Homological theory of orthogonal modules

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Contents

Global notation

Joint work with Changchang Xi (CNU).

In the talk:
A: Artin algebra (e.g. finite-dim. k-algebra over a field k);
A-Mod: the category of left A-modules;
A-mod: the category of finitely generated (left) A-modules.

§ 1. Classical homological conjectures

(NC) **Nakayama Conjecture** [Nakayama, 1958]:
If A has infinite dominant dimension $\Longrightarrow$ A: self-injective.

Dominant dimension of A:

$$\mathrm{domdim}(A) := \sup\{n \mid I^j : \text{projective } \ \forall\, 0 \leq j < n\}$$

where $0 \to {}_A A \to I^0 \to I^1 \to \cdots \to I^{n-1} \to I^n \to \cdots$ is a minimal injective coresolution.

A is **self-injective** if projectives = injectives.

How to describe infinite dominant dimension?

Morita-Tachikawa; Mueller(1968):

Theorem

Let Λ *be an algebra. Then* $\mathrm{domdim}(\Lambda) = \infty \Longleftrightarrow \Lambda = \mathrm{End}_A(M)$, *where* M *is*

- *generator (i.e.* $A \in \mathrm{add}(M)$*);*
- *cogenerator (i.e.* $D(A_A) \in \mathrm{add}(M)$*);*
- *orthogonal (i.e.* $\mathrm{Ext}^n_A(M, M) = 0$ *for any* $n \geq 1$*).*

D: the usual duality over A-mod (e.g. $D = \mathrm{Hom}_k(-, k)$).

Tachikawa's conjectures

(TC1) **Tachikawa's First Conjecture** [Tachikawa, 1973]:
If $\mathrm{Ext}_A^n(D(A), A) = 0 \;\; \forall\, n \geq 1 \Longrightarrow A$: self-injective.

In (TC1), the A-module $A \oplus D(A)$ is orthogonal.

(TC2) **Tachikawa's Second Conjecture** [Tachikawa, 1973]:
If A: self-injective and ${}_AM$: finitely generated, orthogonal
$\Longrightarrow M$: projective.

Proposition

(NC) holds for all algebras $\Leftrightarrow$ (TC1)+(TC2) hold for all algebras.

Tachikawa's Second Conjecture (TC2)

TC2

If A: self-injective, ${}_AM$: finitely generated, orthogonal
$\Longrightarrow M$: projective.

In (TC2), we can assume M is a generator (e.g. ${}_AM = A \oplus M_0$).

Lemma

The pair (A, M) *satisfies* (TC2) $\Leftrightarrow$ $\mathrm{End}_A(M)$ satisfies (NC).

(TC2) holds for (A, M) where ${}_AM$ is **arbitrary**, but A is

- symmetric alg./local self-injective alg. with radical$^3 = 0$ [Hoshino, 1984];
- group alg. of a finite group [Schulz, 1986];
- self-injective alg. of finite represent. type [Schulz, 1986].

Related conjectures

- Generalized Nakayama Conjecture

  [Auslander-Reiten, 1975]

- Auslander-Reiten Conjecture in commutative algebra

  [Avramov, Iyengar, Nasseh, Sather-Wagstaff, Takahashi, Yoshino, $\cdots\cdots$, 2017-2022.]

§ 2. Tachikawa's Second Conjecture

Consider (TC2) for the pair (A, M) where

- A: arbitrary self-injective alg.
- $M \in A$-mod: generator.

Aim of the talk

Try to understand (TC2) by studying homological properties of orthogonal modules over self-injective algebras.

- Provide equivalent characterizations of (TC2).
- Introduce new homological conditions and Gorenstein-Morita algebras.
- Show that Gorenstein-Morita algebras satisfy the Nakayama Conjecture.

Gorenstein-projective modules

B: (arbitrary) Artin algebra

Definition

A module Y over B is **Gorenstein-projective** *if*
$\exists$ *exact complex of projective B-modules*

$$P^\bullet: \quad \cdots \to P^{-2} \to P^{-1} \to P^0 \xrightarrow{d^0} P^1 \to P^2 \to \cdots$$

such that $\mathrm{Im}\,(d^0) = Y$ *and the complex* $\mathrm{Hom}^\bullet_B(P^\bullet, B)$ *is exact.*

Gorenstein-injective modules can be defined dually.

