Classifying torsion classes of Noetherian algebras

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Notation

- R : commutative Noetherian ring
- Λ : Noetherian R-algebra i.e. ${}_R\Lambda$ is a finitely generated R-module
- $\operatorname{mod}\Lambda$: the category of finitely generated left Λ-modules

- $\Lambda = R \Rightarrow \Lambda$ is a commutative Noetherian ring
- R is a field $\Rightarrow \Lambda$ is a finite dimensional R-algebra

Goal

Classify torsion (free) classes, Serre subcategories of $\operatorname{mod}\Lambda$.

Definitions of subcategories

Let $\mathcal{A} = \mathrm{mod}\,\Lambda$ and $\mathcal{C} \subset \mathcal{A}$ a subcategory.

- $\mathcal{C}$ is closed under <u>extensions</u> :⇔ for a short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$, if $X, Z \in \mathcal{C}$, then $Y \in \mathcal{C}$.
- $\mathcal{C}$ is closed under <u>quotients</u> :⇔ 「$Y \in \mathcal{C}$, $Y \twoheadrightarrow Z \in \mathcal{A} \Rightarrow Z \in \mathcal{C}$」
- $\mathcal{C}$ is closed under <u>submodules</u> :⇔ 「$Y \in \mathcal{C}$, $X \rightarrowtail Y \in \mathcal{A} \Rightarrow X \in \mathcal{C}$」

Definition

(1) $\mathcal{C}$: torsion class :⇔ closed under quotients and extensions.

(2) $\mathcal{C}$: torsionfree class :⇔ closed under submodules and extensions.

(3) $\mathcal{C}$: Serre subcategory :⇔ $\mathcal{C}$ is a torsion class and a torsionfreee class.

$\mathrm{tors}\,\Lambda = \{$torsion classes of $\mathcal{A}\}$, $\mathrm{torf}\,\Lambda = \{$torsionfree classes of $\mathcal{A}\}$

$\mathrm{serre}\,\Lambda = \{$Serre subcategories of $\mathcal{A}\}$

serre R

- $\operatorname{Spec} R = \{\text{prime ideals of } R\}$
- $\operatorname{Supp} M = \{\mathfrak{p} \in \operatorname{Spec} R \mid M_{\mathfrak{p}} \neq 0\}$ for $M \in \operatorname{mod} R$
- $\mathcal{W} \subseteq \operatorname{Spec} R$: specialization closed

  $:\Leftrightarrow$ 「$\mathfrak{p} \in \mathcal{W}, \quad \mathfrak{p} \subseteq \mathfrak{q} \in \operatorname{Spec} R \Rightarrow \mathfrak{q} \in \mathcal{W}$」

Theorem (Gabriel '62)

For a subcategory $\mathcal{C}$ of $\operatorname{mod} R$, let $\operatorname{Supp} \mathcal{C} := \bigcup_{M \in \mathcal{C}} \operatorname{Supp} M$. Then this induces an isomorphism of posets:

$$\operatorname{serre} R \xrightarrow{\operatorname{Supp}(-)} \{\text{specialization closed subsets of } \operatorname{Spec} R\}$$

tors R and torf R

For $M \in \operatorname{mod} R$, $\operatorname{Ass} M := \{\mathfrak{p} \in \operatorname{Spec} R \mid {}^{\exists} R/\mathfrak{p} \hookrightarrow M\}$.

Theorem (Takahashi '08)

For a subcategory $\mathcal{C}$ of $\operatorname{mod} R$, let $\operatorname{Ass} \mathcal{C} := \bigcup_{M \in \mathcal{C}} \operatorname{Ass} M$. Then this induces an isomorphism of posets:

$$\operatorname{torf} R \xrightarrow{\operatorname{Ass}(-)} \mathsf{P}(\operatorname{Spec} R) := \{\text{subsets of } \operatorname{Spec} R\}.$$

Theorem (Stanley-Wang '11)

$\operatorname{serre} R = \operatorname{tors} R$ holds for a commutative Noetherian ring R.

