Graded extensions of Verma modules

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Standard modules

Let $\mathbb{k}$ be an algebraically closed field.

Let A be a finite dimensional associative $\mathbb{k}$-algebra.

Let Λ be an indexing set for simple A-modules.

Let $\leq$ be a fixed linear order on Λ.

Usual notation:

- L_λ — the simple module corresponding to $\lambda \in \Lambda$;
- P_λ — the indecomposable projective cover of L_λ;
- I_λ — the indecomposable injective envelope of L_λ;

Definition. For $\lambda \in \Lambda$, the standard module Δ_λ is the maximum quotient of P_λ such that $[\Delta_\lambda : L_\mu] = 0$ for all $\lambda < \mu$.

Quasi-hereditary algebras

Dual notion: costandard modules ∇_λ.

Let A-mod be the category of all left A-modules.

Notation: $\mathcal{F}(\Delta)$ is the full subcategory of A-mod consisting of all modules with standard filtration, that is a filtration whose subquotients are standard modules.

Similarly: $\mathcal{F}(\nabla)$.

Definition. We say that $(A, \leq)$ is quasi-hereditary provided that ${}_A A \in \mathcal{F}(\Delta)$ and $[\Delta_\lambda : L_\lambda] = 1$, for all λ.

Classical examples:

- hereditary algebras;
- directed algebras;
- Schur algebras;
- blocks of category $\mathcal{O}$.

Grothendieck group

Consider the Grothendieck group $\mathbf{Gr}(A\text{-mod})$ of A-mod.

Important property: all quasi hereditary algebras have finite global dimension.

In fact, $\mathbf{Gr}(A\text{-mod})$ has the following natural bases:

- the basis $\{[L_\lambda] : \lambda \in \Lambda\}$;
- the basis $\{[P_\lambda] : \lambda \in \Lambda\}$;
- the basis $\{[I_\lambda] : \lambda \in \Lambda\}$;
- the basis $\{[\Delta_\lambda] : \lambda \in \Lambda\}$;
- the basis $\{[\nabla_\lambda] : \lambda \in \Lambda\}$.

Reason: by definition, the transformation matrices both from simples to standards and from standards to projectives are upper/lower triangular with "1" on the diagonal.

Main problem for today

Natural question: Given a quasi-hereditary algebra A, determine $\dim \mathrm{Ext}^i_A(\Delta_\lambda, \Delta_\mu)$, for all i, λ and μ.

Why: to understand the Yoneda algebra $\mathrm{Ext}^*_A(\Delta, \Delta)$,

where $\Delta = \bigoplus_{\lambda \in \Lambda} \Delta_\lambda$.

Remark: The algebra $\mathrm{Ext}^*_A(\Delta, \Delta)$ is directed in the sense that $\mathrm{Ext}^i_A(\Delta_\lambda, \Delta_\mu) \neq 0$ implies $\lambda \leq \mu$, moreover, we have $\lambda < \mu$ if $i > 0$.

Naïve hope: Understand $\mathcal{D}^b(A)$ using an "easier" directed algebra $\mathrm{Ext}^*_A(\Delta, \Delta)$ and its derived category.

Remark: standard modules form an exceptional sequence.

Delorme Theorem

Theorem. [Patrick Delorme, 1980]

Let A be quasi-hereditary.

Let $T = (t_{\lambda,\mu})$ be the transformation matrix between the standard and the costandard bases.

Then, for any $\lambda, \mu \in \Lambda$, we have

$$\sum_{i \geq 0} (-1)^i \dim \mathrm{Ext}_A^i(\Delta_\lambda, \Delta_\mu) = t_{\mu,\lambda}.$$

Note: Delrome's result is formulated for Verma modules in category $\mathcal{O}$.

In 1980, quasi-hereditary algebras were not yet defined.

However, the hints for the proof given in the paper

Delorm, M. Extensions in the Bernstein-Gelfand-Gelfand category $\mathcal{O}$. Funktsional. Anal. i Prilozhen. 14 (1980), no. 3, 77–78.

generalize to the quasi-hereditary setting as formulated above.

Proof

Important property: if A is quasi-hereditary, then we have the following ext-orthogonality:

$$\dim \mathrm{Ext}_A^i(\Delta_\lambda, \nabla_\mu) = \delta_{\lambda,\mu}\delta_{i,0}1.$$

Corollary. For any $M \in A$-mod, in $\mathbf{Gr}(A\text{-mod})$, we have:

$$[M] = \sum_{i \geq 0} \sum_{\lambda \in \Lambda} (-1)^i \dim \mathrm{Ext}_A^i(\Delta_\lambda, M) \cdot [\nabla_\lambda].$$

Proof. For $M = \nabla_\mu$, this follows from the ext-orthogonality above.