Notation: B-GProj (resp., B-Gproj): the cat. of (resp., finitely generated) Gorenstein-projective B-modules.

Stable module category

Definition

The stable module category B-$\underline{\mathrm{Mod}}$ of B:
Objects: *all B-modules;*
Morphisms: $\forall\ X, Y \in B\text{-Mod}$,

$$\underline{\mathrm{Hom}}_B(X,Y) := \mathrm{Hom}_B(X,Y)/\mathcal{P}(X,Y)$$

where $\mathcal{P}(X,Y)$ consists of homos. factorizing through projective B-modules.

- B-$\underline{\mathrm{Mod}}$ is triangulated for a self-injective algebra B.
- B-$\underline{\mathrm{GProj}}$ is always triangulated.

Compact objects in categories

$\mathscr{C}$: additive category with set-indexed coproducts.

Definition

An object $X \in \mathscr{C}$ is compact if $\mathrm{Hom}_{\mathcal{C}}(X, -) : \mathscr{C} \to \mathbb{Z}\text{-Mod}$ commutes with coproducts.
$\mathscr{C}^c$: the subcat. of $\mathcal{C}$ consisting of compact objects.

- $B\text{-}\underline{\mathrm{mod}} = B\text{-}\underline{\mathrm{Mod}}^c$ for a self-injective algebra B.
- $B\text{-}\underline{\mathrm{Gproj}} \subseteq B\text{-}\underline{\mathrm{GProj}}^c$.

Notation

A: self-injective algebra (i.e. Projectives = Injectives);

A-Proj: the cat. of projective A-modules;

$M \in A$-mod: generator;

$\mathrm{add}\,({}_AM)$ (resp., $\mathrm{Add}\,({}_AM)$): direct summands of finite (resp., arbitrary) direct sums of copies of M;

${}^{\perp>0}M := \{X \in A\text{-Mod} \mid \mathrm{Ext}^n_A(X, M) = 0, \forall n > 0\}$;

$M^{\perp>0} := \{X \in A\text{-Mod} \mid \mathrm{Ext}^n_A(M, X) = 0, \forall n > 0\}$;

$$\mathscr{G} := {}^{\perp>0}M \cap M^{\perp>0};$$

$\mathscr{G}^{\mathbf{fin}} := \mathscr{G} \cap A$-mod;

$\varinjlim \mathscr{G}^{\mathrm{fin}}$: filtered colimits in A-Mod of modules from $\mathscr{G}^{\mathrm{fin}}$.

- $\varinjlim \mathscr{G}^{\mathrm{fin}} \subseteq \mathscr{G}$.
- If $M = A$, then $\mathscr{G}^{\mathrm{fin}} = A\text{-mod}$ and $\mathscr{G} = A\text{-Mod} = \varinjlim \mathscr{G}^{\mathrm{fin}}$.

Relative stable category

Definition

The M-stable category $A\text{-Mod}/[M]$ *of* $A\text{-Mod}$*:*
Objects: *all A-modules;*
Morphisms: $\forall\, X, Y \in A\text{-Mod}$,

$$\underline{\mathrm{Hom}}_{M}(X,Y) := \mathrm{Hom}_{A}(X,Y)/\mathcal{M}(X,Y)$$

where $\mathcal{M}(X,Y)$ consists of homos. factorizing through objects in $\mathrm{Add}\,(M)$*.*

$A\text{-Mod}/[A] = A\text{-}\underline{\mathrm{Mod}}$ (stable module category of A)

Pretriangulated category

In general, $\mathscr{D} := A\text{-Mod}/[M]$ is **not** a triangulated category, but a **pretriangulated** category.

$X \in A$-Mod;
$\ell_X : X \to M^X$: minimal left $\mathrm{Add}\,(M)$-approximation of X;
$r_X : M_X \to X$: minimal right $\mathrm{Add}\,(M)$-approximation of X.

Remark

The M-cosyzygy and M-syzygy functors

$$\Omega_M : \mathscr{D} \to \mathscr{D}, \quad X \mapsto \mathrm{Ker}\ (r_X),$$

$$\Omega_M^- : \mathscr{D} \to \mathscr{D}, \quad X \mapsto \mathrm{Coker}\ (\ell_X)$$

are **not** necessarily equivalences.

