Rational homotopy type of the moduli space of stable rational curves

Vladimir Dotsenko (Université de Strasbourg)

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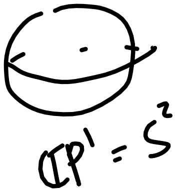
Motivation

For each $n \geq 3$, one can define $\overline{\mathcal{M}}_{0,n}$ the Deligne–Mumford compactification of the moduli space $\mathcal{M}_{0,n}$ of smooth rational curves with n distinct marked points labelled $1, \ldots, n$.



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Recollections: Koszul algebras

$\mathbb{k}$ a field

Let A be a weight graded (commutative or noncommutative) associative $\mathbb{k}$-algebra. We assume the weight grading to be *standard*, in other words, we assume A to be generated by elements of weight 1 (in particular, $A_0 = \mathbb{k}$). Such an algebra is automatically augmented, and $\mathbb{k}$ acquires a trivial A-module structure via the augmentation map.

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The algebra A is said to be *Koszul* if the trivial module has a resolution by free A-modules

$$\cdots \to A^{a_n} \xrightarrow{d_n} A^{a_{n-1}} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} A^{a_1} \xrightarrow{d_1} A \xrightarrow{\epsilon} \Bbbk \to 0,$$

where the differentials d_k are "linear", i.e. their matrices consist of elements of weight 1. (This resolution can be used, among other things, to relate the derived category of A-modules to that of $\mathrm{Ext}^\bullet(\Bbbk, \Bbbk)$.)

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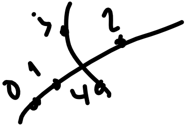
A Koszul algebra is necessarily quadratic, but the converse is not true. However, an algebra with a quadratic Gröbner basis is Koszul.

Recollections: Keel presentation



Keel established that the Chow ring of $\overline{\mathcal{M}}_{0,1+n}$ is generated by elements D_S with $S \subset \{0, 1, \dots, n\}$, $2 \le |S| \le n-1$, subject to the relations

- $D_S = D_{S^c}$ for all S,
- $\sum_{i,j \in S,\, k,l \notin S} D_S = \sum_{i,k \in S,\, j,l \notin S} D_S = \sum_{i,l \in S,\, j,k \notin S} D_S$ for all pairwise distinct i, j, k, l,
- $D_S D_T = 0$ for all S, T with $S \cap T \neq \emptyset$, $S \not\subseteq T$, $T \not\subseteq S$,

Geometrically, the class D_S correspond to the divisor D^S whose generic element is the curve with two components, the points of S on one branch, the points of the complement S^c on the other.

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the generators are Y_S with $S \subseteq \{1, \dots, n\}$, $2 \leq |S| \leq n$ and the relations between them are

- $\sum_{i,j \in S} Y_S = 0$ for all $i \neq j$,
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It is easy to see that if one eliminates D_S for $0 \in S$ using the symmetry relation $D_S = D_{S^c}$, and also eliminates $Y_{\{1,\ldots,n\}}$ from the De Concini–Procesi presentation, the two presentations become identical.

Need another presentation

ordering of monomials

rels: $\underline{m} = \sum_{m' < m} c_{m,m'} m'$

Proposition. Neither the Keel presentation nor the De Concini–Procesi presentation admits a linear-and-quadratic Gröbner basis (either commutative or noncommutative) for $n \geq 4$, no matter what admissible ordering of generators one chooses.

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the generators are X_S, $S \subseteq \{1,2,\ldots,n\}$, $|S| \geq 3$, and the relations between them are

- $X_S^2 = 0$ for $|S| = 3$,
- $X_S(X_S - X_{S\setminus\{s\}}) = 0$ for $|S| > 3$, and $s \in S$,
- $(X_{S\cup T} - X_S)(X_{S\cup T} - X_T) = 0$ for all S, T with $S \cap T \neq \varnothing$, $S \not\subseteq T$, $T \not\subseteq S$.

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(This is related to the De Concini–Procesi presentation by setting $X_S := \sum_{S\subseteq T} Y_T$.