If Λ is a finite dimensional algebra over a field

$M \in \mathsf{mod}\,\Lambda$

$$\mathsf{Fac}\,M := \{X \in \mathsf{mod}\,\Lambda \mid {}^{\exists}M^{\oplus \ell} \twoheadrightarrow X,\ {}^{\exists}\ell \geq 0\}$$

$$\text{f-}\mathsf{tors}\,\Lambda := \{\mathcal{T} \in \mathsf{tors}\,\Lambda \mid \mathcal{T} \text{ is functorially finite in } \mathsf{mod}\,\Lambda\}$$
$$= \{\mathcal{T} \in \mathsf{tors}\,\Lambda \mid {}^{\exists}M \in \mathsf{mod}\,\Lambda \text{ s.t. } \mathsf{Fac}\,M = \mathcal{T}\}$$

Results for $\mathsf{tors}\,\Lambda$ and f-$\mathsf{tors}\,\Lambda$

(a) [Adachi-Iyama-Reiten '14]

$$\text{f-}\mathsf{tors}\,\Lambda \overset{1-1}{\longleftrightarrow} \{\text{isoclasses of basic support } \tau\text{-tilting } \Lambda\text{-modules}\}$$

(b) [Demonet-Iyama-Jasso '19]

$$|\mathsf{tors}\,\Lambda| < \infty \Leftrightarrow |\text{f-}\mathsf{tors}\,\Lambda| < \infty \Leftrightarrow \mathsf{tors}\,\Lambda = \text{f-}\mathsf{tors}\,\Lambda$$

(c) [Ingalls-Thomas, Mizuno, Chan-Demonet,...]
For some classes of algebras, classification results of $\mathsf{tors}\,\Lambda$ and f-$\mathsf{tors}\,\Lambda$.

- Path algebras of Dynkin quivers.
- Preprojective algebras of Dynkin type.
- Gentle algebras

Notation

- For $\mathfrak{p} \in \operatorname{Spec} R$, let
$$\Lambda_{\mathfrak{p}} := R_{\mathfrak{p}} \otimes_R \Lambda.$$
$\Lambda_{\mathfrak{p}}$ is a Noetherian $R_{\mathfrak{p}}$-algebra.
- For $\mathfrak{p} \in \operatorname{Spec} R$, let $k_{\mathfrak{p}} = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ and
$$k_{\mathfrak{p}}\Lambda := k_{\mathfrak{p}} \otimes_R \Lambda.$$
$k_{\mathfrak{p}}\Lambda$ is a finite dimensional $k_{\mathfrak{p}}$-algebra. We have $\operatorname{mod} k_{\mathfrak{p}}\Lambda \subseteq \operatorname{mod} \Lambda_{\mathfrak{p}}$.

Today

Classify $\operatorname{tors} \Lambda$, $\operatorname{torf} \Lambda$ and $\operatorname{serre} \Lambda$ via $\operatorname{tors}(k_{\mathfrak{p}}\Lambda)$, $\operatorname{torf}(k_{\mathfrak{p}}\Lambda)$ and $\operatorname{serre}(k_{\mathfrak{p}}\Lambda)$.

Localization

For a subcategory $\mathcal{C} \subseteq \operatorname{mod} \Lambda$ and $\mathfrak{p} \in \operatorname{Spec} R$, let

$$\mathcal{C}_\mathfrak{p} := \{ M_\mathfrak{p} \mid M \in \mathcal{C} \} \subseteq \operatorname{mod} \Lambda_\mathfrak{p}.$$

Lemma

(a) $\mathcal{C} \in \operatorname{tors} \Lambda \Longrightarrow \mathcal{C}_\mathfrak{p} \in \operatorname{tors} \Lambda_\mathfrak{p}$

(b) $\mathcal{C} \in \operatorname{torf} \Lambda \Longrightarrow \mathcal{C}_\mathfrak{p} \in \operatorname{torf} \Lambda_\mathfrak{p}$

(c) $\mathcal{C} \in \operatorname{serre} \Lambda \Longrightarrow \mathcal{C}_\mathfrak{p} \in \operatorname{serre} \Lambda_\mathfrak{p}$

An assignment $\mathcal{C} \mapsto \mathcal{C}_\mathfrak{p} \cap \operatorname{mod} k_\mathfrak{p}\Lambda$ gives three maps

$$\operatorname{tors} \Lambda \longrightarrow \operatorname{tors}(k_\mathfrak{p}\Lambda), \quad \operatorname{torf} \Lambda \longrightarrow \operatorname{torf}(k_\mathfrak{p}\Lambda), \quad \operatorname{serre} \Lambda \longrightarrow \operatorname{serre}(k_\mathfrak{p}\Lambda)$$