Minimal left approximations

$\mathcal{C}$: additive category, $\mathcal{B}$: full subcat. of $\mathcal{C}$.

Definition

A morphism $f : X \to B$ *(or the object* B*) in* $\mathcal{C}$ *is a* **minimal left $\mathcal{B}$-approximation** *of* X *if*

- $B \in \mathcal{B}$,
- $\operatorname{Hom}_{\mathcal{C}}(f, Y) : \operatorname{Hom}_{\mathcal{C}}(B, Y) \to \operatorname{Hom}_{\mathcal{C}}(X, Y)$ *is surjective* $\forall Y \in \mathcal{B}$,
- $g \in \operatorname{End}_{\mathcal{C}}(B)$ *is an isomorphism whenever* $f = fg$.

Minimal right $\mathcal{B}$-approximations can be defined dually.

What happen if M is orthogonal?

Proposition

Suppose ${}_AM$: orthogonal (i.e. $M \in {}^{\perp>0}M$). Then:

(1) $\mathscr{G}$ *(resp., $\mathscr{G}^{\text{fin}}$) is a* **Frobenius category**. *Its full subcategory of projective-injective objects equals* $\operatorname{Add}(M)$ *(resp.,* $\operatorname{add}(M)$*).*

(2) *Let* $\Lambda := \operatorname{End}_A(M)$. *Then* $\operatorname{Hom}_A(M, -) : A\text{-Mod} \to \Lambda\text{-Mod}$ *induces triangle equivalences*

$$\mathscr{G}/[M] \xrightarrow{\simeq} \Lambda\text{-}\underline{\text{GProj}} \quad \text{and} \quad \mathscr{G}^{\text{fin}}/[M] \xrightarrow{\simeq} \Lambda\text{-}\underline{\text{Gproj}}.$$

$\mathscr{G}$ is called the M**-Gorenstein subcategory** of A-Mod.

Nakayama-stable genertators

Nakayama functor:

$$\nu_A = {}_A D(A) \otimes_A - : A\text{-Mod} \xrightarrow{\simeq} A\text{-Mod}.$$

If A: symmetric algebra (i.e. $D(A) \simeq {}_A A_A$), then $\nu_A \simeq \mathrm{Id}$.

Definition

A generator ${}_A M$ *is* **Nakayama-stable** *if*

$$\mathrm{add}\,({}_A M) = \mathrm{add}\,(\nu_A(M)).$$

Auslander-Reiten formula:

$$D\underline{\mathrm{Hom}}_A(M, -) \simeq \underline{\mathrm{Hom}}_A(-, \nu_A(M)[-1]).$$

where $[-1] := \Omega_A$ (autoequivalence of A-$\underline{\mathrm{Mod}}$).

Minimal left $\mathscr{G}$-approximations of modules

${}_AM$: orthogonal generator.
$\Omega_A^-(M) \to W$: minimal left $\mathscr{G}$-approximation of $\Omega_A^-(M)$;

M-$resdim(X) < \infty$ if $\exists$ exact sequence in A-mod

$$0 \longrightarrow M_n \longrightarrow \cdots \longrightarrow M_1 \longrightarrow M_0 \longrightarrow X \longrightarrow 0$$

with $M_i \in \mathrm{add}\,(M)$ for $0 \le i \le n \in \mathbb{N}$;

$\mathscr{M} := \{X \in A\text{-mod} \mid M\text{-}resdim(X) < \infty\}$.

Remk: $\mathscr{M} \subseteq \mathscr{G}^{\perp>0} \subseteq W^{\perp 1}$.

Equivalent characterizations of (TC2)

Assumptions:
A: self-injective Artin algebra;
$M \in A$-mod: orthogonal, Nakayama-stable generator.

Theorem

The following statements are equivalent:
(1) ${}_A M$ *is projective.*
(2) $\mathscr{G} = \varinjlim \mathscr{G}^{\mathrm{fin}}$.
(3) $W \in \varinjlim \mathscr{G}^{\mathrm{fin}}$.
(4) $\mathrm{Ext}^1_A(W, \bigoplus_{i \in \mathbb{N}} M_i) = 0$ *for all* $M_i \in \mathscr{M}$.