Statement of the main result

Theorem. There exist an ordering of monomials for which the Etingof–Henriques–Kamnitzer–Rains–Singh presentation has a quadratic Gröbner basis of relations. Consequently, the Chow ring of $\overline{\mathcal{M}}_{0,1+n}$ is Koszul for all n.

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Choice of ordering. Consider the following binary relation $\prec'$ on the set $2^{\{1,\ldots,n\}}$: we say that $I \prec' J$ if either $J = I \setminus \{a\}$ where $a \in I$, $a \neq \max(I)$, or $I = J \setminus \{\max(J)\}$. Let $\prec$ be the transitive closure of $\prec'$. Then $\prec$ is a partial order. We extend it to a total order in ~~certain~~ any way, and use it to order the generators X_S, $S \subseteq \{1, 2, \ldots, n\}$, $|S| \geq 3$. The associated graded lexicographic order of monomials is our order of choice.

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For example, for $n = 4$, $\{1, 2, 3\} \prec' \{1, 2, 3, 4\}$, and $\{1, 2, 3, 4\} \prec' S$ if $|S| = 3$, $S \neq \{1, 2, 3\}$. One suitable extension is the lexicographic order of subsets (represented as ordered sequences):

$$\{1, 2, 3\} < \{1, 2, 3, 4\} < \{1, 2, 4\} < \{1, 3, 4\} < \{2, 3, 4\}.$$

Sketch of a proof, part 1 (technical)

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- X_S^2, $|S| = 3$,
- $X_S^2 + X_{\partial(S)}X_S$, $|S| > 3$,
- $X_{S\setminus\{s\}}X_S - X_S^2$, $|S| > 3$, $\max(S) \neq s \in S$,
- $(X_S - X_{S\cup T})(X_T - X_{S\cup T})$, $S \cap T \neq \varnothing$, $S \not\subset T$, $T \not\subset S$, $\max(S) = \max(T)$,
- $(X_S - X_{S\cup T})(X_T - U)$, $S \cap T \neq \varnothing$, $S \not\subset T$, $T \not\subset S$, $\max(S) > \max(T)$, T not an initial interval of $S \cup T$, U is the shortest initial interval of $S \cup T$ containing T,
- $(X_S - X_{S\cup T})(X_{\partial^{p-1}(S\cup T)} - X_{\partial^p(S\cup T)})$, $|(S \cup T) \setminus S| > 1$, $p \geq 1$, $S \cap \partial^p(S \cup T) \neq \varnothing$, $S \not\subset \partial^p(S \cup T)$, $\partial^p(S \cup T) \not\subset S$,
- $(X_S - X_{S\cup T})X_{\partial^p(S\cup T)}$, $|(S \cup T) \setminus S| = 1$, $p \geq 1$, $S \cap \partial^p(S \cup T) \neq \varnothing$, $S \not\subset \partial^p(S \cup T)$, $\partial^p(S \cup T) \not\subset S$.

Here $\partial(S) = S \setminus \{\max(S)\}$.

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Let us describe all quadratic monomials that are normal with respect to our modified set of relations, that is all quadratic monomials that do not occur among the leading terms of those relations. For that, we introduce a combinatorial notion.

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Suppose that S and T are two proper subsets of $\{1,\ldots,n\}$ with $|S|,|T|\geq 3$. We shall say that T is an *MI-complement* (minimal interval complement) of S if the following conditions hold simultaneously:

- the intersection of S and T is not empty,
- T is an initial interval of $S\cup T$ (that is, if $s\in S\cup T$ and $s\leq\max(T)$, then $s\in T$),
- among all the initial intervals of $S\cup T$ of cardinality at least three satisying the above conditions, T is the shortest one.

We shall say that T is an *essential MI-complement* of S if it is an MI-complement of S and $|T\setminus S|>1$.

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On the contrary, $T' = \{1, 2, 3, 4\}$ is not an essential MI-complement of S even though T' is an initial interval of $S \cup T' = \{1, 2, 3, 4, 6\}$: the problem is that $S \cup \partial(T') = \{1, 2, 3, 4, 6\} = S \cup T'$, so there is a shorter initial interval that can be taken.

