n-cluster tilting subcategories for truncated path algebras

Laertis Vaso

NTNU
Institutt for matematiske fag

Joint work in progress with Steffen Oppermann (NTNU)

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Introduction

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Λ — a finite-dimensional $\mathbf{k}$-algebra.

$\operatorname{mod}\Lambda$ — category of finitely generated right Λ-modules.

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Definition [Iyama]

A functorially finite subcategory $\mathcal{C} \subseteq \operatorname{mod}\Lambda$ is called an n-cluster tilting (CT) subcategory if

$$\begin{aligned} \mathcal{C} &= \{X \in \operatorname{mod}\Lambda \mid \operatorname{Ext}^i(\mathcal{C}, X) = 0 \text{ for } 0 < i < n\} \\ &= \{X \in \operatorname{mod}\Lambda \mid \operatorname{Ext}^i(X, \mathcal{C}) = 0 \text{ for } 0 < i < n\}. \end{aligned}$$

If $\mathcal{C} = \operatorname{add}(M)$ for some $M \in \operatorname{mod}\Lambda$, then we call M an n-cluster tilting module.

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- $\mathcal{C} = \operatorname{mod}\Lambda$ is the unique 1-cluster tilting subcategory.
- $\exists$ 1-cluster tilting module M $\iff$ $\operatorname{mod}\Lambda = \operatorname{add}(M)$
  $\iff$ Λ is representation-finite.

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- If $M \in \operatorname{mod}\Lambda$, then $\operatorname{add}(M)$ is functorially finite.
- $\mathcal{C} = \operatorname{mod}\Lambda$ is the unique 1-cluster tilting subcategory.
- $\exists$ 1-cluster tilting module $M \iff \operatorname{mod}\Lambda = \operatorname{add}(M)$
  $\iff \Lambda$ is representation-finite.
- $n \leq \operatorname{gl.dim.}(\Lambda)$.

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We denote by $\tau_n := \tau\Omega^{n-1}$ and $\tau_n^{-} := \tau^{-}\Omega^{-(n-1)}$ the n-Auslander–Reiten translations.

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For a subcategory $\mathcal{C} \subseteq \operatorname{mod}\Lambda$ we set

$$\mathcal{C}_P := \{\text{isoclasses of indecomposable non projective } \Lambda\text{-modules in } \mathcal{C}\}$$

$$\mathcal{C}_I := \{\text{isoclasses of indecomposable non injective } \Lambda\text{-modules in } \mathcal{C}\}.$$

Introduction

$$C = f X \ 2 \ \text{mod} \quad j \ \text{Ext}^i \ (C, X) = 0 \text{ for } 0 < i < n g$$
$$= f X \ 2 \ \text{mod} \quad j \ \text{Ext}^i \ (X, C) = 0 \text{ for } 0 < i < n g.$$

Proposition [Iyama, V]

Let C mod Λ be n-CT. Then the following hold.

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Let C mod Λ be n-CT. Then the following hold.

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(b) $\tau_n : C_P \ ! \ C_I$ and $\tau_n : C_I \ ! \ C_P$ are mutually inverse bijections.

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Let $\mathcal{C} \subseteq \operatorname{mod}\Lambda$ be n-CT. Then the following hold.

(a) $\mathcal{C}$ contains all projective and all injective Λ-modules.

(b) $\tau_n : \mathcal{C}_P \to \mathcal{C}_I$ and $\tau_n^- : \mathcal{C}_I \to \mathcal{C}_P$ are mutually inverse bijections.

(c) Let $M \in \mathcal{C}_P$. Then $\Omega^i(M)$ is indecomposable for $1 \leq i \leq n-1$.

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Let $\mathcal{C} \subseteq \text{mod}\,\Lambda$ be n-CT. Then the following hold.

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(b) $\tau_n : \mathcal{C}_P \to \mathcal{C}_I$ and $\tau_n^- : \mathcal{C}_I \to \mathcal{C}_P$ are mutually inverse bijections.

(c) Let $M \in \mathcal{C}_P$. Then $\Omega^i(M)$ is indecomposable for $1 \leq i \leq n-1$.

(d) Let $M \in \mathcal{C}_I$. Then $\Omega^{-i}(M)$ is indecomposable for $1 \leq i \leq n-1$.

