A_∞ deformations of extended Khovanov arc algebras and Stroppel's Conjecture

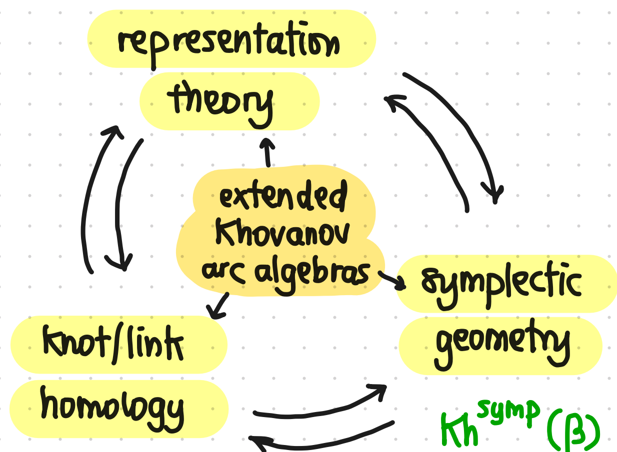
joint with Zhengfang Wang

arXiv: 2211.03354

Big picture

perverse sheaves on Grassmannians

parabolic category $\mathcal{O}$



$Kh(\beta)$

Khovanov homology

$Kh^{symp}(\beta) := HF^*(\varphi_\beta(L_{⋒}), L_{⋒})$

symplectic Khovanov homology

Floer cohomology in a Fukaya-Seidel category

Outline

I extended Khovanov arc algebras K_m^n & Stroppel's Conjecture

II Koszul duality & Hochschild cohomology

III A_∞ deformations

IV Further directions

I Extended Khovanov arc algebras

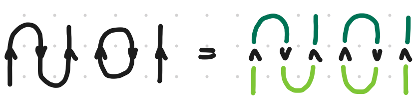
Let $m, n \geq 1$ be two natural numbers

A **weight** of type $\binom{n}{m}$ is a sequence of m $\vee$'s & n $\wedge$'s.

$\vee\vee\wedge\wedge\vee\wedge\vee\wedge\vee\vee$

An **arc diagram** of type $\binom{n}{m}$ is obtained by drawing open/closed arcs through $\wedge$'s & $\vee$'s inducing a well-defined orientation on each arc subject to the condition that the following shapes do not appear

<u>Rmk.</u>



$\widehat{\beta}$ cap diagram
λ weight
$\underset{\smile}{\alpha}$ cup diagram

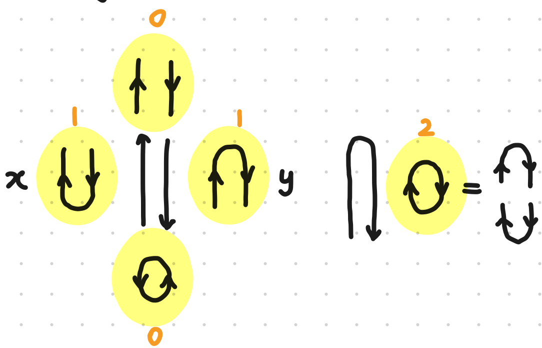
$\Big\}$ arc diagram $\underset{\smile}{\alpha}\lambda\widehat{\beta}$

Def. The extended Khovanov arc algebra K^n_m is the algebra with $\Bbbk$-basis given by arc diagrams of type $\substack{n\\m}$ graded by the number of clockwise cups & caps in each arc diagram.

