n-cluster tilting subcategories for truncated path algebras

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Introduction

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$\operatorname{mod} \Lambda$ — category of finitely generated right Λ-modules.

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Definition [Iyama]

A functorially finite subcategory $\mathcal{C} \subseteq \operatorname{mod}\Lambda$ is called an n-cluster tilting (CT) subcategory if

$$\begin{aligned}\mathcal{C} &= \{X \in \operatorname{mod}\Lambda \mid \operatorname{Ext}^i_\Lambda(\mathcal{C}, X) = 0 \text{ for } 0 < i < n\} \\ &= \{X \in \operatorname{mod}\Lambda \mid \operatorname{Ext}^i_\Lambda(X, \mathcal{C}) = 0 \text{ for } 0 < i < n\}.\end{aligned}$$

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- $n \leq \operatorname{gl.dim.}(\Lambda)$.

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For a subcategory $\mathcal{C} \subseteq \operatorname{mod}\Lambda$ we set

$$\mathcal{C}_{\mathcal{P}} := \{\text{isoclasses of indecomposable non projective } \Lambda\text{-modules in } \mathcal{C}\}$$

$$\mathcal{C}_{\mathcal{I}} := \{\text{isoclasses of indecomposable non injective } \Lambda\text{-modules in } \mathcal{C}\}.$$

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(d) Let $M \in \mathcal{C}_\mathcal{I}$. Then $\Omega^{-i}(M)$ is indecomposable for $1 \leq i \leq n-1$.

Introduction

Examples where n-cluster tilting subcategories exist:

- tensor products of l-homogeneous n-representation-finite algebras (if **k** is perfect) [Herschend–Iyama]
- n-APR tilts of n-representation-finite algebras [Iyama–Oppermann]
- higher Nakayama algebras [Jasso–Külshammer]
- many more...

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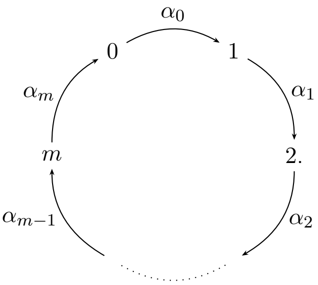
Question

For which Q, L and n does there exist an n-CT subcategory/module of $\operatorname{mod}\Lambda$?

The quivers A_m and $\tilde{A}_m$

$$A_m \coloneqq 1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{m-1}} m.$$

$\tilde{A}_m \coloneqq$



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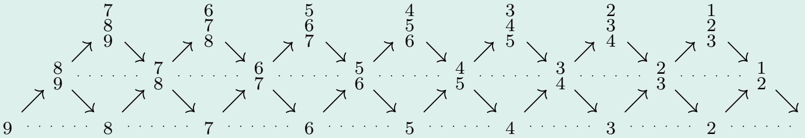
- $\Lambda = \mathbf{k}Q/J^L$ and $n = \text{gl.dim.}(\Lambda)$ [Sandøy–Thibault],
- $\Lambda = \mathbf{k}Q/J^2$ [V].

$L = 2$ and $L \geq 3$

Example for $L \geq 3$

The Auslander–Reiten quiver of $\mathbf{k}A_9/J^3$ is



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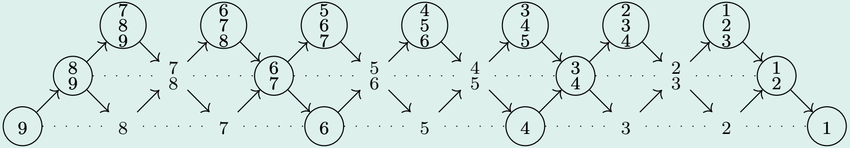

and the additive closure of the encircled modules is a 2-CT subcategory.

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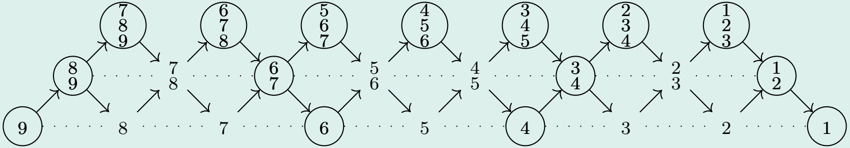

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In general, if $Q = A_m$ and $L \geq 3$, and if there exists an n-CT subcategory, then n is even.