Maps Φ

Definition

$$\mathbb{T}_R(\Lambda) := \prod_{\mathfrak{p}\in \mathsf{Spec}\, R} \mathsf{tors}(k_\mathfrak{p}\Lambda), \qquad \mathbb{F}_R(\Lambda) := \prod_{\mathfrak{p}\in \mathsf{Spec}\, R} \mathsf{torf}(k_\mathfrak{p}\Lambda)$$

$$\mathbb{S}_R(\Lambda) := \prod_{\mathfrak{p}\in \mathsf{Spec}\, R} \mathsf{serre}(k_\mathfrak{p}\Lambda)$$

Definition

For a subcategory $\mathcal{C}$ of $\mathsf{mod}\,\Lambda$, let

$$\Phi(\mathcal{C}) := \{\mathcal{C}_\mathfrak{p} \cap \mathsf{mod}\, k_\mathfrak{p}\Lambda\}_{\mathfrak{p}\in \mathsf{Spec}\, R}$$

By restricting Φ, we have the following three maps

$$\mathsf{tors}\,\Lambda \xrightarrow{\Phi_\mathrm{t}} \mathbb{T}_R(\Lambda), \quad \mathsf{torf}\,\Lambda \xrightarrow{\Phi_\mathrm{f}} \mathbb{F}_R(\Lambda), \quad \mathsf{serre}\,\Lambda \xrightarrow{\Phi_\mathrm{s}} \mathbb{S}_R(\Lambda)$$

Maps Ψ

For $\mathcal{X} = \{\mathcal{X}^{\mathfrak{p}}\}_{\mathfrak{p}} \in \mathbb{T}_R(\Lambda)$, let

$$\Psi_{\mathrm{t}}(\mathcal{X}) := \{M \in \operatorname{mod}\Lambda \mid k_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} M \in \mathcal{X}^{\mathfrak{p}}, {}^{\forall}\mathfrak{p} \in \operatorname{Spec} R\}.$$

For $\mathcal{Y} = \{\mathcal{Y}^{\mathfrak{p}}\}_{\mathfrak{p}} \in \mathbb{F}_R(\Lambda)$, let

$$\widetilde{\mathcal{Y}^{\mathfrak{p}}} = \{M \in \operatorname{mod}\Lambda \mid M_{\mathfrak{p}} \in \mathcal{Y}^{\mathfrak{p}},\ \operatorname{Ass}_R M \subseteq \{\mathfrak{p}\}\}$$

$$\Psi_{\mathrm{f}}(\mathcal{Y}) := \operatorname{Filt}\left(\widetilde{\mathcal{Y}^{\mathfrak{p}}} \middle| \mathfrak{p} \in \operatorname{Spec} R\right) \subset \operatorname{mod}\Lambda$$

Proposition

We have three maps

$$\mathbb{T}_R(\Lambda) \xrightarrow{\Psi_{\mathrm{t}}} \operatorname{tors}\Lambda, \quad \mathbb{F}_R(\Lambda) \xrightarrow{\Psi_{\mathrm{f}}} \operatorname{torf}\Lambda, \quad \mathbb{S}_R(\Lambda) \xrightarrow{\Psi_{\mathrm{s}} = \Psi_{\mathrm{t}}|_{\mathbb{S}_R(\Lambda)}} \operatorname{serre}\Lambda$$

Theorem

Theorem 1 (Iyama-Kimura)

(a) Φ_{f} is an isomorphism of posets with an inverse Ψ_{f}.

(b) $\Psi_{\mathrm{t}} \circ \Phi_{\mathrm{t}} = \mathrm{id}_{\mathsf{tors}\,\Lambda}$ and $\Psi_{\mathrm{s}} \circ \Phi_{\mathrm{s}} = \mathrm{id}_{\mathsf{serre}\,\Lambda}$ hold.

(c) Φ_{t} and Φ_{s} are embeddings of posets.

$$\begin{array}{ccccc}
\mathsf{serre}\,\Lambda & \hookrightarrow & \mathsf{tors}\,\Lambda & \xhookrightarrow{(-)^{\perp}} & \mathsf{torf}\,\Lambda \\
{\scriptstyle \Psi_{\mathrm{s}}}\twoheaduparrow \; \downarrow{\scriptstyle \Phi_{\mathrm{s}}} & & {\scriptstyle \Psi_{\mathrm{t}}}\twoheaduparrow \; \downarrow{\scriptstyle \Phi_{\mathrm{t}}} & & {\scriptstyle \Psi_{\mathrm{f}}}\uparrow \wr \downarrow{\scriptstyle \Phi_{\mathrm{f}}} \\
\mathbb{S}_R(\Lambda) & \hookrightarrow & \mathbb{T}_R(\Lambda) & \xrightarrow[\simeq]{(-)^{\perp}} & \mathbb{F}_R(\Lambda)
\end{array}$$

where $\mathcal{T}^{\perp} = \{X \in \mathsf{mod}\,\Lambda \mid \mathrm{Hom}_{\Lambda}(T, X) = 0, {}^{\forall}T \in \mathcal{T}\}$.