Modules of finite projective dimension

$$\mathscr{P}^{<\infty}(B) := \{Y \in B\text{-mod} \mid \text{proj.dim}\,(Y) < \infty\}$$

$$\text{fin.dim}\,(B) := \sup\{\text{proj.dim}\,(Y) \mid Y \in \mathscr{P}^{<\infty}(B)\}.$$

What is a "compact" version of the equality

$$\mathscr{P}^{<\infty}(B) \cap B\text{-Gproj} = B\text{-proj}?$$

Recall: $B\text{-}\underline{\text{Gproj}} \subseteq B\text{-}\underline{\text{GProj}}^{c}$.

Finitely generated to infinitely generated modules

Definition

*A B-module X is compactly filtered if it has a countable filtration in B-*Mod

$$0 = X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots \subseteq X_n \subseteq \cdots \subseteq X$$

such that $X = \bigcup_{n=0}^{\infty} X_n$ *and* $X_{n+1}/X_n \in \mathscr{P}^{<\infty}(B)$, $\forall n \in \mathbb{N}$.

Compactly filtered modules $\Longrightarrow$ countably generated.

Definition

A B-module X is **compactly Gorenstein-projective** *if it is compact in B-$\underline{\mathrm{GProj}}$.*

Finitely generated Gorenstein-projective modules $\Longrightarrow$ compactly Gorenstein-projective.

Two homological conditions

Notation:

B-CF: compactly filtered B-modules.

B-Proj_ω: countably generated projective B-modules.

B-GProj^c: compactly Gorenstein-projective B-modules.

B-GProj_ω: countably generated Gorenstein-projective B-modules.

(HC1): $\mathrm{Ext}_B^{>0}\big(B\text{-}\mathrm{GProj}_\omega, \bigoplus_{i\in\mathbb{N}} M_i\big) = 0$ for all $M_i \in \mathscr{P}^{<\infty}(B)$.

(HC2): $B\text{-CF} \cap B\text{-}\mathrm{GProj}^c = B\text{-}\mathrm{Proj}_\omega$.

Remk: $\mathscr{P}^{<\infty}(B) \subseteq B\text{-}\mathrm{GProj}^{\perp>0}$.

Implication and invariance of properties

Lemma

(1) $\mathrm{fin.dim}\,(B) < \infty \Longrightarrow$ (HC1) $\Longrightarrow$ (HC2).
(2) B-$\mathrm{GProj}^{\perp>0} \subseteq B$-$\mathrm{Mod}$: *closed under direct sums* $\Longrightarrow$ (HC1).
(3) *If B is* **virtually Gorenstein**, *then* (HC1) *holds.*
(4) (HC2) *is preserved under*

- *derived equivalences,*
- *stable equivalences of Morita type,*
- *certain singular equivalences of Morita type with level.*

Open question:
Does (HC1) (resp., HC2) hold for all Artin algebras?

- **Finitistic Dimension Conjecture** [Rosenberg, Zelinsky; Bass, 1960]: $\mathrm{fin.dim}\,(B) < \infty$, $\forall B$.

Virtually Gorenstein algebras

Definition (Beligiannis, 2005)

B is **virtually Gorenstein** *if* $B\text{-GProj}^{\perp>0} = {}^{\perp>0}B\text{-GInj}$.

B-GInj: the cat. of Gorenstein-injective B-modules.

Theorem (Beligiannis, 2005)

Virtually Gorenstein algebras satisfy the Gorenstein Symmetric Conjecture.

(GSC): If $\text{inj.dim}\,({}_BB) < \infty$, then $\text{inj.dim}\,(B_B) < \infty$.

[A. Beligiannis, Cohen-Macaulay modules, (co)torsion pairs and virtually Gorenstein algebras, *J. Algebra*. 288 (2005), 137-211.]

Gorenstein-Morita algebras

Definition

An algebra B is called a **Gorenstein-Morita** *algebra if*

- $B = \text{End}_A(M)$ *where*
  - A*: self-injective algebra;*
  - M*: Nakayama-stable generator for* A-mod.
- B *satisfies the condition* (HC2) :

$$B\text{-CF} \cap B\text{-GProj}^c = B\text{-Proj}_\omega.$$

Corollary

Let B be a Gorenstein-Morita algebra.
If $\text{domdim}(B) = \infty$*, then B is self-injective.*
Thus B satisfies the Nakayama Conjecture.