Introduction

Examples where n-cluster tilting subcategories exist:

- tensor products of l-homogeneous n-representation-finite algebras (if $\mathbf{k}$ is perfect) [Herschend–Iyama]
- n-APR tilts of n-representation-finite algebras [Iyama–Oppermann]
- higher Nakayama algebras [Jasso–Külshammer]
- many more...

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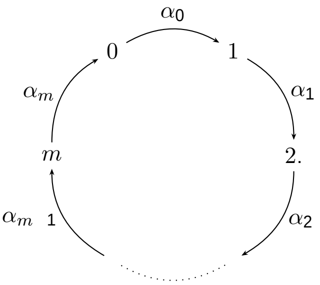
Question

For which Q, L and n does there exist an n-CT subcategory/module of $\operatorname{mod}\Lambda$?

The quivers A_m and $\tilde{A}_m$

$$A_m := 1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{m-1}} m.$$

$\tilde{A}_m :=$



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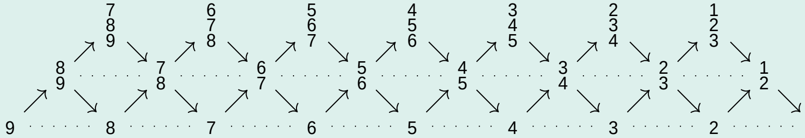
$\Lambda = \mathbf{k}Q/J^L$ and $n = \mathrm{gl.dim.}(\Lambda)$ [Sandøy–Thibault],

$\Lambda = \mathbf{k}Q/J^2$ [V].

L = 2 and L 3

Example for L 3

The Auslander–Reiten quiver of $\mathbf{k}A_9/J^3$ is



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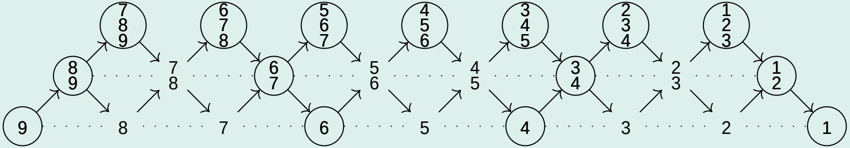

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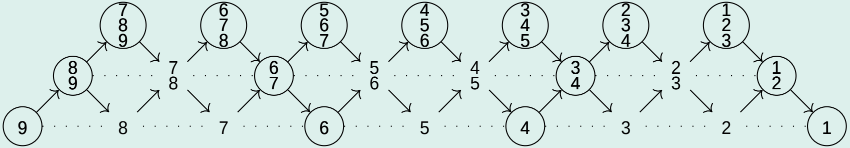

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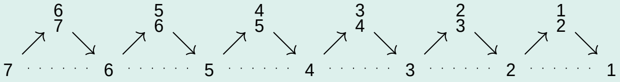
In general, if $Q = A_m$ and L 3, and if there exists an n-CT subcategory, then n is even.

L = 2 and L 3

Example for $L = 2$

The Auslander–Reiten quiver of $\mathbf{k}A_7/J^2$ is



and there exist

L = 2 and L 3

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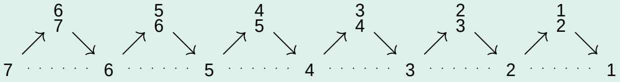

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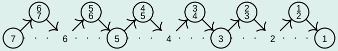

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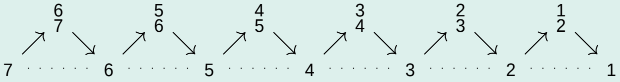

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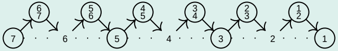

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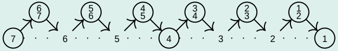

a 3-CT subcategory

L = 2 and L 3

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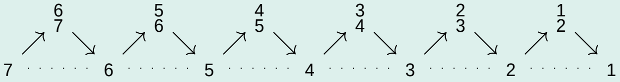

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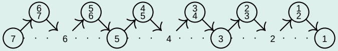

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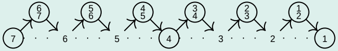

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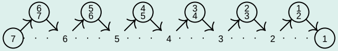

a 6-CT subcategory

L = 2 and L 3

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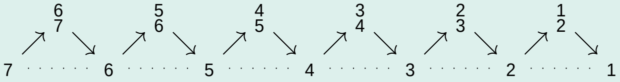

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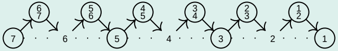
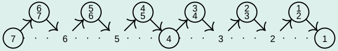


a 2-CT subcategory



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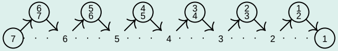

a 6-CT subcategory

In general, if $L = 2$, there is no restriction on the parity of n.