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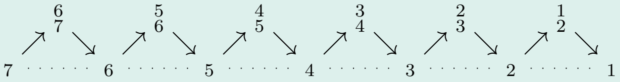

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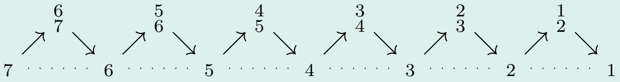

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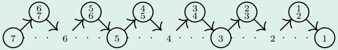

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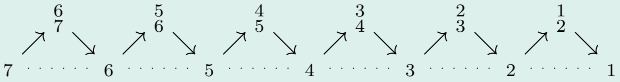

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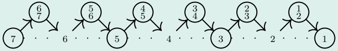

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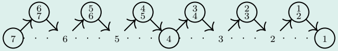

a 3-CT subcategory

$L = 2$ and $L \geq 3$

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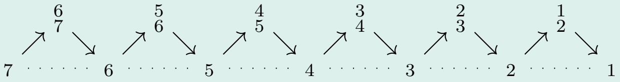

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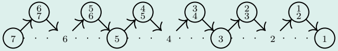

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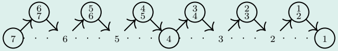

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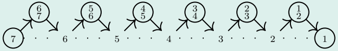

a 6-CT subcategory

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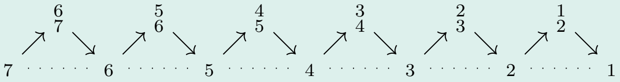

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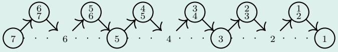

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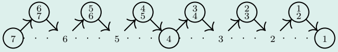

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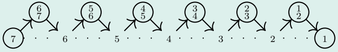

a 6-CT subcategory

In general, if $L = 2$, there is no restriction on the parity of n.

Answer for any Q, L, n

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Two ingredients:

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The case $Q = \tilde{A}_m$ and the case $L = 2$ are special.

Theorem [Darpö–Iyama]

Let $\Lambda = \mathbf{k}\tilde{A}_m / J^L$. There exists an n-CT subcategory of $\operatorname{mod}\Lambda$ if and only if one of the following two conditions holds:

(i) $\left(2\left(\frac{n-1}{2}L+1\right)\right) \mid 2(m+1)$, or

(ii) $\left(2\left(\frac{n-1}{2}L+1\right)\right) \mid t(m+1)$, where $t = \gcd(n+1, 2(L-1))$.

There are many different n-CT subcategories, all of the form $\operatorname{add}(M)$ for some $M \in \operatorname{mod}\Lambda$.

Shape of Q

For a vertex v in Q we denote

- $\delta^-(v)$:=number of arrows terminating at v (incoming degree)
- $\delta^+(v)$:=number of arrows starting at v (outgoing degree)
- $\delta(v) := (\delta^-(v), \delta^+(v))$ (degree)

Shape of Q

Proposition [Oppermann-V]

Let $\Lambda = \mathbf{k}Q/J^L$. Assume there exists an n-CT subcategory $\mathcal{C} \subseteq \operatorname{mod}\Lambda$. Then for every $v \in Q_0$ we have

$$\delta(v) \in \{(0,0),(0,1),(1,0),(1,1),(1,2),(2,1),(2,2)\}.$$

Moreover, if $L \geq 3$ or $n \geq 3$, then $\delta(v) \neq (2,2)$.

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Proof sketch

Assume that there are at least 3 arrows terminating at v. Show that $\Omega(I(v))$ has at least two indecomposable summands using results of Huisgen-Zimmermann.

Shape of Q

Definition

Let Q be a quiver, let $n \geq 2$ and let $L \geq 2$. We say that Q is (n, L)-pre-admissible if

(i) every vertex of Q has at most two incoming and at most two outgoing arrows,

(ii) no vertex of Q has degree $(0, 2)$ or $(2, 0)$, and

(iii) if $L \geq 3$ or $n \geq 3$, then no vertex of Q has degree $(2, 2)$.