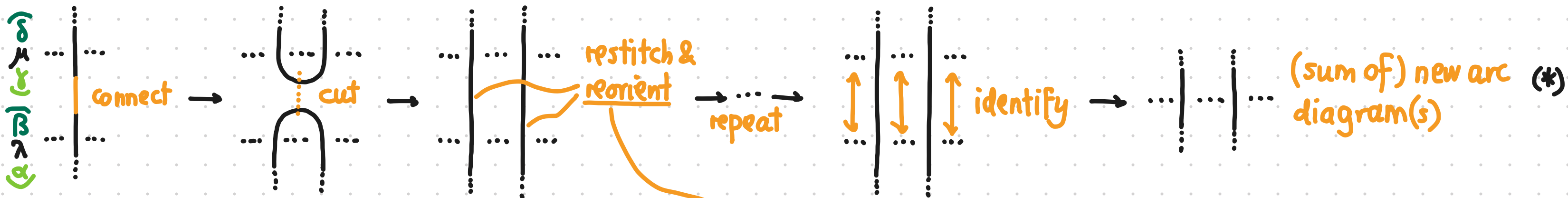
$\dim K^n_m$ = # arc diagrams

$m \backslash n$	1	2	3	ℓ
1	5	9	13	$4\ell+1$
2	9	47	101	$8\ell^2+14\ell-13$

Ex. $K^1_1 \simeq \Bbbk\left(\bullet \underset{y}{\overset{x}{\rightleftarrows}} \bullet\right)/(yx) \qquad |x| = |y| = 1.$



The multiplication can be defined via a 2D TQFT $\longleftrightarrow$ Frobenius algebra structure on $\Bbbk[\varepsilon]/(\varepsilon^2)$



$1\otimes 1 \longmapsto 1$ $\quad$ $1\otimes\varepsilon \longmapsto \varepsilon$

$\varepsilon\otimes 1 \longmapsto \varepsilon$ $\quad$ $\varepsilon\otimes\varepsilon \longmapsto 0$

$\varepsilon\otimes\zeta \longmapsto 0$ $\quad$ $\cdots$

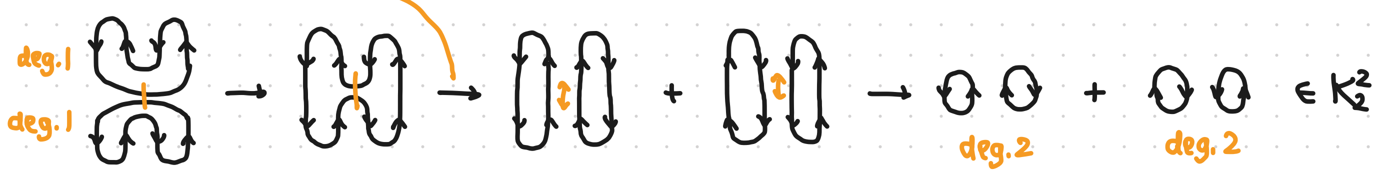
$1 \longmapsto 1\otimes\varepsilon + \varepsilon\otimes 1$

$\varepsilon \longmapsto \varepsilon\otimes\varepsilon$

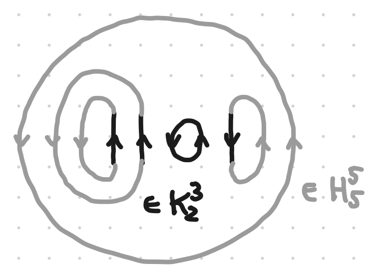
$\zeta \longmapsto \varepsilon\otimes\zeta$ $\quad\cdots$

$$\underline{\alpha}\lambda\overline{\beta}\cdot\underline{\gamma}\mu\overline{\delta} = \begin{cases} (*) & \text{if } \beta=\gamma \\ 0 & \text{else.}\end{cases}$$

e.g.



Rmk. K^n_m can be viewed as a quotient of the classical Khovanov arc algebra H^{m+n}_{m+n} — same # of $\vee$'s & $\wedge$'s; only closed arcs



Extended Khovanov arc algebras in representation theory & symplectic geometry

Thm. 1. (Stroppel 2009)

$$\operatorname{mod} K_m^n \simeq \underbrace{\mathcal{O}_0^{\mathfrak{p}}}_{} \simeq \underbrace{\operatorname{Perv}(\operatorname{Gr}(m,m+n))}_{}$$

principal block of parabolic category $\mathcal{O}$ assoc. to $\mathfrak{gl}_m(\mathbb{C}) \oplus \mathfrak{gl}_n(\mathbb{C}) \subset \mathfrak{gl}_{m+n}(\mathbb{C})$

category of perverse sheaves on Grassmannian

2. (Stroppel-Webster 2012)