Answer for any Q, L, n

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Two ingredients:

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Two ingredients:

the shape of Q, and

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the shape of Q, and

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Remark

The case $Q = \tilde{A}_m$ and the case $L = 2$ are special.

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Remark

The case $Q = \tilde{A}_m$ and the case $L = 2$ are special.

Theorem [Darpö–Iyama]

Let $\Lambda = \mathbf{k}\tilde{A}_m/J^L$. There exists an n-CT subcategory of mod Λ if and only if one of the following two conditions holds:

(i) $\left(2\left(\frac{n\ \ 1}{2}L+1\right)\right)$ *j* $2(m+1)$, or

(ii) $\left(2\left(\frac{n\ \ 1}{2}L+1\right)\right)$ *j* $t(m+1)$, where $t = \mathrm{gcd}(n+1, 2(L\ \ \ 1))$.

There are many different n-CT subcategories, all of the form add(M) for some M *2* mod Λ.

Shape of Q

For a vertex v in Q we denote

$\delta\ \ (v)$:=number of arrows terminating at v (incoming degree)

$\delta^{+}(v)$:=number of arrows starting at v (outgoing degree)

$\delta(v) := (\delta\ \ (v), \delta^{+}(v))$ (degree)

Shape of Q

Proposition [Oppermann-V]

Let $\Lambda = \mathbf{k}Q/J^L$. Assume there exists an n-CT subcategory $\mathcal{C}$ $\quad$ mod Λ. Then for every $v \in Q_0$ we have

$$\delta(v) \in \{(0,0),(0,1),(1,0),(1,1),(1,2),(2,1),(2,2)\}.$$

Moreover, if $L \quad 3$ or $n \quad 3$, then $\delta(v) \neq (2,2)$.

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Moreover, if L 3 or n 3, then $\delta(v) \neq (2,2)$.

Proof sketch

Assume that there are at least 3 arrows terminating at v. Show that $\Omega(I(v))$ has at least two indecomposable summands using results of Huisgen-Zimmermann.

Shape of Q

Definition

Let Q be a quiver, let n 2 and let L 2. We say that Q is (n, L)-pre-admissible if

(i) every vertex of Q has at most two incoming and at most two outgoing arrows,

(ii) no vertex of Q has degree $(0, 2)$ or $(2, 0)$, and

(iii) if L 3 or n 3, then no vertex of Q has degree $(2, 2)$.

Flow paths

Definition

Let $k \geq 2$. A k-flow path **v** in Q is a path

$$\mathbf{v} = v_1 \xrightarrow{\alpha_1} v_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{k-2}} v_{k-1} \xrightarrow{\alpha_{k-1}} v_k$$

such that

$\delta(v_1) \neq (1,1)$,

$\delta(v_k) \neq (1,1)$, and

$\delta(v_i) = (1,1)$ for all $1 < i < k$.

We define the degree of **v** to be $\delta(\mathbf{v}) = (\delta^-(\mathbf{v}), \delta^+(\mathbf{v})) := (\delta^-(v_1), \delta^+(v_k))$.

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Note: if Q is (n, L)-pre-admissible, then there exists a k-flow path if and only if $Q \neq A_1$ and $Q \neq \tilde{A}_m$.

Length of flow paths

Let Q be (n, L)-pre-admissible and let $\mathbf{v}$ be a k-flow path in Q. We define $r(\mathbf{v}, L)$ depending on the degrees of v_1 and v_2 as in the following table:

$\delta(v_1)$ \ $\delta(v_k)$	(1,0)	(2,1)	(1,2)	(2,2)
(0,1)	$\frac{L}{2}$	1	0	1
(1,2)	1	2 $\quad \frac{L}{2}$	1 $\quad \frac{L}{2}$	1
(2,1)	0	1 $\quad \frac{L}{2}$	$\frac{L}{2}$	0
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(1,2)	1	$2 \quad \frac{L}{2}$	$1 \quad \frac{L}{2}$	1
(2,1)	0	$1 \quad \frac{L}{2}$	$\frac{L}{2}$	0
(2,2)	1	1	0	1

Example

Let **v** be a k-flow path with $\delta(v_1) = (1, 2)$ and $\delta(v_k) = (2, 1)$. Then $r(\mathbf{v}, 4) = 2 \quad \frac{4}{2} = 0$.

