Relative dominant dimension and quasi-hereditary covers

<u>Motivation</u>

- terminology of q.h. covers [Rouquier 2008]
- For any finite-dimensional algebra B

$A = \mathrm{End}_B(M)^{op}$, $M = \bigoplus_{i\geq 1}^{m} B/\mathrm{rad}^i B$, $m \gg 0$ [Auslander '71, Dlab-Ringel '89]

$(A, \mathrm{Hom}_A(M, A))$ is a q.h. cover of B

- Iyama used q.h. covers to prove finiteness of representation dimension
- q.h. covers can be seen as "resolutions"

Quasi-hereditary algebras determine all finite-dimensional algebras q.h. covers can be used to obtain information about B

- simple B-modules
- A finite type $\Rightarrow$ B finite type
- $Z(B)$

<u>Def</u> R regular Noetherian ring

A projective Noetherian R-algebra ($A \in R\text{-proj}$)

$P \in A\text{-proj}$ $\quad (A, P)$ is split q.h. cover if

(1) $F := \mathrm{Hom}_A(P, -) : A\text{-mod} \longrightarrow B\text{-mod}$ is fully faithful on $A\text{-proj}$

(2) (A, Δ) is split q.h.

(i) $\bigoplus_{\lambda \in \Lambda} P(\lambda)$ progenerator $\quad \underbrace{X(\lambda)}_{\in \mathcal{F}(\tilde{\Delta}_{\mu > \lambda})} \hookrightarrow P(\lambda) \twoheadrightarrow \Delta(\lambda)$

(ii) $\mathrm{End}_A(\Delta(\lambda)) = R$, $\mathrm{Hom}_A(\Delta(\lambda), \Delta(\mu)) \neq 0 \Rightarrow \lambda < \mu$

$\Delta(\lambda) \in R\text{-proj}$

$$\Rightarrow A\text{-proj} \subset \mathcal{F}(\tilde{\Delta})$$

- $K = \overline{K}$ split q.h. = q.h.
- A split q.h. $\Longrightarrow$ gldim $A < +\infty$

(gldim $R < +\infty$)

Assume that (A, P) split q.h. cover of B

<u>Quality of a split q.h. cover</u> $\mathrm{HNdim}_F \mathcal{F}(\tilde{\Delta}) \geq i$ if $i \in \mathbb{Z}_{\geq -1}$

$i = -1$ F is faithful on $\mathcal{F}(\tilde{\Delta})$

$(i \geq 0)$ $\cancel{F}$ $\mathrm{Ext}_A^j(M, N) \simeq \mathrm{Ext}_B^j(FM, FN)$ $0 \leq j \leq i$ $\forall M, N \in \mathcal{F}(\tilde{\Delta})$

<u>Prop</u> [C.] If $\mathrm{HNdim}_F \mathcal{F}(\tilde{\Delta}) \geq \#$ simple B-modules and B finite-dimensional k-algebra then F is an equivalence of categories

<u>Prop</u> [Rouquier '08]
If $\mathrm{HNdim}_F \mathcal{F}(\tilde{\Delta}) \geq 1 \Longrightarrow \mathcal{F}(\tilde{\Delta}) \xrightarrow{\simeq} \mathcal{F}(F\tilde{\Delta})$

(A, P) split q.h. cover of B
(k field)

deform

$(\tilde{A}, \tilde{P})$ split q.h. cover of $\tilde{B}$ (over a ring R)

compute HN dim

if low

compute HN dim

Return to the finite-dimensional realm with more information

Find new q.h. covers for related algebras to B and $\tilde{B}$ (e.g. quotient algebras)