$K_m^n \simeq$ cohomology of intersections of irreducible components of 2-block Springer fibres

Flags in $\mathbb{C}^{m+n}$ fixed by nilpotent N with Jordan type (m,n)

3. (Mak-Smith 2022)

$$\operatorname{perf} K_m^n \simeq D(\mathcal{FS}(\pi_m^n))$$

Fukaya-Seidel category

where $\pi_m^n : \operatorname{Hilb}^m(A_{m+n-1}) \setminus D \longrightarrow \mathbb{C}$

symplectic Lefschetz fibration

Algebraic properties

Thm. (Brundan-Stroppel 2011)

1. K_m^n is
   - cellular
   - Koszul
   - quasi-hereditary

2. K_m^n has a double centralizer property

$$K_m^n \simeq \operatorname{End}_{H_m^n}(eK_m^n)^{op} \;\; \& \;\; eK_m^n e \simeq H_m^n$$

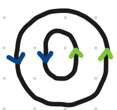
H_n^n classical Khovanov arc algebra

Rmk. Can recover Khovanov homology of knots/links from representation theory & symplectic geometry.

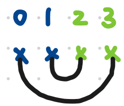
Geometric interpretation of arc diagrams

$A_3 = \{(x,y,z) \in \mathbb{C}^3 \mid x^2+y^2+\prod_{i=0}^{3}(z-i)=0\}$

K_2^2

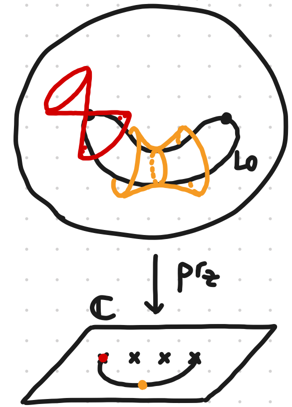


$\longleftrightarrow$



degree 0
arc diagram

paths in $\mathbb{C}$



$\longleftrightarrow$

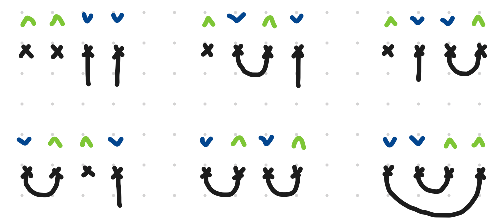
$\mathcal{FS}(\pi_2^2)$

Lagrangian thimbles/
matching spheres in
$\mathrm{Hilb}^2(A_{2+2-1}) \setminus D$

$\downarrow \pi_2^2$

$\mathbb{C}$

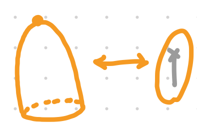
Mak–Smith show that the Lagrangians in $\mathrm{Hilb}^n(A_{m+n-1}) \setminus D$ associated to the degree 0 arc diagrams give a <u>formal</u> generator of $D(\mathcal{FS}(\pi_m^n))$.

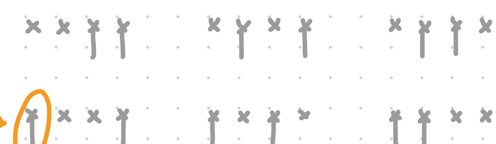


formal generator of $D(\mathcal{FS}(\pi_2^2))$

The "standard" generator given by Lagrangian thimbles is <u>not</u> formal [Klamt–Stroppel].

Verma modules in parabolic category $\mathcal{O}$





generator but not formal

Thm. (Seidel–Thomas 2001) $\underbrace{K^1_m, K^n_1}_{\text{zigzag algebras of type A}}$ are intrinsically formal.

i.e. K^1_m, K^n_1 admit no interesting A_∞ structure at all

Conj. (Stroppel ICM 2010) $HH^2_{i-2}(K^n_m, K^n_m) = 0$ for all $i \neq 0$. $\Rightarrow$ intrinsic formality of K^n_m

[Kadeishvili, Seidel–Thomas]

Thm. (B–Wang 2022) (1.) $HH^2_{i-2}(K^n_m, K^n_m) \neq 0$ for $i = 2mn-4$ for all $m, n \geq 2$.