Length of flow paths

Definition

Let Q be an (n, L)-pre-admissible quiver and **v** be a k-flow path in Q. We say that **v** is (n, L)-admissible if there exists an integer $p_{\mathbf{v}} \quad 0$ such that

$$k = (p_{\mathbf{v}} + 1)\left(\frac{n \;\; 1}{2} L + 1\right) + r(\mathbf{v}, L)$$

and one of the following conditions holds:

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(ii) $L \quad 3$, n and $p_{\mathbf{v}}$ are both even and $\delta(\mathbf{v}) = (0, 0)$,

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Definition

Let Q be an (n, L)-pre-admissible quiver and $\mathbf{v}$ be a k-flow path in Q. We say that $\mathbf{v}$ is (n, L)-admissible if there exists an integer $p_{\mathbf{v}} \geq 0$ such that

$$k = (p_{\mathbf{v}} + 1)\left(\frac{n-1}{2}L + 1\right) + r(\mathbf{v}, L)$$

and one of the following conditions holds:

(i) $L = 2$,

(ii) $L \geq 3$, n and $p_{\mathbf{v}}$ are both even and $\delta(\mathbf{v}) = (0, 0)$,

(iii) $L \geq 3$, n and $p_{\mathbf{v}}$ are both even, $n + p_{\mathbf{v}} > 2$ and $\delta(\mathbf{v}) \in \{(1,1), (1,2), (2,1), (2,2)\}$, or

Length of flow paths

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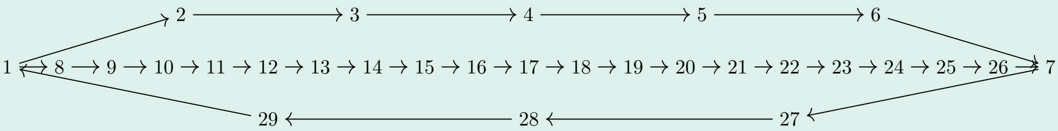
Length of flow paths

Example

Let $n = 4$, $L = 4$ and Q be the quiver



Length of flow paths

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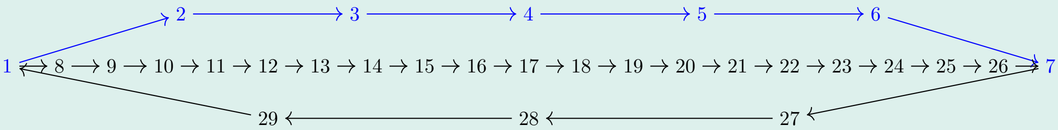

We have $r(\mathbf{v}, 4) = 2 \quad \frac{4}{2} = 0$. Since $7 = (0+1)\left(\frac{4\;\,1}{2}4+1\right)+0$, $\mathbf{v}$ is $(4,4)$-admissible $(p_{\mathbf{v}} = 0.)$

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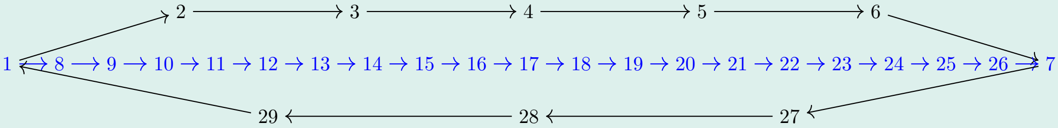

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Length of flow paths

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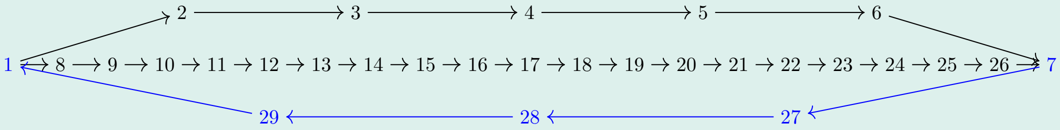

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Length of flow paths

Proposition [Oppermann–V]

Let $\Lambda = \mathbf{k}Q/J^L$. Assume there exists an n-CT subcategory C $\quad \operatorname{mod} \Lambda$. Then every flow path in Q is (n, L)-admissible.