Flow paths

Definition

Let $k \geq 2$. A k-flow path $\mathbf{v}$ in Q is a path

$$\mathbf{v} = v_1 \xrightarrow{\alpha_1} v_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{k-2}} v_{k-1} \xrightarrow{\alpha_{k-1}} v_k$$

such that

- $\delta(v_1) \neq (1,1)$,
- $\delta(v_k) \neq (1,1)$, and
- $\delta(v_i) = (1,1)$ for all $1 < i < k$.

We define the degree of $\mathbf{v}$ to be $\delta(\mathbf{v}) = (\delta^-(\mathbf{v}), \delta^+(\mathbf{v})) \coloneqq (\delta^-(v_1), \delta^+(v_k))$.

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Note: if Q is (n, L)-pre-admissible, then there exists a k-flow path if and only if $Q \neq A_1$ and $Q \neq \tilde{A}_m$.

Length of flow paths

Let Q be (n, L)-pre-admissible and let $\mathbf{v}$ be a k-flow path in Q. We define $r(\mathbf{v}, L)$ depending on the degrees of v_1 and v_2 as in the following table:

$\delta(v_1)$ \ $\delta(v_k)$	(1,0)	(2,1)	(1,2)	(2,2)
(0,1)	$\frac{L}{2}$	1	0	1
(1,2)	1	$2-\frac{L}{2}$	$1-\frac{L}{2}$	1
(2,1)	0	$1-\frac{L}{2}$	$-\frac{L}{2}$	0
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Example

Let **v** be a k-flow path with $\delta(v_1)=(1,2)$ and $\delta(v_k)=(2,1)$. Then $r(\mathbf{v},4)=2-\frac{4}{2}=0$.

Length of flow paths

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Let Q be an (n, L)-pre-admissible quiver and $\mathbf{v}$ be a k-flow path in Q. We say that $\mathbf{v}$ is (n, L)-admissible if there exists an integer $p_{\mathbf{v}} \geq 0$ such that

$$k = (p_{\mathbf{v}} + 1)\left(\frac{n-1}{2}L + 1\right) + r(\mathbf{v}, L)$$

and one of the following conditions holds:

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$$k = (p_{\mathbf{v}} + 1)\left(\frac{n-1}{2}L + 1\right) + r(\mathbf{v}, L)$$

and one of the following conditions holds:

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Length of flow paths

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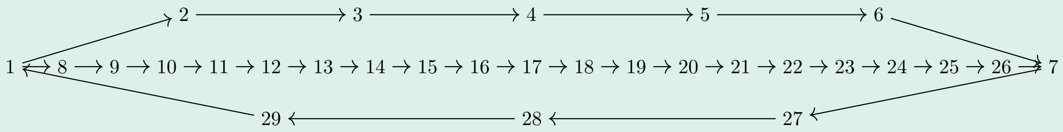
(iv) $L \geq 3$, n and $p_{\mathbf{v}}$ are not both even and $\delta(\mathbf{v}) \in \{(0,1), (0,2), (1,0), (2,0)\}$.

Length of flow paths

Example

Let $n = 4$, $L = 4$ and Q be the quiver



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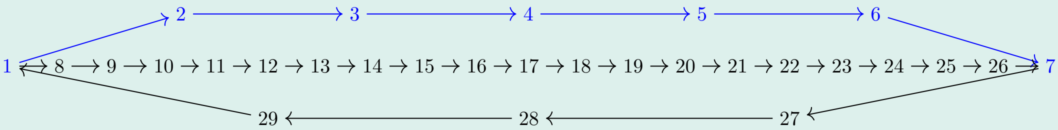

We have $r(\mathbf{v}, 4) = 2 - \frac{4}{2} = 0$. Since $7 = (0+1)\left(\frac{4-1}{2}4 + 1\right) + 0$, $\mathbf{v}$ is $(4,4)$-admissible ($p_{\mathbf{v}} = 0$.)