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The set $T'' = \{1, 2, 4\}$ is an MI-complement of S because it is an initial interval of $S \cup T'' = \{1, 2, 4, 6\}$, and it is of length three, so there is nothing shorter; however, this MI-complement is not essential because $T'' \setminus S = \{2\}$ is a set of cardinality one.

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The set $U = \{1,2,4\}$ is an essential MI-complement of each of the sets $V' = \{1,5,6\}$, $V'' = \{2,5,6\}$, and $V''' = \{4,5,6\}$.

Sketch of a proof, part 2 (combinatorial, continued)

Lemma. Suppose that $S, T \subseteq \{1, \ldots, n\}$ with $|S|, |T| \geq 3$. A commutative quadratic monomial $X_S X_T$ is normal with respect to the modified set of generators if and only if $\max(S) \neq \max(T)$ and one of the following three conditions hold:

- the subsets S and T are disjoint,
- the subsets S and T are comparable,
- one of them is an essential MI-complement of the other.

Sketch of a proof, part 3 (the dual space)

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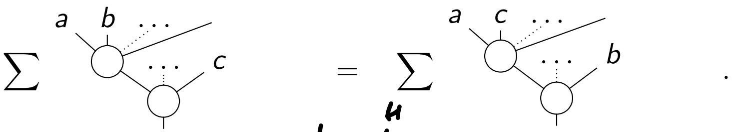

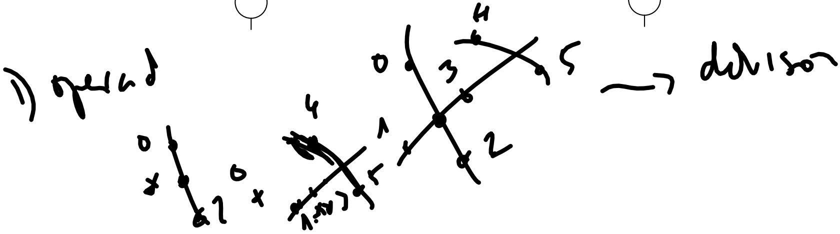

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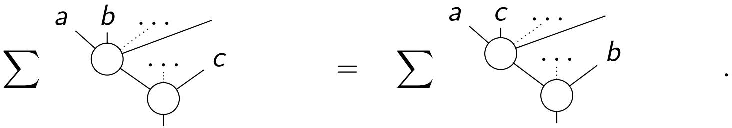
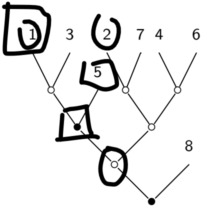


Using "shuffle operads" and Gröbner bases for operads (developed by myself and Khoroshkin in 2008), one obtains a basis B of the corresponding operad.

Digression: shuffle tree monomials

This is a shuffle tree monomial for which the internal vertices of the underlying tree are labelled by two binary operations $\{\circ, \bullet\}$:



For each internal vertex, the minimal leaves of the subtrees growing at it increase from the left to the right.

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- the tree $\overset{a}{|}$ without internal vertices belongs to B,
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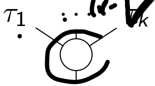

belongs to B if and only if $\tau_1, \ldots, \tau_k$ belong to B, and the root vertex of each of them but τ_k has either just one input or at least three ones.

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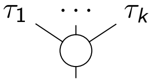
- a shuffle tree monomial 

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To build things inductively, we allow labels to come from any totally ordered set, not just $\{1, \ldots, n\}$.

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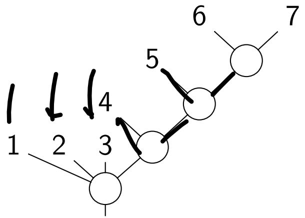
This can be expressed slightly less recursively as follows. One takes a "right comb" tree $\gamma_{k,\ell}$, e.g.

$\gamma_{4,3} =$ 

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and grafts at its leaves certan elements of B, say τ_1, ..., τ_k, τ_{k+1}, ..., $\tau_{k+\ell}$, each of which is either the trivial tree or a tree whose root vertex has strictly more than two children.

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This can be expressed slightly less recursively as follows. One takes a "right comb" tree $\gamma_{k,\ell}$, e.g.