Corollary : $\Lambda_{\mathfrak{p}}$ is Morita equivalent to a local ring

For $\mathcal{X} = \{\mathcal{X}^{\mathfrak{p}}\}_{\mathfrak{p}} \in \mathbb{T}_R(\Lambda)$(or $\mathbb{F}_R(\Lambda)$), let $\mathcal{S}(\mathcal{X}) = \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathcal{X}^{\mathfrak{p}} \neq 0\}$.

Corollary

Assume that $\Lambda_{\mathfrak{p}}$ is Morita equivalent to a local ring for all $\mathfrak{p} \in \operatorname{Spec} R$.

(a) We have $\mathcal{S} \circ \Phi_{\mathrm{t}} = \operatorname{Supp}(-)$. This gives an isomorphism of posets.

$$\operatorname{tors}\Lambda \xrightarrow{\Phi_{\mathrm{t}}} \operatorname{Im}\Phi_{\mathrm{t}} \xrightarrow{\mathcal{S}} \{\text{specialization closed subsets of } \operatorname{Spec} R\}$$

(b) We have $\mathcal{S} \circ \Phi_{\mathrm{f}} = \operatorname{Ass}(-)$. This gives an isomorphism of posets.

$$\operatorname{torf}\Lambda \xrightarrow{\Phi_{\mathrm{f}}} \mathbb{F}_R(\Lambda) \xrightarrow{\mathcal{S}} \mathsf{P}(\operatorname{Spec} R).$$

If $\Lambda_{\mathfrak{p}}$ is Morita equivalent to a local ring, then we have

$$\operatorname{tors}(k_{\mathfrak{p}}\Lambda) = \operatorname{torf}(k_{\mathfrak{p}}\Lambda) = \operatorname{serre}(k_{\mathfrak{p}}\Lambda) = \{0, \operatorname{mod} k_{\mathfrak{p}}\Lambda\}.$$

If $\Lambda = R$, then we have results by [Gabriel] and [Takahashi].

serre Λ

(1) sim $k_\mathfrak{p}\Lambda$: the set of isomorphism classes of simple $k_\mathfrak{p}\Lambda$-modules

(2) $^\exists$ isomorphisms of posets between serre$(k_\mathfrak{p}\Lambda)$ and the power set P(sim $k_\mathfrak{p}\Lambda$):

$$\mathsf{serre}(k_\mathfrak{p}\Lambda) \simeq \mathsf{P}(\mathsf{sim}\, k_\mathfrak{p}\Lambda)$$

(3) Let Sim $:= \bigcup_{\mathfrak{p}\in \mathsf{Spec}\, R} \mathsf{sim}(k_\mathfrak{p}\Lambda)$, then

$$\mathbb{S}_R(\Lambda) \simeq \mathsf{P}(\mathsf{Sim})$$

(4) We have

$$\mathsf{serre}\,\Lambda \xrightarrow[\simeq]{\Phi_{\mathrm{s}}} \mathrm{Im}\,\Phi_{\mathrm{s}} \subset \mathbb{S}_R(\Lambda) \simeq \mathsf{P}(\mathsf{Sim})$$

Characterize serre Λ inside P(Sim)

serre Λ inside P(Sim)

Sim $= \bigcup_{\mathfrak{p} \in \mathrm{Spec}\, R} \mathrm{sim}(k_{\mathfrak{p}}\Lambda)$

Definition (Poset structure on Sim)

(1) For $S, T \in$ Sim we write $S \leq T$ if :

- $S \in \mathrm{sim}(k_{\mathfrak{p}}\Lambda)$, $T \in \mathrm{sim}(k_{\mathfrak{q}}\Lambda)$, $\mathfrak{p} \supseteq \mathfrak{q}$. We regard T as a $\Lambda_{\mathfrak{p}}$-module by $\Lambda_{\mathfrak{p}} \to \Lambda_{\mathfrak{q}}$. Then $S \leq T$ if S is a subfactor of T as a $\Lambda_{\mathfrak{p}}$-module.

Then (Sim, $\leq$) is a poset.