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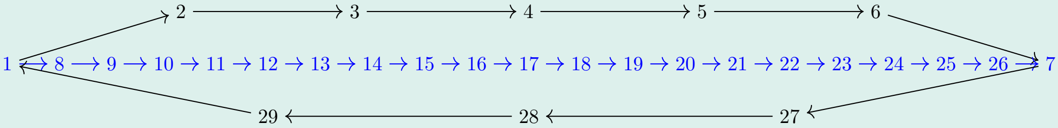

We have $r(\mathbf{v}, 4) = 2 - \frac{4}{2} = 0$. Since $21 = (2+1)\left(\frac{4-1}{2}4 + 1\right) + 0$, $\mathbf{v}$ is $(4, 4)$-admissible ($p_{\mathbf{v}} = 2$.)

Length of flow paths

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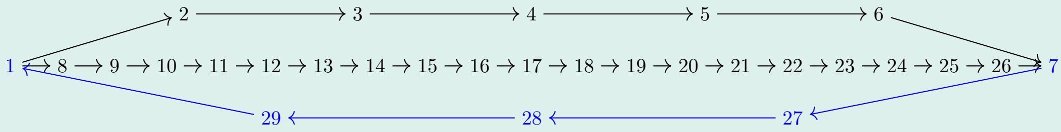

We have $r(\mathbf{v}, 4) = -\frac{4}{2} = -2$. Since $5 = (0+1)\left(\frac{4-1}{2}4 + 1\right) - 2$, $\mathbf{v}$ is $(4,4)$-admissible ($p_{\mathbf{v}} = 0$.)

Length of flow paths

Proposition [Oppermann–V]

Let $\Lambda = \mathbf{k}Q/J^L$. Assume there exists an n-CT subcategory $\mathcal{C} \subseteq \operatorname{mod}\Lambda$. Then every flow path in Q is (n, L)-admissible.

Length of flow paths

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To prove this, first we show the following.

Length of flow paths

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To prove this, first we show the following.

Lemma [Oppermann–V]

Let $\Lambda = \mathbf{k}Q/J^L$ and let $L \geq 3$. Assume there exists an n-CT subcategory $\mathcal{C} \subseteq \operatorname{mod}\Lambda$. If $\mathbf{v}$ is a k-flow path in Q, then $k \geq L + 1$.

Injective non-projective indecomposables

Injective non-projective indecomposables

Now let

$$\mathbf{v} = v_1 \xrightarrow{\alpha_1} v_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{k-2}} v_{k-1} \xrightarrow{\alpha_{k-1}} v_k$$

be a k-flow path in Q.

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Then

$$\delta(v_1) \in \{(0,1),(1,2),(2,1),(2,2)\} \text{ and } \delta(v_k) \in \{(1,0),(2,1),(1,2),(2,2)\},$$

and $k \geq L+1$.

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and $k \geq L+1$.

We want to define $L-1$ indecomposable injective non-projective Λ-modules which depend on $\delta(v_1)$.

Injective non-projective indecomposables

Case $\delta(v_1) = (0, 1)$: then we have

$$v_1 \xrightarrow{\alpha_1} v_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{L-2}} v_{L-1} \xrightarrow{\alpha_{L-1}} \cdots \xrightarrow{\alpha_{k-1}} v_k$$

and we set

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Injective non-projective indecomposables

Case $\delta(v_1) \in \{(1,2),(2,2)\}$: then we have

$$\begin{array}{ccccccccc} & \overset{\alpha_1}{\nearrow} & v_2 & \xrightarrow{\alpha_2} & \cdots & \xrightarrow{\alpha_{L-1}} & v_L & \xrightarrow{\alpha_L} & \cdots \\ v_1 & & & & & & & & \\ & \underset{\beta_1}{\searrow} & u_2 & \xrightarrow{\beta_2} & \cdots & \xrightarrow{\beta_{L-1}} & u_L & \xrightarrow{\beta_L} & \cdots \end{array}$$

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Injective non-projective indecomposables

Case $\delta(v_1) = (2,1)$: then we have

$$\begin{array}{l}
\cdots \xrightarrow{\alpha_{1-L}} v_{2-L} \xrightarrow{\alpha_{2-L}} \cdots \xrightarrow{\alpha_{-1}} v_0 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \searrow^{\alpha_0} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad v_1 \xrightarrow{\alpha_1} v_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{k-2}} v_{k-1} \xrightarrow{\alpha_{k-1}} v_k \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \nearrow_{\beta_0} \\
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$$I_{\mathsf{v}}(1) = I(v_1), I_{\mathsf{v}}(2) = I(v_2), \ldots, I_{\mathsf{v}}(L-1) = I(v_{L-1}).$$

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Dually we define $P_{\mathsf{v}}(i)$ for $1 \leq i \leq L-1$.