Length of flow paths

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Let $\Lambda = \mathbf{k}Q/J^L$. Assume there exists an n-CT subcategory C $\quad \operatorname{mod}\Lambda$. Then every flow path in Q is (n, L)-admissible.

To prove this, first we show the following.

Length of flow paths

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Let $\Lambda = \mathbf{k}Q/J^L$. Assume there exists an n-CT subcategory $\mathcal{C}$ $\quad \operatorname{mod}\Lambda$. Then every flow path in Q is (n, L)-admissible.

To prove this, first we show the following.

Lemma [Oppermann–V]

Let $\Lambda = \mathbf{k}Q/J^L$ and let $L \quad 3$. Assume there exists an n-CT subcategory $\mathcal{C}$ $\quad \operatorname{mod}\Lambda$. If $\mathbf{v}$ is a k-flow path in Q, then $k \quad L + 1$.

Injective non-projective indecomposables

Injective non-projective indecomposables

Now let

$$\mathbf{v} = v_1 \xrightarrow{\alpha_1} v_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{k-2}} v_{k-1} \xrightarrow{\alpha_{k-1}} v_k$$

be a k-flow path in Q.

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be a k-flow path in Q.

Then

$$\delta(v_1) \in \{(0,1),(1,2),(2,1),(2,2)\} \text{ and } \delta(v_k) \in \{(1,0),(2,1),(1,2),(2,2)\},$$

and $k \quad L+1$.

Injective non-projective indecomposables

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and $k \geq L+1$.

We want to define $L-1$ indecomposable injective non-projective Λ-modules which depend on $\delta(v_1)$.

Injective non-projective indecomposables

Case $\delta(v_1) = (0, 1)$: then we have

$$v_1 \xrightarrow{\alpha_1} v_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{L-2}} v_{L-1} \xrightarrow{\alpha_{L-1}} \cdots \xrightarrow{\alpha_{k-1}} v_k$$

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$$I_{\mathbf{v}}(1) = I(v_1)$$

Injective non-projective indecomposables

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Injective non-projective indecomposables

Case $\delta(v_1) \in \{(1,2),(2,2)\}$: then we have

$$\begin{array}{ccccccccc} & & v_2 & \xrightarrow{\alpha_2} & \cdots & \xrightarrow{\alpha_{L-1}} & v_L & \xrightarrow{\alpha_L} & \\ & \overset{\alpha_1}{\nearrow} & & & & & & & \\ v_1 & & & & & & & & \\ & \underset{\beta_1}{\searrow} & & & & & & & \\ & & u_2 & \xrightarrow{\beta_2} & \cdots & \xrightarrow{\beta_{L-1}} & u_L & \xrightarrow{\beta_L} & \end{array}$$

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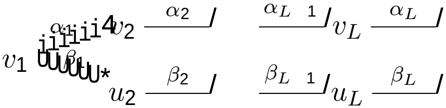

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and we set

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Injective non-projective indecomposables

Case $\delta(v_1) = (2,1)$: then we have

$$\begin{array}{ccc} \xrightarrow{\alpha_{1}\ L} v_{2}\ L \xrightarrow{\alpha_{2}\ L} \quad \xrightarrow{\alpha\ 1} v_{0} & \xrightarrow{\alpha_0} & \\ & & v_1 \xrightarrow{\alpha_1} v_2 \xrightarrow{\alpha_2} \quad \xrightarrow{\alpha_{k\ 2}} v_{k\ 1} \xrightarrow{\alpha_{k\ 1}} v_k \\ \xrightarrow{\beta_{1}\ L} u_{2}\ L \xrightarrow{\beta_{2}\ L} \quad \xrightarrow{\beta\ 1} u_{0} & \xrightarrow{\beta_0} & \end{array}$$

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Injective non-projective indecomposables

Case $\delta(v_1) = (2,1)$: then we have

$$
\begin{array}{l}
\xrightarrow{\alpha_1} v_2 \cdots \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{-1}} v_0 \searrow^{\alpha_0} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad v_1 \xrightarrow{\alpha_1} v_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{k-2}} v_{k-1} \xrightarrow{\alpha_{k-1}} v_k \\
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Injective non-projective indecomposables