For any M it follows from the linearity, in M, of both parts w.r.t. distinguished triangles in $\mathcal{D}^b(A)$ and the fact that costandard modules generate the latter. □

To prove Delorme Theorem, take $M = \Delta_\mu$. □

Simple preserving duality

Assume, additionally, that A-mod has a simple preserving duality $\star$.

This is the case in many examples (e.g. category $\mathcal{O}$).

Then $\Delta_\lambda^\star \cong \nabla_\lambda$, for all λ.

In this case $[\Delta_\lambda] = [\nabla_\lambda]$, for all λ.

Consequence. In this case, we have:

$$\sum_{i \geq 0} (-1)^i \dim \mathrm{Ext}^i(\Delta_\lambda, \Delta_\mu) = \delta_{\lambda,\mu}.$$

References: some general and special results

Explicit formulae for extensions between standard modules in special cases:

B. Shelton, Extensions between generalized Verma modules: the Hermitian symmetric cases, Math. Z. 197 (3) (1988) 305–318.

R. Biagioli, Closed product formulas for extensions of generalized Verma modules, Trans. Amer. Math. Soc. 356 (1) (2004) 159–184.

A. Klamt, C. Stroppel, On the Ext algebras of parabolic Verma modules and A infinity-structures, English, J. Pure Appl. Algebra 216 (2) (Feb. 2012) 323–336.

Thuresson, Markus, The Ext-algebra of standard modules over dual extension algebras. J. Algebra 606 (2022), 519–564.

For deeper role of the extension algebra of standard modules, see:

Koenig, Steffen; Külshammer, Julian; Ovsienko, Sergiy, Quasi-hereditary algebras, exact Borel subalgebras, A_∞-categories and boxes. Adv. Math. 262 (2014), 546–592.

Category $\mathcal{O}$

Let $\mathfrak{g}$ be a complex finite dimensional semi-simple Lie algebra.

Typical example: $\mathfrak{g} = \mathfrak{sl}_n$.

Fix a triangular decomposition: $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$.

For example, for $\mathfrak{sl}_n$ we can take:

- $\mathfrak{h}$ — diagonal matrices;
- $\mathfrak{n}_+$ — strictly upper triangular matrices;
- $\mathfrak{n}_-$ — strictly lower triangular matrices.

BGG category $\mathcal{O}$: full subcategory of $\mathfrak{g}$-mod consisting of all $\mathfrak{h}$-diagonalizable modules on which $U(\mathfrak{n}_+)$ acts locally finitely.

Simple objects in $\mathcal{O}$ are simple highest weight modules L_λ, where $\lambda \in \mathfrak{h}^*$.

Verma modules

For $\lambda \in \mathfrak{h}^*$,

let $\mathbb{C}_\lambda$ be the simple $\mathfrak{h}$-module on which elements of $\mathfrak{h}$ act via the scalars given by λ.

Set $\mathfrak{n}_+\mathbb{C}_\lambda = 0$.

Verma module: $\Delta_\lambda := U(\mathfrak{g}) \bigotimes_{U(\mathfrak{h}\oplus\mathfrak{n}_+)} \mathbb{C}_\lambda$.

Note: L_λ is the unique simple top of Δ_λ.

Main problem for today: describe $\mathrm{Ext}^i_{\mathcal{O}}(\Delta_\lambda, \Delta_\mu)$.

Remark: category $\mathcal{O}$ has a simple preserving duality $\star$.

Dual Verma modules: $\nabla_\lambda := \Delta_\lambda^\star$.

Blocks

Let W be the Weyl group of $(\mathfrak{g}, \mathfrak{h})$.

It acts on $\mathfrak{h}^*$ naturally and via the dot action (the natural action shifted by the half of the sum of all positive roots).

Harish-Chandra Theorem. Two Verma modules have the same central characters iff their highest weights belong to the same dot-orbit of W.

Corollary. Category $\mathcal{O}$ decomposes into a direct sum of indecomposable subcategories (a.k.a. blocks) each of which is equivalent to the module category over a finite dimensional associative algebra.

BGG Theorem. The associative algebra underlying each block of $\mathcal{O}$ is quasi-hereditary with Verma modules being standard.

Corollary. (The original Delorme Formula).

We have: $\sum_{i\geq 0}(-1)^i \dim \mathrm{Ext}^i_{\mathcal{O}}(\Delta_\lambda, \Delta_\mu) = \delta_{\lambda,\mu}$.

The principal block

Principal block: $\mathcal{O}_0$ containing the trivial $\mathfrak{g}$-module L_0 (the later has dimension 1 and $\mathfrak{g}L_0 = 0$).

The block $\mathcal{O}_0$ is the "biggest" or "most complicated" block of $\mathcal{O}$.