(2.) K^n_m admits explicit nontrivial A_∞ deformations for all $m, n \geq 2$.

$\Rightarrow$ cannot give a purely algebraic proof of Mak & Smith's result

To prove this, we need some algebraic tools, since the smallest candidate is already too large for naive computation. $\dim K^2_2 = 47$

$$HH^2_{i-2} = H^2\left(\cdots \to C^1_{i-2} \longrightarrow C^2_{i-2} \longrightarrow C^3_{i-2} \longrightarrow \cdots \right)$$

$$C^2_{i-2} = \prod_{j \geq 0} \underbrace{\mathrm{Hom}(A^{\otimes j}, A)^{2-j}_{i-2}}_{\dim.\ (47)^{j+1}}$$

Algebraic description of K_m^n via quivers with relations

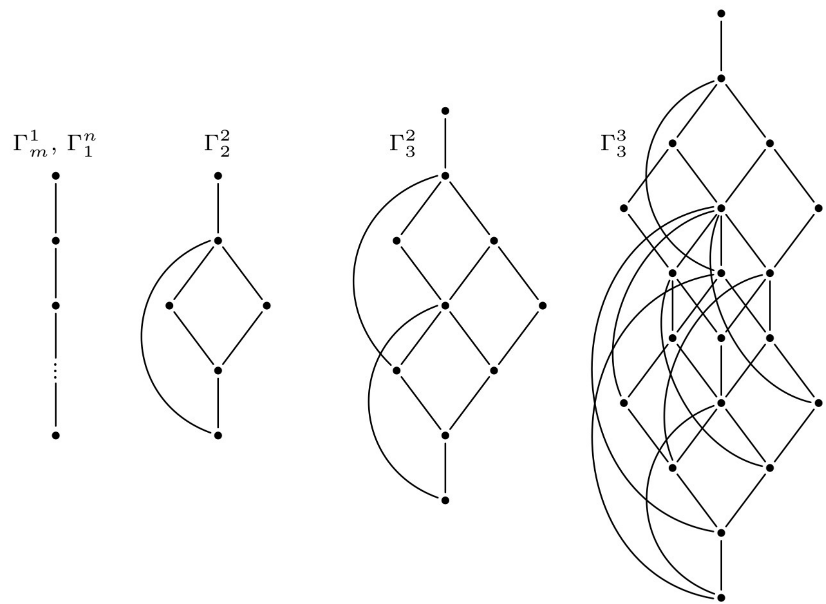
<u>Prop.</u> (B–Wang) $K_m^n \simeq \Bbbk Q_m^n / I_m^n$ where

- Q_m^n is the double quiver associated to a bipartite graph Γ_m^n
- I_m^n is generated by quadratic relations
  - monomial relations
  - relations at vertices
  - commutativity relations across all squares

K_m^n is Koszul, generated in degree 1

vertices $\longleftrightarrow$ deg. 0 arc diagrams

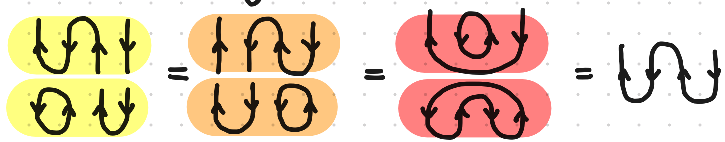
arrows $\longleftrightarrow$ deg. 1 arc diagrams



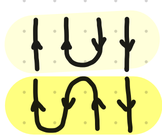
Ex. K_2^2 $\quad \dim_{\Bbbk} K_2^2 = 47 \qquad K_2^2 \simeq \Bbbk Q_2^2 / I_2^2$