§ 3. Recollements of (relative) stable categories

A: (arbitrary) self-injective algebra,
$M \in A$-mod: generator.

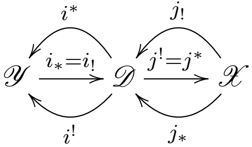
- Given (A, M), we construct two pairs of triangle endofunctors of the **stable module category** of A:

$$(\diamondsuit) \quad (\Phi, \Psi) \text{ and } (\Phi', \Psi') : \quad A\text{-}\underline{\mathrm{Mod}} \longrightarrow A\text{-}\underline{\mathrm{Mod}}.$$

- If ${}_AM$ is **orthogonal** or **Ω-periodic**, then these functors can be embedded into a **recollement** of A-$\underline{\mathrm{Mod}}$.
  (M is called **Ω-periodic** if $\Omega_A^n(M) \simeq M$ in A-$\underline{\mathrm{Mod}}$ for some $n \geq 1$.)
- If ${}_AM$ is orthogonal and Nakayama-stable, then this recollement can be restricted to a recollement of the M**-stable category** of A-Mod.

Recollements of triangulated categories

Beilinson, Bernstein and Deligne [1982]:
A recollement $(\mathscr{Y}, \mathscr{D}, \mathscr{X})$:



- 6 triangle functors;
- 4 adjoint pairs: $(i^*, i_*), (i_!, i^!), (j_!, j^!)$ and (j^*, j_*);
- 3 fully faithful functors (pointing to $\mathscr{D}$, e.g. i_*);
- 3 zeros of composition (along the same level, e.g. $i^* j_! = 0$);
- 2 triangles: $\forall\, X \in \mathscr{D}$, $\exists$ triangles in $\mathscr{D}$:

$$j_! j^!(X) \xrightarrow{counit} X \xrightarrow{unit} i_* i^*(X) \longrightarrow j_! j^!(X)[1],$$
$$i_! i^!(X) \xrightarrow{counit} X \xrightarrow{unit} j_* j^*(X) \longrightarrow i_! i^!(X)[1].$$

Thick subcategories of module categories

Definition

A full subcat. $\mathcal{U} \subseteq A\text{-Mod}$ *is* **thick** *if*

- *it is closed under direct summands in* A-Mod;
- $\forall\ 0 \to X \to Y \to Z \to 0$ *exact seq. in* A-Mod *with two terms in* $\mathcal{U}$, *the third term also belongs to* $\mathcal{U}$.

Notation:

$\mathscr{S}$: the smallest **thick** subcat. of A-Mod containing M and being closed under **direct sums**.

- $\underline{\mathscr{S}}$: localizing subcat. of A-$\underline{\text{Mod}}$ containing M.
- If $M = A$, then $\mathscr{S} = A$-Proj.

A recollement of A-$\underline{\mathrm{Mod}}$ with explicit functors

Theorem

If $_AM$ is **orthogonal** *or Ω-periodic, then $\exists$ a recollement:*

$$\xymatrix{ \underline{M}^{\perp} \ar[rr]|{\mathrm{inc}} && A\text{-}\underline{\mathrm{Mod}} \ar@/_2pc/[ll]_{\widetilde{\Psi}} \ar@/^2pc/[ll]^{\widetilde{\Psi'}} \ar[rr]|{\widetilde{\Phi}} && \underline{\mathscr{S}} \ar@/_2pc/[ll]_{\mathrm{inc}} \ar@/^2pc/[ll]^{\Phi''} }$$

where $\underline{M}^{\perp} := \{X \in A\text{-}\underline{\mathrm{Mod}} \mid \underline{\mathrm{Hom}}_A(M, X[n]) = 0 \; \forall\, n \in \mathbb{Z}\}$,

$\mathbf{\Phi} = \mathrm{inc} \circ \widetilde{\Phi}, \quad \mathbf{\Psi} = \mathrm{inc} \circ \widetilde{\Psi}, \quad \mathbf{\Psi}' = \mathrm{inc} \circ \widetilde{\Psi'}$ and $\Phi'' = \mathbf{\Phi}' \circ \mathrm{inc}.$

Explicit constructions of the functors (Φ, Ψ) and $(\Phi', \Psi') : A\text{-}\underline{\mathrm{Mod}} \to A\text{-}\underline{\mathrm{Mod}}$.