(2) A subset $\mathcal{W}$ of Sim is down-set if 「$T \in \mathcal{W}, \quad S \leq T \in \mathrm{Sim} \Rightarrow S \in \mathcal{W}$」 holds.

Theorem 2 (Iyama-Kimura)

Φ_{s} : serre $\Lambda \to$ P(Sim) induces an isomorphism of posets:

$$\mathrm{serre}\,\Lambda \simeq \{\mathcal{W} \subseteq \mathrm{Sim} \mid \mathcal{W} \text{ is a down-set}\}$$

Example

Let k be a field, $R = k[[x]]$, $\mathfrak{m} = (x)$. $\Lambda = \begin{pmatrix} R & R \\ \mathfrak{m} & R \end{pmatrix}$.

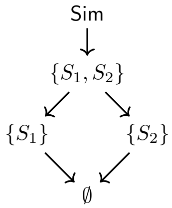
- $k_{\mathfrak{m}}\Lambda = \Lambda/\mathfrak{m}\Lambda = \begin{pmatrix} R/\mathfrak{m} & R/\mathfrak{m} \\ \mathfrak{m}/\mathfrak{m}^2 & R/\mathfrak{m} \end{pmatrix} \simeq k(1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2)/\langle \alpha\beta, \beta\alpha \rangle$

  This algebra has two simple modules $S_1 = \begin{pmatrix} k \\ 0 \end{pmatrix}$ and $S_2 = \begin{pmatrix} 0 \\ k \end{pmatrix}$.

- $k_0\Lambda = \Lambda_0 = \begin{pmatrix} K & K \\ K & K \end{pmatrix}$, where $K = R_0 = k((x))$.

  This algebra has one simple module $T = \begin{pmatrix} K \\ K \end{pmatrix}$.

Sim $= \{T, S_1, S_2\}$ with $T \geq S_1$ and $T \geq S_2$. The Hasse diagram of serre Λ is



tors Λ

For $\mathfrak{p} \in \operatorname{Spec} R$ and $\mathcal{T} \in \operatorname{tors}(k_{\mathfrak{p}}\Lambda)$, the following $\overline{\mathcal{T}}$ is a torsion class of $\operatorname{mod} \Lambda_{\mathfrak{p}}$:

$$\overline{\mathcal{T}} = \{X \in \operatorname{mod} \Lambda_{\mathfrak{p}} \mid k_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} X \in \mathcal{T}\} \in \operatorname{tors} \Lambda_{\mathfrak{p}}.$$

For $\mathfrak{p} \supseteq \mathfrak{q} \in \operatorname{Spec} R$, define a map $\mathrm{r}_{\mathfrak{p},\mathfrak{q}}$ by

$$\mathrm{r}_{\mathfrak{p},\mathfrak{q}} : \operatorname{tors}(k_{\mathfrak{p}}\Lambda) \xrightarrow{\overline{(-)}} \operatorname{tors} \Lambda_{\mathfrak{p}} \xrightarrow{(-)_{\mathfrak{q}}} \operatorname{tors} \Lambda_{\mathfrak{q}} \xrightarrow{(-)\cap \operatorname{mod} k_{\mathfrak{q}}\Lambda} \operatorname{tors}(k_{\mathfrak{q}}\Lambda)$$

Definition

We say that $\mathcal{X} = \{\mathcal{X}^{\mathfrak{p}}\}_{\mathfrak{p}} \in \mathbb{T}_R(\Lambda)$ is compatible if $\mathrm{r}_{\mathfrak{p},\mathfrak{q}}(\mathcal{X}^{\mathfrak{p}}) \supseteq \mathcal{X}^{\mathfrak{q}}$ holds for any pair $\mathfrak{p} \supseteq \mathfrak{q}$ of prime ideals of R.

Proposition

$$\operatorname{Im} \Phi_{\mathrm{t}} \subset \{\text{compatible elements of } \mathbb{T}_R(\Lambda)\}$$

Compatible elements

Question

For which Λ, tors $\Lambda \simeq \operatorname{Im} \Phi_{\mathrm{t}} = \{$compatible elements of $\mathbb{T}_R(\Lambda)\}$?

Partial answer.

Theorem 3 (Iyama-Kimura)

Assume that R is semi-local with $\dim R = 1$. Then we have

$$\text{tors}\,\Lambda \simeq \operatorname{Im} \Phi_{\mathrm{t}} = \{\text{compatible elements of } \mathbb{T}_R(\Lambda)\}.$$

Question

How to calculate $\mathrm{r}_{\mathfrak{p},\mathfrak{q}}$?