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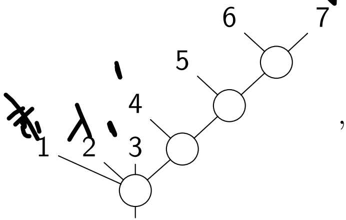

,

and grafts at its leaves certan elements of B, say τ_1, ..., τ_k, τ_{k+1}, ..., $\tau_{k+\ell}$, each of which is either the trivial tree or a tree whose root vertex has strictly more than two children.

For such tree, we let

$S_p = \left\{ a \in \mathrm{Leaves}(\tau) \colon a \leq \min(\mathrm{Leaves}(\tau_p)) \right\}$, and define

$$\Phi(\tau) = \Phi(\tau_1) \cdots \Phi(\tau_{k+\ell}) \prod_{j=3}^{k} X_{S_{j+\ell}}.$$

Sketch of a proof, part 4 (surjection)

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First, we check that Φ constructs normal monomials. Second, we construct a right inverse Ψ, inductively. Essentially, either a normal monomial is a product of several ones for disjoint sets, or it is divisible by $X_{\{1,\ldots,n\}}$, or it has a unique divisor X_S with $n \in S$, and $S \neq \{1, \ldots, n\}$; depending on this, we construct an inductive description.

Sketch of a proof, conclusion

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Of course, we know that they are equal, so the estimate itself is not a big deal: what is a big deal is that the estimate is sharp if and only if our quadratic relations form a Gröbner basis, completing the proof.

Application

According to the theory of Koszul spaces (Berglund 2014), for a formal space whose rational cohomology algebra is Koszul, the Koszul dual algebra is isomorphic $H_\bullet(\Omega X, \mathbb{Q})$ with the Pontryagin product, and the latter, thanks to Milnor–Moore, is the universal envelope $U(\pi_*(\Omega\overline{\mathcal{M}}_{0,1+n}) \otimes \mathbb{Q})$.

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As a consequence, we can compute the rational homotopy Lie algebras of our spaces.

Theorem. The rational homotopy Lie algebra $\pi_*(\Omega\overline{\mathcal{M}}_{0,1+n})\otimes\mathbb{Q}$ is isomorphic to the graded Lie algebra generated by odd elements Y_S, where $S\subseteq\{1,\dots,n\}$, $|S|\geq 3$, subject to relations

$$[Y_S,Y_T]=0,\quad \text{for } S \text{ and } T \text{ with } S\cap T=\varnothing,$$

$$\sum_{\substack{\{T_1,T_2\}\subset 2^S:\\ T_1\cap T_2\neq\varnothing,\, T_1\cup T_2=S}}[Y_{T_1},Y_{T_2}]=0,\quad \text{for } |S|>3,$$

$$\left[Y_T,\sum_{T\cup K=S}Y_K\right]=0,\quad \text{for } S \text{ and } T \text{ with } T\subset S, |S\setminus T|>1.$$

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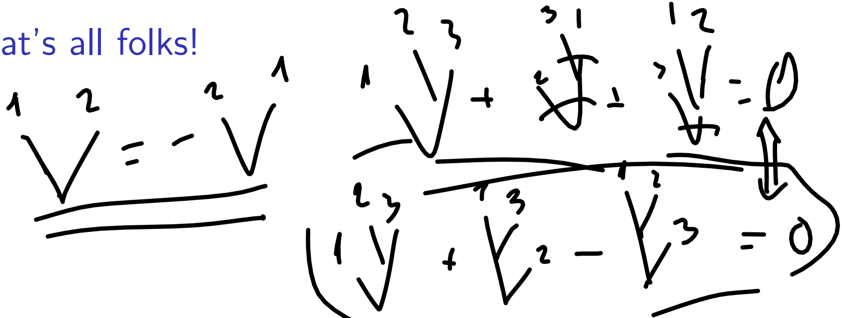
Theorem (Matthew Mastroeni and Jason McCullough, arXiv:2111.00393). The Chow ring of any matroid is Koszul.

That's all folks!



Thank you for your attention!