Orthogonal generators over self-injective algebras

A: self-injective Artin algebra;
M: orthogonal, Nakayama-stable generator for A-mod.
In this case,

$$\mathscr{G} = \{X \in A\text{-Mod} \mid \underline{\mathrm{Hom}}_A(M[n], X) = 0, \forall\, n \neq 0, 1\}.$$

Now, let
$\Lambda := \mathrm{End}_A(M)$: the endomorphism algebra of M in A-Mod;
$\Gamma := \underline{\mathrm{End}}_A(M)$: the endomorphism algebra of M in A-$\underline{\mathrm{Mod}}$;

$\mathscr{E} := \{X \in \mathscr{G} \mid \underline{\mathrm{Hom}}_A(M, X), \underline{\mathrm{Hom}}_A(M[1], X) \in \Gamma\text{-mod}\}$;

$\pi : \underline{\mathscr{G}} = \mathscr{G}/[A] \to \mathscr{G}/[M]$: the quotient functor.

Recollements of relative stable categories

Theorem

∃ a recollement of triangulated categories:

$$\xymatrix{ \underline{M}^{\perp} \ar[r]|{\pi\circ\mathrm{inc}} & \mathscr{G}/[M] \ar@/_1.5pc/[l]_{\widetilde{\Psi}} \ar@/^1.5pc/[l]^{\widetilde{\Psi'}} \ar[r]|{\widetilde{\Phi}} & (\mathscr{G}\cap\mathscr{S})/[M] \ar@/_1.5pc/[l]_{\mathrm{inc}} \ar@/^1.5pc/[l]^{\Phi''} }$$

which restricts to a recollement

$$\xymatrix{ \underline{M}^{\perp} \ar[r] & \mathscr{E}/[M] \ar@/_1.5pc/[l] \ar@/^1.5pc/[l] \ar[r] & (\mathscr{E}\cap\mathscr{S})/[M]. \ar@/_1.5pc/[l] \ar@/^1.5pc/[l] }$$

Remk: ${}_AM$ is projective $\Leftrightarrow$ $(\mathscr{G}\cap\mathscr{S})/[M]=0$ $\Leftrightarrow$ $(\mathscr{E}\cap\mathscr{S})/[M]=0$.

Compact objects and generating sets

Proposition

(1) *Each nonzero object of* $(\mathscr{E} \cap \mathscr{S})/[M]$ *is compact in* $\mathscr{G}/[M]$ *and infinitely generated as an* A*-module.*

(2) *Let* $\mathcal{S}$ *be the set of isomorphism classes of simple objects of the heart of a torsion pair in* A-$\underline{\text{Mod}}$ *determined by* M*.*
Then

$$\big(\mathscr{G} \cap \mathscr{S}\big)/[M] = \langle \text{Add}\,(\mathcal{S})\rangle_{2n}^{\{0,1\}},$$

$$\big((\mathscr{G} \cap \mathscr{S}\big)/[M])^{c} = (\mathscr{E} \cap \mathscr{S})/[M] = \langle \mathcal{S}\rangle_{2n}^{\{0,1\}}.$$

where n *is the Loewy length of* Γ*.*
(3) $\dim\big((\mathscr{E} \cap \mathscr{S})/[M]\big) \leq \min\{2n-1, 2m+1\} < \infty$, *where* m *is the global dimension of* Γ*.*

Heart of a torsion pair in A-$\underline{\mathrm{Mod}}$ defined by M

Let

$$\mathscr{Y} := \{Y \in A\text{-}\underline{\mathrm{Mod}} \mid \underline{\mathrm{Hom}}_A(M[n], Y) = 0 \ \ \forall n \geq 0\},$$

$$\mathscr{X} := \{X \in A\text{-}\underline{\mathrm{Mod}} \mid \underline{\mathrm{Hom}}_A(X, Y) = 0 \ \ \forall Y \in \mathscr{Y}\}.$$

$\Longrightarrow$ $(\mathscr{X}, \mathscr{Y})$: torsion pair in A-$\underline{\mathrm{Mod}}$.

