K_A \cong \mathcal{F}(A)$$

  such that $\Phi(\mathrm{Coker}_R(-, f)) = \mathrm{Coker}_A(-, F_\lambda(f))$:

$$\begin{array}{ccc} \mathcal{H}(R) & \xrightarrow{\mathrm{Ck}_R} & \mathcal{F}(R) \\ \downarrow{\scriptstyle F_\lambda^{\mathcal{H}}} & & \downarrow{\scriptstyle \Phi} \\ \mathcal{H}(A) & \xrightarrow{\mathrm{Ck}_A} & \mathcal{F}(A) \end{array}$$

- Φ exact, because $F_\lambda^{\mathcal{H}}$ is exact; one can show that Φ is faithful.

Theorem. (– -Pastuszak'23)

Assume $F : R \to A$ is a Galois covering. Then $\mathrm{KG}(R) \leq \mathrm{KG}(A)$.

Remark

If F_λ is dense and G torsion-free, then Pastuszak proved that $\Phi : \mathcal{F}(R) \to \mathcal{F}(A)$ is a **Galois precovering** of functor categories, that is:

- G acts freely on $\mathcal{F}(R)$ as $(gT)(X) = T({}^{g^{-1}}X)$.
- There are natural isomorphisms of vector spaces

$$\bigoplus_{g \in G} {}_{\mathcal{F}(R)}(gT_1, T_2) \to {}_{\mathcal{F}(A)}(\Phi(T_1), \Phi(T_2)).$$

- We have $\Phi(T) \cong \Phi(gT)$ and $\Phi(T_1) \cong \Phi(T_2)$ implies $T_1 \cong hT_2$, for some $h \in G$, if T_1, T_2 have local endomorphism rings.

Remark.

(1) The above theorem can be viewed as some instance of general results of Asashiba from *A generalization of Gabriel's Galois covering functors and derived equivalences* (obtained independently).

(2) For arbitrary $F : R \to A$ and G, the functor $\Phi : \mathcal{F}(R) \to \mathcal{F}(A)$ is not a Galois precovering.

Let T be triangulation of a surface S, $\vec{T}$ an orientation of triangles, $(Q, f) = (Q(S, \vec{T}), f)$ the associated triangulation quiver, $m_\bullet, c_\bullet$ weight and parameter functions on (Q, f).
Weighted surface algebra $\Lambda = \Lambda(S, \vec{T}, m_\bullet, c_\bullet) = KQ/I$, where (Q, f) is a triangulation quiver, generators of I depend on permutation f.
Exceptional families: disc algebras $D(\lambda)$, tetrahedral algebras $\Lambda(\lambda)$, triangle algebras $T(\lambda)$, spherical algebras $S(\lambda)$ for any $\lambda \in K^*$.

Theorem (Erdmann-Skowroński'20)

(1) Weighted surface algebras Λ not isomorphic to $D(\lambda)$, $\Lambda(\lambda)$, $T(\lambda)$, $S(\lambda)$ are tame of non-polynomial growth.

(2) For Λ not isomorphic to $D(\lambda)$, $\Lambda(\lambda)$, $T(\lambda)$, $S(\lambda)$, $D(\lambda)^{(1)}$, $D(\lambda)^{(2)}$, there exists a quotient algebra $\Gamma = \Lambda/L$ of Λ which is a string algebra of non-polynomial growth.

Observe that in (2) there is $\mathrm{mod}\Gamma \to \mathrm{mod}\Lambda$ - faithful, exact $\Rightarrow$ $\mathrm{KG}(\Gamma) \leq \mathrm{KG}(\Lambda)$. Hence $\mathrm{KG}(\Lambda) = \infty$.
For $D(\lambda)^{(1)}$, $D(\lambda)^{(2)}$ we also have $\mathrm{KG}(\Lambda) = \infty$.

If $\lambda \neq 1$ then $D(\lambda)$, $\Lambda(\lambda)$, $T(\lambda)$, $S(\lambda)$ are of polynomial growth and:

- tetrahedral algebras $\Lambda(\lambda) \cong T(B(\lambda))$ for $B(\lambda)$ tubular algebra of type $(2, 2, 2, 2)$,
- disc algebras $D(\lambda) = \Lambda(\lambda)/\mathbb{Z}_3$,
- spherical algebras $S(\lambda)) \cong T(C(\lambda))$ for $C(\lambda)$ tubular algebra of type $(2, 2, 2, 2)$,
- triangle algebras $T(\lambda) \cong S(\lambda)/\mathbb{Z}_2$.

By applying new theorem in this case also $\mathrm{KG}(\Lambda) = \infty$.

Theorem. (– -Pastuszak'23)

Periodic weighted surface algebras Λ have $\mathrm{KG}(\Lambda) = \infty$.