Let A be the basic associative algebra underlying $\mathcal{O}$.

Simple objects in $\mathcal{O}_0$ are indexed by W: $\{L_w := L_{w\cdot 0} : w \in W\}$.

Quasi hereditary order: any linear extension of the Bruhat order on W.

Observation. (Coulembier)

The quasi-hereditary structure on A is essentially unique (any two formally different structures give the same standard modules).

Remark. This is true for any quasi-hereditary algebra with a simple preserving duality.

Let H be the Hecke algebra of W.

It is an algebra over $\mathbb{Z}[v, v^{-1}]$.

It has the standard basis $\{H_w \, : \, w \in W\}$

and the Kazhdan-Lusztig basis $\{\underline{H}_w \, : \, w \in W\}$.

Entries of the transformation matrix between these two bases are called the Kazhdan-Lusztig polynomials $\{p_{x,y} \, : \, x, y \in W\}$.

Remark: KL polynomials are genuine polynomials in v.

Kazhdan-Lusztig Conjecture (theorem). $[\Delta_x : L_y] = p_{y,x}(1)$.

BGG Reciprocity. We have $P_x \in \mathcal{F}(\Delta)$ and

$$[P_x : \Delta_y] = [\Delta_y : L_x].$$

This determines the Cartan matrix of $\mathcal{O}_0$.

Grading

Theorem. (Soergel) The algebra A is Koszul.

In particular, it is (positively!) $\mathbb{Z}$-graded.

Let $A^{\mathbb{Z}}$ be the corresponding $\mathbb{Z}$-cover of A (note: $A^{\mathbb{Z}}$ is infinite dimensional with a free action of $\mathbb{Z}$).

Let $\mathcal{O}_0^{\mathbb{Z}}$ be the category of finite dimensional $A^{\mathbb{Z}}$-modules.

The category $\mathcal{O}_0^{\mathbb{Z}}$ is usually called the graded lift of $\mathcal{O}_0$.

Remark. The forgetful functor $\mathcal{O}_0^{\mathbb{Z}} \to \mathcal{O}_0$ is not dense in general.

However, all simple, projective, injective, standard and costandard modules admit graded lifts (unique up to shift).

Graded combinatorics: the Groth. group of $\mathcal{O}_0^{\mathbb{Z}}$ is isomorphic to $\mathbf{H}$

by sending $[\Delta_w]$ (for the standard graded lift) to H_w.

This sends $[P_w]$ to $\underline{H}_w$.

R-polynomials

Definition. For $x, y \in W$ and a simple reflection s, the polynomial $r_{x,y} \in \mathbb{Z}[v, v^{-1}]$ is recursively defined via:

$$r_{x,w_0} = \begin{cases} 1, & x = w_0; \\ 0, & x \neq w_0; \end{cases} \qquad r_{x,ys} = \begin{cases} r_{xs,y}, & xs < x; \\ r_{xs,y} + (v - v^{-1}) r_{x,y}, & xs > x. \end{cases}$$

Remark: $r_{x,y}$ are genuine polynomials.

Denote by $r_{x,y}^{(k)}$ the coefficient at v^k in $r_{x,y}$.

Gabber-Joseph Conjecture. $\dim \mathrm{Ext}^k_{\mathcal{O}}(\Delta_x, \Delta_y) = (-1)^k r_{x,y}^{(k)}$.

Disproved by Boe (turns out that the coefficients of R-polynomials are not alternating, in general).

Question. What is the role of R-polynomials for extensions of Verma modules?

Another question. Why $(-1)^k$?

Ungraded vs graded

Notation: $\mathrm{ext} := \mathrm{Ext}_{\mathcal{O}_0^{\mathbb{Z}}}$

Also: $\langle k \rangle : \mathcal{O}_0^{\mathbb{Z}} \to \mathcal{O}_0^{\mathbb{Z}}$ — shift of grading.

We have: $\mathrm{Ext}^i_{\mathcal{O}_0}(\Delta_x, \Delta_y) = \bigoplus_{k \in \mathbb{Z}} \mathrm{ext}^i(\Delta_x\langle k \rangle, \Delta_y)$.

Note: different $\Delta_x\langle k \rangle$ are non-isomorphic in $\mathcal{O}_0^{\mathbb{Z}}$.

The algebra $A^{\mathbb{Z}}$ is "quasi-hereditary" (but infinite dimensional).

The duality $\star$ lifts to $\mathcal{O}_0^{\mathbb{Z}}$,

where it is no longer simple preserving since $\langle k \rangle \circ \star = \star \circ \langle -k \rangle$.

In particular, $[\Delta_x] \neq [\nabla_x]$, in general!

So, the standard and the costandard bases of **H** are very different.