[Beilinson, Bernstein and Deligne(1982)]:
The category $\mathscr{H} := \mathscr{X} \cap \mathscr{Y}[1]$, called the **heart** of $(\mathscr{X}, \mathscr{Y})$, is an abelian category.

Lemma

$\exists$ *equivalence of abelian categories:*

$$\mathscr{H} \xrightarrow{\simeq} \Gamma\text{-Mod}.$$

[M. Hoshino, Y. Kato and J-I. Miyachi, On t-structures and torsion theories induced by compact objects, *J. Pure Appl. Algebra* **167** (2002) 15-35.]

Construction of $(\Phi, \Psi) : A\text{-}\underline{\mathrm{Mod}} \to A\text{-}\underline{\mathrm{Mod}}$

Let ${}_A M = A \oplus M_0$, $\Lambda := \mathrm{End}_A(M)$,
$e^2 = e \in \Lambda$ corresponding to the direct summand A of M,
$S_e : \Lambda\text{-Mod} \to A\text{-Mod}$, $Y \mapsto eY$ the Schur functor $(A = e\Lambda e)$.
For $\mathcal{X} \subseteq A\text{-Mod}$ and $\mathcal{Y} \subseteq \Lambda\text{-Mod}$, define

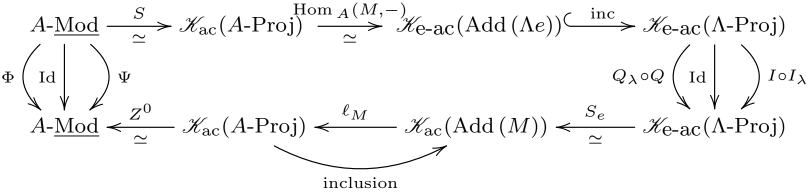
$$\mathscr{K}_{\mathrm{ac}}(\mathcal{X}) := \{X^\bullet \in \mathscr{K}(\mathcal{X}) \mid X^\bullet \text{ is exact}\},$$

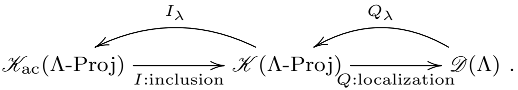
$$\mathscr{K}_{\text{e-ac}}(\mathcal{Y}) := \{Y^\bullet \in \mathscr{K}(\mathcal{Y}) \mid S_e(Y^\bullet) \text{ is exact}\}.$$

If $S_e : \mathcal{Y} \xrightarrow{\simeq} \mathcal{X}$ as additive categories, then

$$S_e : \mathscr{K}_{\text{e-ac}}(\mathcal{Y}) \xrightarrow{\simeq} \mathscr{K}_{\mathrm{ac}}(\mathcal{X}).$$

Construction of $(\Phi, \Psi) : A\text{-}\underline{\mathrm{Mod}} \to A\text{-}\underline{\mathrm{Mod}}$





$\mathscr{K}$: homotopy category;
$\mathscr{D}$: derived category;
Q_λ: taking homotopically projective resolutions;
ℓ_M: taking total complexes of Cartan-Eilenberg injective coresolutions of complexes.

[H.X.Chen, Applications of hyperhomology to adjoint functors, *Comm.Algebra* **50** (1) (2022) 19-32.]

Related papers

[1] H.X.Chen and C.C.Xi, Homological theory of orthogonal modules, 1-40, arXiv:2208.14712.

[2] H.X.Chen, Ming Fang and C.C.Xi, Mirror-reflective algebras and Tachikawa's second conjecture, 1-27, arXiv:2211.08037.

Theorem (Chen-Fang-Xi)

The following are equivalent for a field k.
(1) Tachikawa's Second Conjecture holds for all symmetric algebras over k.
(2) Each indecomposable symmetric algebra over k has no stratifying ideal apart from itself and 0.
(3) The supremum of stratified ratios of all indecomposable symmetric algebras over k is less than 1.

$I := AeA$ with $e^2 = e$ is a *stratifying* ideal of A if $Ae \otimes_{eAe} eA \simeq AeA$ and $\mathrm{Tor}_n^{eAe}(Ae, eA) = 0$, $\forall n > 0$.

Thank you very much!