Case $\delta(v_1) = (2,1)$: then we have

$$\begin{array}{l} \xrightarrow{\alpha_{1-L}} v_{2-L} \xrightarrow{\alpha_{2-L}} \cdots \xrightarrow{\alpha_{-1}} v_0 \overset{\alpha_0}{\searrow} \\ \qquad\qquad\qquad\qquad\qquad\qquad\quad v_1 \xrightarrow{\alpha_1} v_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{k-2}} v_{k-1} \xrightarrow{\alpha_{k-1}} v_k \\ \xrightarrow{\beta_{1-L}} u_{2-L} \xrightarrow{\beta_{2-L}} \cdots \xrightarrow{\beta_{-1}} u_0 \underset{\beta_0}{\nearrow} \end{array}$$

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Dually we define $P_{\mathbf{v}}(i)$ for $1 \leq i \leq L-1$.

Length of flow paths

Now to show that a k-flow path **v** must be (n, L)-admissible, we compute

$$\tau_n^p(I_{\mathbf{v}}(i))$$

for $1 \quad i \quad L \quad 1$ and $p \quad 0$.

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An explicit computation of the above isomorphism gives the condition on the length of **v** .

(n, L)-admissible quivers

Definition

Let $n \geq 2$ and $L \geq 2$. Let Q be an (n, L)-pre-admissible quiver. We say that Q is (n, L)-admissible if one of the following conditions holds:

(a) $Q = \tilde{A}_m$ and $\left(2\left(\frac{n-1}{2}L + 1\right)\right) \mid 2(m+1)$, or

(b) $Q = \tilde{A}_m$ and $\left(2\left(\frac{n-1}{2}L + 1\right)\right) \mid t(m+1)$, where $t = \gcd(n+1, 2(L-1))$, or

(c) $Q \neq \tilde{A}_m$ and every k-flow path **v** in Q is (n, L)-admissible.

(n, L)-admissible quivers

Theorem [case Q = $\tilde{A}_m$ Darpö–Iyama, case L = 2 V, case L 3 Oppermann–V]

The algebra $\Lambda = \mathbf{k}Q/J^L$ admits an n-CT subcategory if and only if Q is an (n, L)-admissible quiver.

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Theorem [case $Q = \tilde{A}_m$ Darpö–Iyama, case $L = 2$ V, case $L \geq 3$ Oppermann–V]

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Proof sketch

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For $Q \neq \tilde{A}_m$: ($\Rightarrow$) has been motivated.

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Proof sketch

For $Q \neq \tilde{A}_m$: $(\Leftarrow)$ has been motivated. For the other direction, we first show existence of an n-CT in a universal cover of Q via a direct computation. Then we use a result of Darpö–Iyama to induce an n-cluster tilting subcategory in $\mathsf{mod}\,\Lambda$.

(n, L)-admissible quivers

Proposition

Let $Q \not\cong \tilde{A}_m$ be an (n, L)-admissible quiver.

(n, L)-admissible quivers

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either they are supported only on vertices with degree $(1, 1)$ (interval modules):

$$0 \xrightarrow{0} \mathbf{k} \xrightarrow{\varphi} \mathbf{k} \xrightarrow{\varphi} \cdots \xrightarrow{\varphi} \mathbf{k} \xrightarrow{0} 0$$

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they are supported in exactly one vertex with degree different than $(1, 1)$.

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they are supported in exactly one vertex with degree different than $(1, 1)$. If that vertex has degree $(2, 1)$ then an indecomposable has the form

$$\begin{array}{l} 0 \xrightarrow{\bullet} M_{v_{2_L}} \xrightarrow{\bullet} \cdots \xrightarrow{\bullet} M_{v_0} \searrow \\ \qquad\qquad\qquad\qquad\qquad\qquad M_{v_1} \longrightarrow M_{v_2} \longrightarrow \cdots \longrightarrow M_{v_L} \longrightarrow 0, \\ 0 \xrightarrow{\bullet} M_{u_{2_L}} \xrightarrow{\bullet} \cdots \xrightarrow{\bullet} M_{u_0} \nearrow \end{array}$$

and similarly in other cases.