If B is self-injective, (A, P) q.h. cover of B

Then domdim $A \geq 2$ by Mueller-Tachikawa correspondence

and $\mathrm{HNdim}_{\mathcal{F}}\, \mathcal{F}(\Delta) = \mathrm{domdim}_A T - 2$

where T is a characteristic tilting module

$$\mathrm{add}\, T = \mathcal{F}(\tilde{\Delta}) \cap \underbrace{\mathcal{F}(\tilde{\nabla})}_{= \mathcal{F}(\tilde{\Delta})^{\perp}}$$

where $\mathcal{F}(\tilde{\Delta})^{\perp} = \{ N : \mathrm{Ext}_A^{i>0}(\mathcal{F}(\tilde{\Delta}), N) = 0 \}$

Example

$S_R(n,d) \circlearrowright (R^n)^{\otimes d} \circlearrowleft RS_d$ place permutation

$\| $

$\mathrm{End}_{RS_d}(V^{\otimes d})$

one version of SW duality $\Longrightarrow$

$\mathrm{End}_{S_R(n,d)}(V^{\otimes d})^{op} \cong \mathrm{im}\left(RS_d \longrightarrow \mathrm{End}_R(V^{\otimes d}) \right)$

Assume that $n \geq d$: $(S_R(n,d), V^{\otimes d})$ is a split q.h. cover of RS_d

Thm [HN 2004, FK 2010) $F_k = \mathrm{Hom}_{S_k(n,d)}(V^{\otimes d}, -)$

$\mathrm{HNdim}_{F_k} \mathcal{F}(\Delta) = \mathrm{domdim}\, S_k(n,d) - 2$

$$= \begin{cases} \mathrm{char}\, k - 3 & 0 < \mathrm{char}\, k \leq d \\ +\infty & \text{otherwise} \end{cases}$$

$p \neq 2,3$ $\quad \mathcal{F}(\Delta) \xrightarrow{\cong} \mathcal{F}(F_k \Delta)$

Def A projective Noetherian R-algebra

$M, Q \in A\text{-mod} \cap R\text{-proj}$

$Q\text{-domdim}_{(A,R)} M \geq n \iff \exists$ an (A,R)-exact sequence

$$0 \to M \to P_1 \to \cdots \to P_n, \quad P_i \in \operatorname{add} Q$$

which remains exact under $\operatorname{Hom}_A(-, Q)$

(A,R)-exact sequences $=$ exact sequences of A-modules which split as sequence of R-modules

$D = \operatorname{Hom}_R(-, R)$

Interesting cases:

(1) $\operatorname{add} P = \operatorname{add} Q = \operatorname{add} A \cap \operatorname{add} DA$

$\operatorname{domdim}_{(A,R)} \Omega := P\text{-}\operatorname{domdim}_{(A,R)} \Omega$

- there exists an integral version of the Morita–Tachikawa correspondence [C. 2022]

(2) (A, Δ) split q.h.

$Q \in \operatorname{add} T = \mathcal{F}(\tilde{\Delta}) \cap \mathcal{F}(\tilde{\nabla})$

<u>Thm</u> A [C.] $\Omega \in \mathcal{F}(\tilde{\Delta})$

$$Q\text{-}\operatorname{domdim}_{(A,R)} \Omega = \inf_{m \in \operatorname{maxSpec} R} \{ Q(m)\text{-}\operatorname{domdim}_{A(m)} \Omega(m) \}$$

where $\Omega(m) = R/m \otimes_R \Omega$.

So all computations can be reduced to computations over finite-dimensional algebras over $k = \bar{k}$

Thm B [C.] (A, Δ) split q.h. $\quad Q \in \operatorname{add} T = \mathcal{F}(\tilde{\Delta}) \cap \mathcal{F}(\tilde{\nabla})$

If $Q\text{-codomdim}_{(A,R)} T := DQ\text{-domdim}_{(A^{op},R)} DT \geq n \geq 2$

then $\operatorname{HNdim}_{\mathcal{F}} \mathcal{F}(\tilde{\Delta}_{R_A}) \geq n-2$, $\quad R_A = \operatorname{End}_A(T)^{op}$