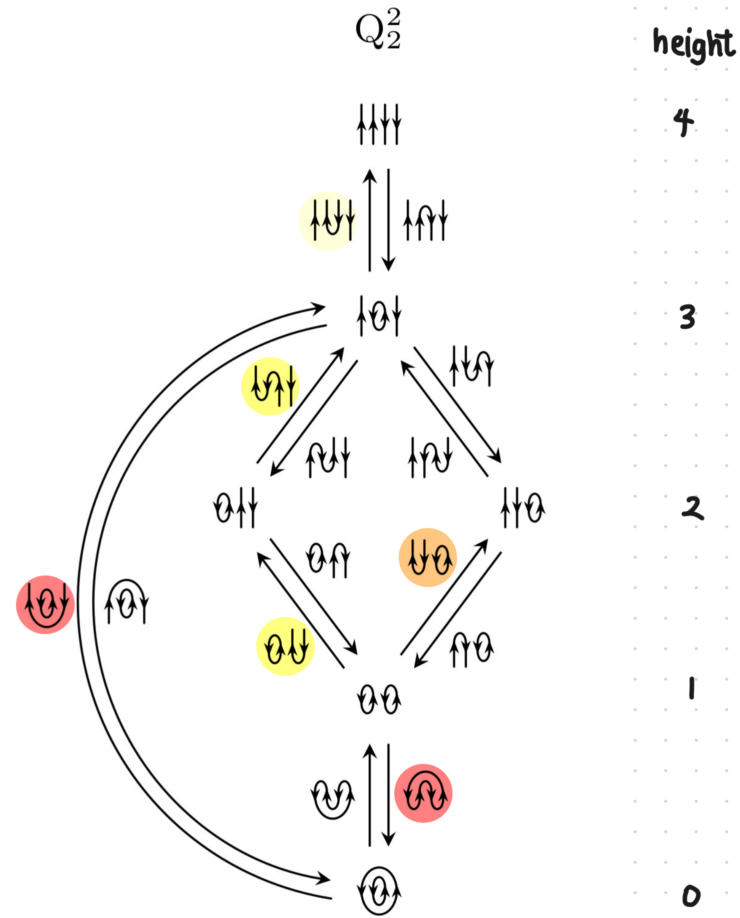
Examples of relations

- a "commutativity relation"



- a monomial relation

 $= 0$



Bruhat order $\longleftrightarrow$ $\text{height}(\underbrace{\lambda}_{\text{weight}}) = \sum_{\vee \in \lambda} \#\wedge\text{'s left of } \vee$

II Hochschild cohomology

Let $A = \bigoplus_{k \in \mathbb{Z}} A^k$ be any graded algebra with an additional Adams grading, i.e. $A^k = \bigoplus_{\ell \in \mathbb{Z}} A^k_\ell$

Then Hom & $\otimes$ and the Hochschild complex are bigraded

$$C^p_q(A,A) = \prod_{i \geq 0} \mathrm{Hom}(A^{\otimes i}, A)^{p-i}_q$$

$$d : C^p_q \longrightarrow C^{p+1}_q$$

$$(U \otimes V)^p_q := \bigoplus_{i,j \in \mathbb{Z}} U^i_j \otimes V^{p-i}_{q-j}$$

$$\mathrm{Hom}(U,V)^p_q := \prod_{i,j \in \mathbb{Z}} \mathrm{Hom}(U^i_j, V^{p+i}_{q+j})$$

$$A = \bigoplus_{k\in\mathbb{Z}} A^k \quad \text{graded vector space}$$

assign $a\in A^k$ bidegree $(0,k)$

assign $a\in A^k$ bidegree $(k,-k)$

associative structures on A

A_∞ structures on A

$$\downarrow$$

associativity

$$\Big\{\mu\in\prod_q \underbrace{C^2_q(A,A)}_{\prod_{i\geq 0}\mathrm{Hom}(A^{\otimes i},A)^{2-i}_q} \text{ satisfying } \underbrace{[\mu,\mu]}_{\text{Gerstenhaber bracket}}=0\Big\}$$

$$= \mathrm{Hom}(A^{\otimes 2},A)^0_q$$

since A is trivially graded w.r.t. first "differential" grading

$$\mu(a,b) = \sum_q \mu_q(a,b) \qquad \text{arity} = 2$$

graded components of multiplication

$$\downarrow$$

A_∞ relations

$$\Big\{\mu\in\prod_q \underbrace{C^2_q(A,A)}_{=\mathrm{Hom}(A^{\otimes q+2},A)^{-q}_q} \text{ satisfying } [\mu,\mu]=0\Big\}$$