(n, L)-admissible quivers

Assume $Q \neq \tilde{A}_m$ is (n, L)-admissible. To find an n-CT module M:

(n, L)-admissible quivers

Assume $Q \not\cong \tilde{A}_m$ is (n, L)-admissible. To find an n-CT module M:

All projective and all injective indecomposable modules are direct summands of M.

(n, L)-admissible quivers

Assume $Q \neq \tilde{A}_m$ is (n, L)-admissible. To find an n-CT module M:

- All projective and all injective indecomposable modules are direct summands of M.
- If $\mathbf{v}$ is a k-flow path in Q, then

$$k = (p_{\mathbf{v}} + 1)\left(\frac{n-1}{2}L + 1\right) + r(\mathbf{v}, L)$$

  and there are exactly $p_{\mathbf{v}}(L-1)$ interval modules supported in $\mathbf{v}$ which are direct summands of M.

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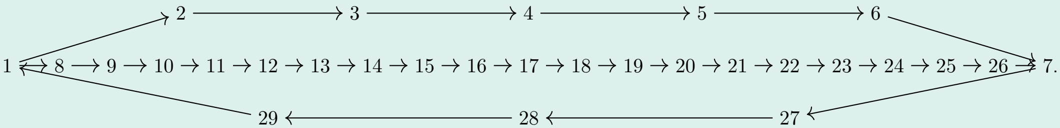
These are all the direct summands of M.

(n, L)-admissible quivers

Example

Let Q be the quiver



Then Q is $(4, 4)$-admissible. Hence the algebra $\Lambda = \mathbf{k}Q/J^4$ admits a unique 4-CT subcategory $\mathcal{C}$.

(n, L)-admissible quivers

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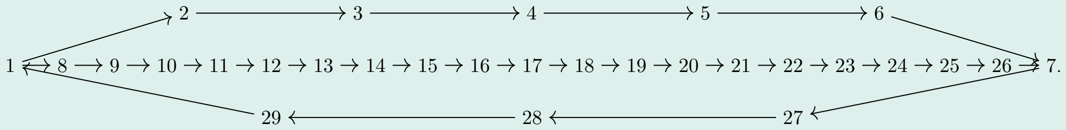

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Moreover, $\mathsf{C} = \mathsf{add}(M)$ where M is the direct sum of the projective modules, the injective modules, and the interval modules (13), $(13, 14)$, $(13, 14, 15)$, $(19, 20, 21)$, $(20, 21)$, (21).

How to find examples

It is easy to find (n, L)-admissible quivers such that $\Lambda = \mathbf{k}Q/J^L$ is a wild algebra and admits an n-cluster tilting subcategory.

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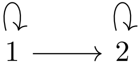

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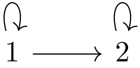

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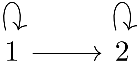

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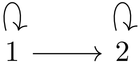

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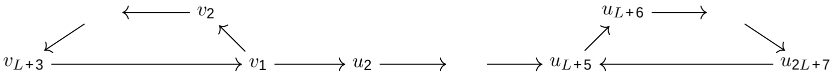

Picking L large enough, gives a wild algebra.

$n\mathbb{Z}$-cluster tilting subcategories

Definition [Iyama–Jasso]

An n-cluster tilting subcategory $\mathcal{C}$ mod Λ is called $n\mathbb{Z}$-cluster tilting if it is closed under Ω^n.

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Theorem [Herschend–Kvamme-V, Oppermann-V]

Let $\Lambda = \mathbf{k}Q/J^L$. Then Λ admits an $n\mathbb{Z}$-cluster tilting subcategory if and only if one of the following conditions holds:

(i) $Q = A_m$ and $L = 2$ or L j $(m \quad 1)$, and $n = 2\frac{m \quad 1}{L}$, or

(ii) $Q = \tilde{A}_m$ and $L = 2$ or $L = n + 2$, and n j $(m + 1)$.

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Corollary [Sandøy–Thibault]

Let $\Lambda = \mathbf{k}Q/J^L$ and $d = \operatorname{gl.dim.}(\Lambda)$. There exists a d-CT subcategory of $\operatorname{mod}\Lambda$ if and only if $Q = A_m$ and either of $L = 2$ or L j $(m \quad 1)$ holds.

A nice property for $L = 2$

Theorem [V]

Let $\Lambda = \mathbf{k}Q/J^2$ and let N be the largest integer for which Q is $(N, 2)$-admissible. Then the following hold.