Length of flow paths

Now to show that a k-flow path $\mathbf{v}$ must be (n, L)-admissible, we compute

$$\tau_n^p(I_{\mathbf{v}}(i))$$

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A case by case analysis shows that the existence of an n-CT subcategory, implies that there exists $p_{\mathbf{v}}$ such that

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An explicit computation of the above isomorphism gives the condition on the length of **v** .

(n, L)-admissible quivers

Definition

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(a) $Q = \tilde{A}_m$ and $\left(2\left(\frac{n-1}{2}L + 1\right)\right) \mid 2(m+1)$, or

(b) $Q = \tilde{A}_m$ and $\left(2\left(\frac{n-1}{2}L + 1\right)\right) \mid t(m+1)$, where $t = \text{gcd}(n+1, 2(L-1))$, or

(c) $Q \neq \tilde{A}_m$ and every k-flow path **v** in Q is (n, L)-admissible.

(n, L)-admissible quivers

Theorem [case $Q = \tilde{A}_m$ Darpö–Iyama, case $L = 2$ V, case $L \geq 3$ Oppermann–V]

The algebra $\Lambda = \mathbf{k}Q/J^L$ admits an n-CT subcategory if and only if Q is an (n, L)-admissible quiver.

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Proof sketch

For $Q \neq \tilde{A}_m$: ($\Longrightarrow$) has been motivated. For the other direction, we first show existence of an n-CT in a universal cover of Q via a direct computation. Then we use a result of Darpö–Iyama to induce an n-cluster tilting subcategory in $\operatorname{mod}\Lambda$.

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- either they are supported only on vertices with degree $(1, 1)$ (interval modules): $0 \xrightarrow{0} \mathbf{k} \xrightarrow{\simeq} \mathbf{k} \xrightarrow{\simeq} \cdots \xrightarrow{\simeq} \mathbf{k} \xrightarrow{0} 0$

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- they are supported in exactly one vertex with degree different than $(1, 1)$. If that vertex has degree $(2, 1)$ then an indecomposable has the form

$$\begin{array}{ccccccccccccccc} 0 & \hookrightarrow & M_{v_{2-L}} & \hookrightarrow & \cdots & \hookrightarrow & M_{v_0} & \searrow & & & & & & & \\ & & & & & & & & M_{v_1} & \twoheadrightarrow & M_{v_2} & \twoheadrightarrow & \cdots & \twoheadrightarrow & M_{v_L} \twoheadrightarrow 0, \\ 0 & \hookrightarrow & M_{u_{2-L}} & \hookrightarrow & \cdots & \hookrightarrow & M_{u_0} & \nearrow & & & & & & & \end{array}$$

and similarly in other cases.

(n, L)-admissible quivers

Assume $Q \neq \tilde{A}_m$ is (n, L)-admissible. To find an n-CT module M:

(n, L)-admissible quivers

Assume $Q \neq \tilde{A}_m$ is (n, L)-admissible. To find an n-CT module M:

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Assume $Q \neq \tilde{A}_m$ is (n, L)-admissible. To find an n-CT module M:

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$$k = (p_{\mathbf{v}} + 1)\left(\tfrac{n-1}{2}L + 1\right) + r(\mathbf{v}, L)$$

  and there are exactly $p_{\mathbf{v}}(L-1)$ interval modules supported in $\mathbf{v}$ which are direct summands of M.