since A is trivially graded w.r.t. total degree

arity + degree $=2$

$$\mu_{q-2} : A^{\otimes q} \longrightarrow A$$

of degree $2-q$

m_q

Thm. (Keller 2003) If A is a Koszul algebra, then

$$C^\bullet_\bullet(A,A) \simeq C^\bullet_\bullet(A^!,A^!)$$

in the homotopy category of B_∞ algebras

$$A^! \simeq \mathrm{Ext}^\bullet_A(\Bbbk Q_0, \Bbbk Q_0)$$

In particular, $HH^p_q(A,A) \simeq HH^p_q(A^!,A^!)$.

Write $\bar{K}^n_m := (K^n_m)^! \simeq \Bbbk\underbrace{\bar{Q}^n_m}_{\text{opposite quiver}} / \bar{I}^n_m$ — $(I^n_m)^\perp$ linear dual of I^n_m w.r.t. $\langle -,- \rangle : \Bbbk Q \times \Bbbk\bar{Q} \longrightarrow \Bbbk$

$$\langle a_1\cdots a_n, \bar{b}_n\cdots\bar{b}_1\rangle = \delta_{a_1,b_1}\cdots\delta_{a_n,b_n}$$

Assigning

- arrows in Q^n_m bidegree $(1,-1)$ — grading counting clockwise cups/caps
- arrows in $\bar{Q}^n_m$ bidegree $(0,1)$ — path length

Keller's theorem gives an isomorphism $HH^p_q(K^n_m,K^n_m) \simeq HH^p_q(\bar{K}^n_m,\bar{K}^n_m)$.

III Main results: A_∞ deformations of K_m^n

Thm. (B–Wang) For any $m, n \geq 2$ we have $\dim HH^2_{2mn-6}(K_m^n, K_m^n) = 1$

Idea of proof Use Keller's isomorphism $HH^2_{2mn-6}(K_m^n, K_m^n) \simeq HH^2_{2mn-6}(\bar{K}_m^n, \bar{K}_m^n)$

Encode the relations of the Koszul dual $\bar{K}_m^n \simeq \Bbbk \bar{Q}_m^n / \bar{I}_m^n$ into a reduction system $\bar{R}_m^n$ satisfying the diamond condition for $\bar{I}_m^n$.

use Kazhdan–Lusztig polynomials to show this

systematic method of deforming relations

$$HH^2(\bar{K}_m^n, \bar{K}_m^n) \simeq \{\text{first order deformations of } \bar{R}_m^n\}/\sim$$

[B–Wang] "Deformations of path algebras of quivers with relations"

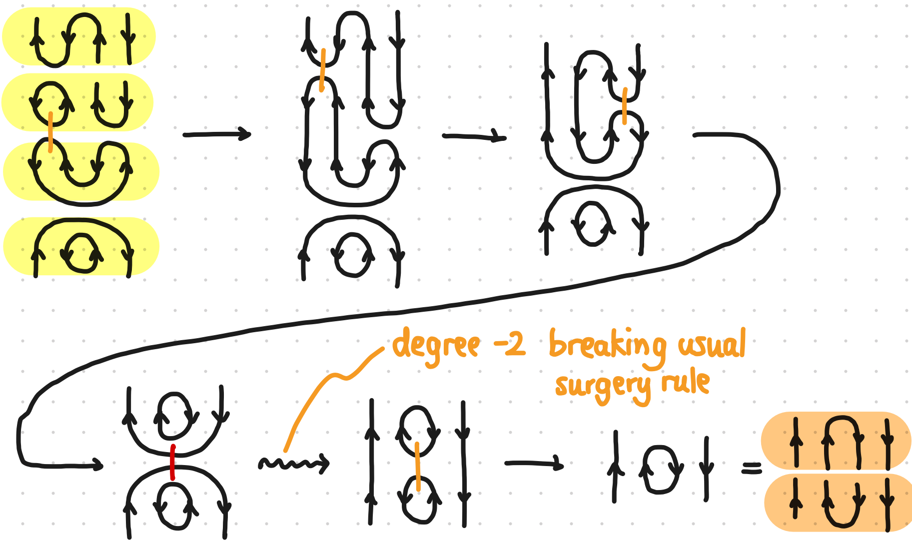
$$HH^2_{2mn-6}(\bar{K}_m^n, \bar{K}_m^n) \simeq {}^{11\text{-dim. space}}/_{10\text{-dim. space}}$$