Principal observation. The R-polynomials are exactly the entries of the transformation matrix between the standard and the costandard bases.

Graded Delorme Formula

Corollary. (Graded Delorme Formula)

For any $x, y \in W$ and $k \in \mathbb{Z}$, we have:

$$\sum_{i \geq 0} (-1)^i \dim \mathrm{ext}^i(\Delta_x\langle k\rangle, \Delta_y) = r_{x,y}^{(k)}.$$

This explains the role of R-polynomials in this story.

This also explains the sign in the formula.

Remark. This formula completely determines all extensions in small ranks (i.e. ranks 1 and 2).

In this case only one of the summands on the LHS is non-zero and it is the correct summand for the Gabber-Joseph formula to work.

How to compute extensions

Tilting modules: objects in $\mathcal{F}(\Delta) \cap \mathcal{F}(\nabla)$.

They are: relative injectives (resp. projectives) in $\mathcal{F}(\Delta)$ (resp. $\mathcal{F}(\nabla)$).

Indecomposable tilting modules: T_w, where $w \in W$.

Each standard module Δ_y has a (minimal) tilting coresolution $\mathcal{T}_y^\bullet$.

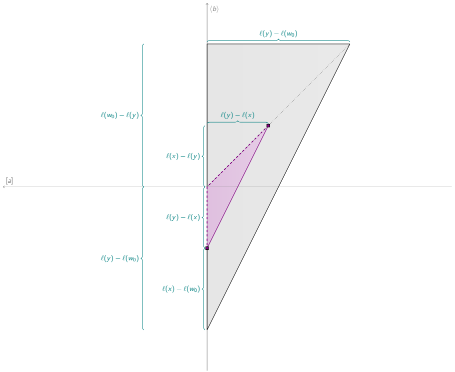
Each extension from Δ_x to Δ_y can be computed as the appropriate homology of the complex $\mathrm{Hom}(\Delta_x, \mathcal{T}_y^\bullet)$.

Combing the following facts:

- The grading of A is positive (KL conjecture);
- A is Koszul and Koszul self-dual (Soergel);
- A is Ringel self-dual (Soergel);

it follows that $\mathcal{T}_y^\bullet$ is linear in the sense that the centers of all it's tilting summands are concentrated on the main diagonal of the coordinate system graded degree/ homological position.

Pictorial description



We only can have $\mathrm{ext}^k(\Delta_x\langle k\rangle, \Delta_y) \neq 0$ in case the top of $\Delta_x\langle k\rangle$ is in the violet undashed area.

Corollary. If all extensions live on the solid violet line, then the Gabber-Joseph formula works.

We call extensions living on the solid violet line expected.

Expected extensions

We prove that all extensions are expected (and hence are given by the Gabber-Joseph formula) in the following cases:

- In ranks 1 and 2.
- If $\ell(x) - \ell(y) \leq 3$.
- If $k \in \{\pm(\ell(x) - \ell(y)), \pm(\ell(x) - \ell(y) - 2)\}$
- If x or $w_0 y$ are boolean.

In case all extensions are expected for all x and y, we prove that $\mathrm{Ext}^*(\Delta, \Delta)$ is Koszul and Koszul self-dual and the bounded derived category of the $\mathbb{Z}$-cover of $\mathrm{Ext}^*(\Delta, \Delta)$ is equivalent to the bounded derived category of $\mathcal{O}_0^{\mathbb{Z}}$.

Interesting special case. In type A_3 (i.e. for $\mathfrak{sl}_4$), it turns out that all extensions, for all x and y, are expected (quite non-trivial ad-hoc proof).

Remark. This is surprising as there are non-trivial Kazhdan-Lusztig polynomials in this type.

Unexpected extensions

Gabber-Joseph conjecture $\Leftrightarrow$ "all extensions are expected".

Boe's counterexample with non-alternating coefficients of the R-polynomials implies that unexpected extensions exist.

An explicit example:

- Abe, Noriyuki, First extension groups of Verma modules and R-polynomials. J. Lie Theory 25 (2015), no. 2, 377–393.
- Carlin, Kevin J. Twisted sequences of extensions. Comm. Algebra 48 (2020), no. 8, 3471–3481.

We construct, in type A, series of examples of unexpected first and second extensions between Verma modules.

We use the description of the socle of the cokernel for the inclusion of two Verma modules from

Ko, Hankyung; Mazorchuk, Volodymyr; Mrđen, Rafael, Bigrassmannian permutations and Verma modules. Selecta Math. (N.S.) 27 (2021), no. 4, Paper No. 55, 24 pp.

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THANK YOU!!!

Check out: Uppsala Algebra on YouTube:
https://www.youtube.com/channel/UCPWnhR29VHTAk7rZUEDQdDQ