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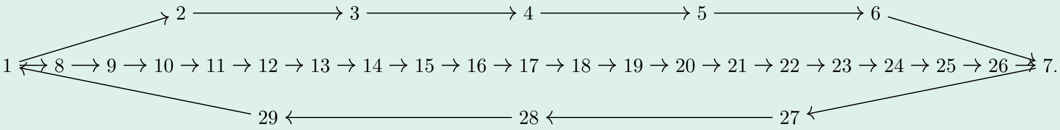
- These are all the direct summands of M.

(n, L)-admissible quivers

Example

Let Q be the quiver



Then Q is $(4,4)$-admissible. Hence the algebra $\Lambda = \mathbf{k}Q/J^4$ admits a unique 4-CT subcategory $\mathcal{C}$.

(n, L)-admissible quivers

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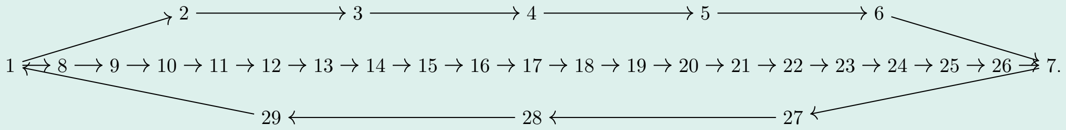

Then Q is $(4,4)$-admissible. Hence the algebra $\Lambda = \mathbf{k}Q/J^4$ admits a unique 4-CT subcategory $\mathcal{C}$.

Moreover, $\mathcal{C} = \mathsf{add}(M)$ where M is the direct sum of the projective modules, the injective modules, and the interval modules (13), $(13, 14)$, $(13, 14, 15)$, $(19, 20, 21)$, $(20, 21)$, (21).

How to find examples

It is easy to find (n, L)-admissible quivers such that $\Lambda = \mathbf{k}Q/J^L$ is a wild algebra and admits an n-cluster tilting subcategory.

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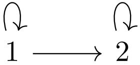

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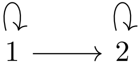

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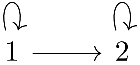

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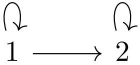

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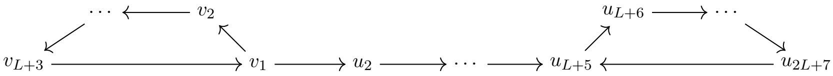

Picking L large enough, gives a wild algebra.

$n\mathbb{Z}$-cluster tilting subcategories

Definition [Iyama–Jasso]

An n-cluster tilting subcategory $\mathcal{C} \subseteq \operatorname{mod}\Lambda$ is called $n\mathbb{Z}$-cluster tilting if it is closed under Ω^n.

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Let $\Lambda = \mathbf{k}Q/J^L$. Then Λ admits an $n\mathbb{Z}$-cluster tilting subcategory if and only if one of the following conditions holds:

(i) $Q = A_m$ and $L = 2$ or $L \mid (m-1)$, and $n = 2\frac{m-1}{L}$, or

(ii) $Q = \tilde{A}_m$ and $L = 2$ or $L = n+2$, and $n \mid (m+1)$.

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Corollary [Sandøy–Thibault]

Let $\Lambda = \mathbf{k}Q/J^L$ and $d = \operatorname{gl.dim.}(\Lambda)$. There exists a d-CT subcategory of $\operatorname{mod} \Lambda$ if and only if $Q = A_m$ and either of $L = 2$ or $L \mid (m-1)$ holds.

A nice property for $L = 2$

Theorem [V]

Let $\Lambda = \mathbf{k}Q/J^2$ and let N be the largest integer for which Q is $(N, 2)$-admissible. Then the following hold.

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(a) For each divisor n of N, the quiver Q is $(n, 2)$-admissible. In particular, there exists an n-cluster tilting subcategory $\mathcal{C}_n \subseteq \operatorname{mod} \Lambda$.

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(a) For each divisor n of N, the quiver Q is $(n, 2)$-admissible. In particular, there exists an n-cluster tilting subcategory $\mathcal{C}_n \subseteq \operatorname{mod} \Lambda$.

(b) The set $\{\mathcal{C}_n \mid n \text{ is a divisor of } N\}$ is a complete lattice with respect to inclusion isomorphic to the opposite of the lattice of divisors of N.