($\Leftarrow$ R field)

$(R_A, \operatorname{Hom}_A(T, Q))$ is a split q.h. cover of $\operatorname{End}_A(Q)^{op}$

Rmk 1 $Q\text{-domdim}_{(A,R)} A$ measures how far $Q \in \operatorname{add} T$ is from being a characteristic tilting module

Rmk 2 Given a split q.h. cover of a finite-dimensional algebra and $\operatorname{HNdim}_F \mathcal{F}(\Delta) \geq 0$ then HNdim is completely determined by this invariant

$$(A, P) = (R_{R_A}, \tilde{P}) = (R_{R_A}, \operatorname{Hom}_{R_A}(T_{R_A}, \tilde{Q}))$$

Example – Part II

By Thm A, $m \geq d$ [C.22]

$$\operatorname{domdim}_{(S_R(m,d),R)} T = \inf\{k \in \mathbb{N} : (k+1)\cdot 1_R \notin U(R),\ k < d\}$$

≥ 1

Corollary [C.] R local regular ring

$$\operatorname{HNdim}_{\operatorname{Hom}_{S_R(m,d)}(V^{\otimes d},-)} \mathcal{F}(\tilde{\Delta}) = \begin{cases} \operatorname{domdim}_{(S_R(m,d),R)} T - 2 & \text{if } R \text{ contains a field} \\ \operatorname{domdim}_{(S_R(m,d),R)} T - 1 & \text{otherwise} \end{cases}$$

Idea: $S_K(m,d)$ has a duality

$$\operatorname{domdim}_{(S_R(m,d),R)} T = \operatorname{Hom}(T, V^{\otimes d}) - \operatorname{domdim}_{(R_{S_R(m,d)},R)} DT$$

proj relative injective

and $\operatorname{HNdim}_F \mathcal{F}(\tilde{\Delta})$ gets completely determined by the values $\operatorname{HNdim}_{Q(R/p)\otimes_R F} \mathcal{F}(Q(R/p)\otimes_R \Delta)$

$\forall p \in \operatorname{Spec} R$, $Q(R/p)$ quotient field of R/p

e.g. $p=3$ we should use instead $R = \mathbb{Z}[\frac{1}{2}]$

what happens if $m < d$?

1. $V^{\otimes d}$ is no longer in general projective but $V^{\otimes d} \in \operatorname{add} \mathcal{F}(\tilde{\Delta}) \cap \mathcal{F}(\tilde{\nabla})$ remains

2. k field. There exists a Schur functor $S_k(d,d)\text{-mod} \longrightarrow S_k(m,d)\text{-mod}$ which preserves the q.h. structure [Green '81, End '94]

$\Longrightarrow (R^m)^{\otimes d}\text{-}\operatorname{domdim}_{(S_R(m,d),R)} \overline{T_m} \geq \operatorname{domdim}_{(S_R(d,d),R)} \overline{T_d} \geq 1$

3. $S_{\mathbb{Q}}(m,d)$ semi-simple

<u>Thm</u> [C.] $(R_{S_R(m,d)}, \operatorname{Hom}_{S_R(m,d)}(T, V^{\otimes d}))$ is a split q.h. cover of $\operatorname{End}_{S_R(m,d)}(V^{\otimes d})^{op}$, $V = R^m$

$\operatorname{HNdim}_{\mathcal{F}} \mathcal{F}(\tilde{\Delta}_{R_A}) \geq V^{\otimes d}\text{-}\operatorname{domdim}_{(S_R(m,d),R)} \overline{T_m} - 2$

- If $n \geq d$ $S_k(n,d)$ is Ringel self-dual [Don 93] and this q.h. cover is equivalent to the previous one
- These arguments also work for q-Schur algebras
- $n = 2$, this argument provides a split q.h. cover for Temperley Lieb algebras

(the precise value of HNdim in the case $n = 2$ on-going work with K. Erdmann)