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(a) For each divisor n of N, the quiver Q is $(n, 2)$-admissible. In particular, there exists an n-cluster tilting subcategory C_n mod Λ.

(b) The set $\{\mathsf{C}_n \mid n \text{ is a divisor of } N\}$ is a complete lattice with respect to inclusion isomorphic to the opposite of the lattice of divisors of N.

A nice property for $L = 2$

Example

Let Q be the quiver

$$\begin{array}{ccccccccccccccccccccccc}
 & & 23 & \longleftarrow & 22 & \longleftarrow & 21 & \longleftarrow & 20 & \longleftarrow & 19 & & & & & & & & & & & & \\
 & \swarrow & & & & & & & & & \uparrow & & & & & & & & & & & & \\
1 & \longrightarrow & 14 & \longrightarrow & 15 & \longrightarrow & 16 & \longrightarrow & 17 & \longrightarrow & 18 & & & & & & & & & & & & \\
\downarrow & & & & & & & & & & & & & & & & & & & & & & \\
2 & \longrightarrow & 3 & \longrightarrow & 4 & \longrightarrow & 5 & \longrightarrow & 6 & \longrightarrow & 7 & \longrightarrow & 8 & \longrightarrow & 9 & \longrightarrow & 10 & \longrightarrow & 11 & \longrightarrow & 12 & \longrightarrow & 13.
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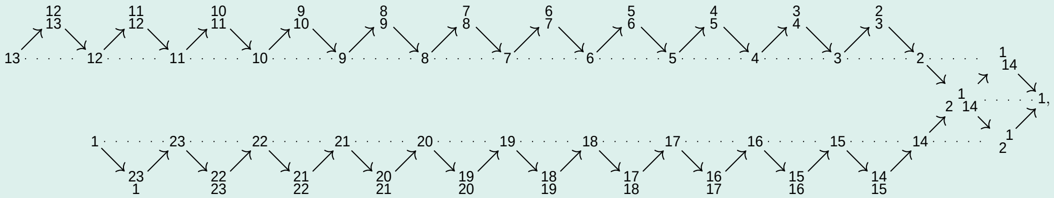
\end{array}
$$

The largest N for which Q is $(N, 2)$-admissible is $N = 12$.

A nice property for $L = 2$

Example

The Auslander–Reiten quiver of $\Lambda = \mathbf{k}Q/J^2$ is



where the simple module $S(1)$ appears twice. Then we have

$C_1 = \mathrm{mod}\,\Lambda,$

$C_2 = \mathrm{add} f\Lambda, 11, 9, 7, 5, 3, \begin{smallmatrix}1\\14\end{smallmatrix}, 23, 21, 19, 17, 15, \begin{smallmatrix}1\\2\end{smallmatrix}g,$

$C_3 = \mathrm{add} f\Lambda, 10, 7, 4, \begin{smallmatrix}1\\14\end{smallmatrix}, 22, 19, 16, \begin{smallmatrix}1\\2\end{smallmatrix}g,$

$C_4 = \mathrm{add} f\Lambda, 9, 5, \begin{smallmatrix}1\\14\end{smallmatrix}, 21, 17, \begin{smallmatrix}1\\2\end{smallmatrix}g,$

$C_6 = \mathrm{add} f\Lambda, 7, \begin{smallmatrix}1\\14\end{smallmatrix}, 19, \begin{smallmatrix}1\\2\end{smallmatrix}g,$

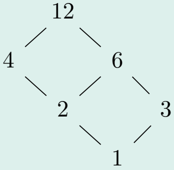
$C_{12} = \mathrm{add} f\Lambda, \begin{smallmatrix}1\\14\end{smallmatrix}, \begin{smallmatrix}1\\2\end{smallmatrix}g,$

and C_n is an n-cluster tilting subcategory of $\mathrm{mod}\,\Lambda$.

A nice property for $L = 2$

Example

Then the lattice



of divisors of 12

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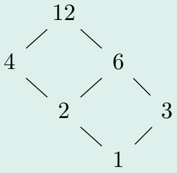
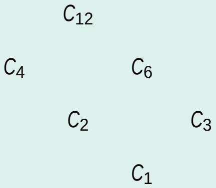


of divisors of 12 corresponds to the lattice



of inclusions of n-cluster tilting subcategories of $\operatorname{mod} \Lambda$.

Thank You!