A nice property for $L = 2$

Example

Let Q be the quiver

$$
\begin{array}{ccccccccccccccccccccccccc}
 & & 23 & \longleftarrow & 22 & \longleftarrow & 21 & \longleftarrow & 20 & \longleftarrow & 19 & & & & & & & & & & & & & & \\
 & \swarrow & & & & & & & & & \uparrow & & & & & & & & & & & & & & \\
1 & \longrightarrow & 14 & \longrightarrow & 15 & \longrightarrow & 16 & \longrightarrow & 17 & \longrightarrow & 18 & & & & & & & & & & & & & & \\
\downarrow & & & & & & & & & & & & & & & & & & & & & & & & \\
2 & \longrightarrow & 3 & \longrightarrow & 4 & \longrightarrow & 5 & \longrightarrow & 6 & \longrightarrow & 7 & \longrightarrow & 8 & \longrightarrow & 9 & \longrightarrow & 10 & \longrightarrow & 11 & \longrightarrow & 12 & \longrightarrow & 13.
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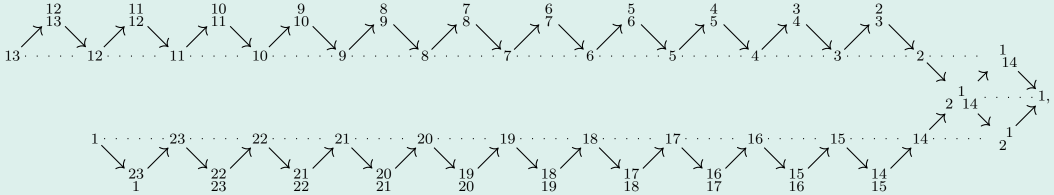
\end{array}
$$

The largest N for which Q is $(N, 2)$-admissible is $N = 12$.

A nice property for $L = 2$

Example

The Auslander–Reiten quiver of $\Lambda = \mathbf{k}Q/J^2$ is



where the simple module $S(1)$ appears twice. Then we have

$\mathcal{C}_1 = \operatorname{mod}\Lambda$,

$\mathcal{C}_2 = \operatorname{add}\{\Lambda, 11, 9, 7, 5, 3, \begin{smallmatrix}1\\14\end{smallmatrix}, 23, 21, 19, 17, 15, \begin{smallmatrix}1\\2\end{smallmatrix}\}$,

$\mathcal{C}_3 = \operatorname{add}\{\Lambda, 10, 7, 4, \begin{smallmatrix}1\\14\end{smallmatrix}, 22, 19, 16, \begin{smallmatrix}1\\2\end{smallmatrix}\}$,

$\mathcal{C}_4 = \operatorname{add}\{\Lambda, 9, 5, \begin{smallmatrix}1\\14\end{smallmatrix}, 21, 17, \begin{smallmatrix}1\\2\end{smallmatrix}\}$,

$\mathcal{C}_6 = \operatorname{add}\{\Lambda, 7, \begin{smallmatrix}1\\14\end{smallmatrix}, 19, \begin{smallmatrix}1\\2\end{smallmatrix}\}$,

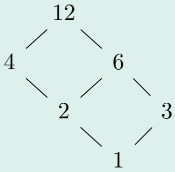
$\mathcal{C}_{12} = \operatorname{add}\{\Lambda, \begin{smallmatrix}1\\14\end{smallmatrix}, \begin{smallmatrix}1\\2\end{smallmatrix}\}$,

and $\mathcal{C}_n$ is an n-cluster tilting subcategory of $\operatorname{mod}\Lambda$.

A nice property for $L = 2$

Example

Then the lattice



of divisors of 12

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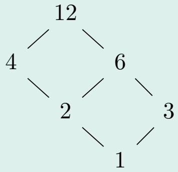
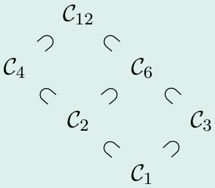


of divisors of 12 corresponds to the lattice



of inclusions of n-cluster tilting subcategories of $\operatorname{mod}\Lambda$.

Thank You!