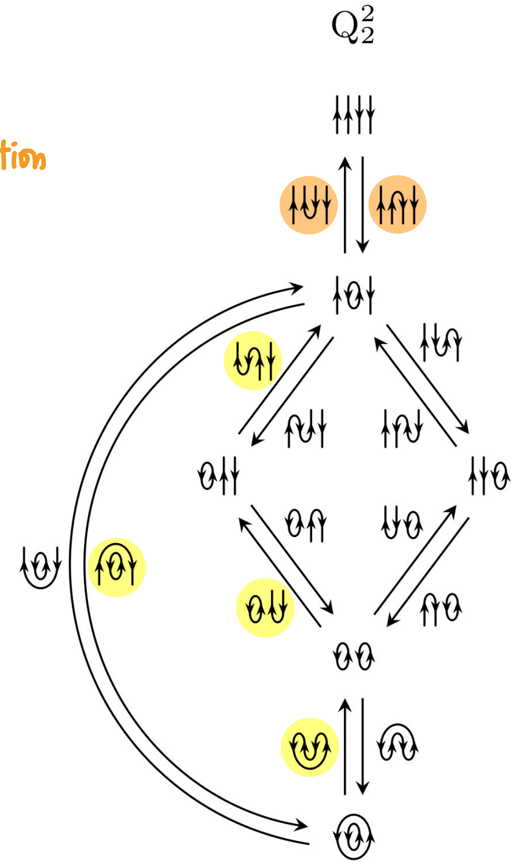
Cor. K_m^n admits an explicit nontrivial A_∞ structure with $\begin{cases} m_i = 0 & \text{for } 2 < i < 2mn-4 \\ m_{2mn-4} \neq 0 & \end{cases}$

Cor. (B–Wang) K_2^2 admits a unique nontrivial A_∞ deformation.

with $m_0 = 0$
$m_1 = 0$
m_2 = original multiplication
$m_4 \neq 0$

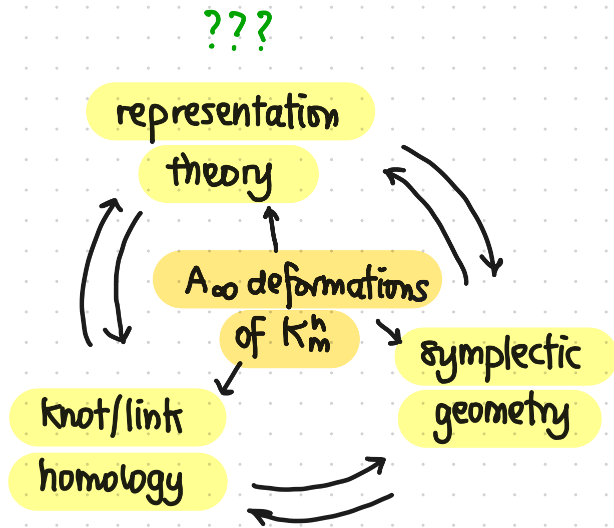
This A_∞ deformation can be described as follows:





Further interpretation & applications

The explicit A_∞ deformations of K_m^n give explicit A_∞ deformations of the Fukaya-Seidel category $\mathcal{FS}(\pi_m^n)$ studied by Mak & Smith.



new link invariants?

Floer theory of a (partial) compactification of $\mathrm{Hilb}^m(A_{m+n-1}) \setminus D$

e.g. $\mathrm{Hilb}^m(A_{m+n-1})$?

cf. Seidel's ICM 2002 address

deformation $\longleftrightarrow$ (partial) compactification