2-term silting complex

- $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)$: the bounded homotopy category of $\mathsf{proj}\,\Lambda$
- $X, Y \in \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)$ are additively equivalent if $\mathsf{add}\,X = \mathsf{add}\,Y$ holds, where

  $\mathsf{add}\,X := \{Z \in \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda) \mid Z$ is a direct summand of $X^{\oplus \ell}$ for some $\ell\}$

Definition

$X \in \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)$ is a 2-term silting complex if

- $X^i = 0$ for $i \neq -1, 0$,
- $\mathrm{Hom}(X, X[1]) = 0$,
- $\mathsf{thick}\,X = \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)$.

$2\text{-}\mathsf{silt}\,\Lambda$: the set of additively equivalent classes of 2-term silting complexes

Remark

- $H^0(X)$ for $X \in 2\text{-}\mathsf{silt}\,\Lambda$ are silting modules [Angeleri Hügel - Marks - Vitória].
- If R is a field and $X \in 2\text{-}\mathsf{silt}\,\Lambda$, $H^0(X)$ is called a support τ-tilting module [Adachi-Iyama-Reiten].

2-term silting complex and the map $\mathrm{r}_{\mathfrak{p},\mathfrak{q}}$

Lemma

$\mathsf{Fac}\, H^0(X)$ is a torsion class of $\mathsf{mod}\, \Lambda$ for $X \in \text{2-silt}\, \Lambda$.

Proposition

Assume that $(R, \mathfrak{m})$ is a local ring. Let $M = H^0(X)$ for $X \in \text{2-silt}\, \Lambda$. Then for each $\mathfrak{q} \in \mathsf{Spec}\, R$, we have

$$\mathrm{r}_{\mathfrak{m},\mathfrak{q}}(\mathsf{Fac}\, M \cap \mathsf{mod}\, \Lambda/\mathfrak{m}\Lambda) = \mathsf{Fac}\, M_{\mathfrak{q}} \cap \mathsf{mod}\, k_{\mathfrak{q}}\Lambda.$$

Note : $\mathsf{Fac}\, M_{\mathfrak{q}} \cap \mathsf{mod}\, k_{\mathfrak{q}}\Lambda = \mathsf{Fac}(M_{\mathfrak{q}}/\mathfrak{q}M_{\mathfrak{q}})$

Example

Let k be a field, $R = k[[x]]$, $\mathfrak{m} = (x)$. $\Lambda = \begin{pmatrix} R & R \\ \mathfrak{m} & R \end{pmatrix}$.

- $k_{\mathfrak{m}}\Lambda = \Lambda/\mathfrak{m}\Lambda \simeq \ k(\ 1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2\ )/\langle \alpha\beta, \beta\alpha \rangle$
- $k_0\Lambda = \Lambda_0 = \mathrm{Mat}_2(K)$, where $K = R_0 = k((x))$.
- We have

$$\begin{aligned} \text{tors}\,\Lambda &\simeq \{\text{compatible elements of } \mathbb{T}_R(\Lambda)\} \qquad\qquad \text{by Theorem 3} \\ &= \{(\mathcal{X}^{\mathfrak{m}}, \mathcal{X}^0) \in \mathbb{T}_R(\Lambda) \mid \mathrm{r}_{\mathfrak{m},0}(\mathcal{X}^{\mathfrak{m}}) \supseteq \mathcal{X}^0\}. \end{aligned}$$

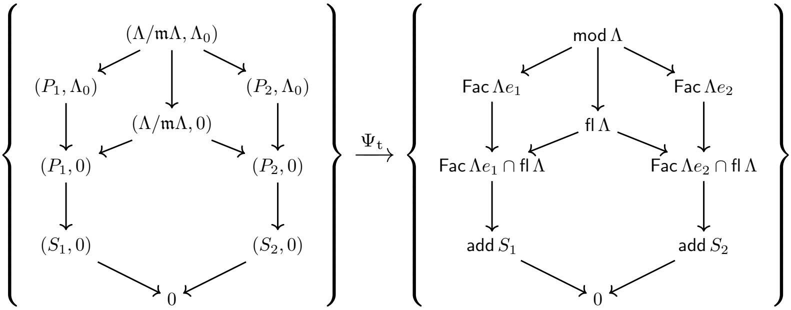
$$\text{tors}(\Lambda/\mathfrak{m}\Lambda) = \left\{ \begin{array}{ccc} & \text{mod}\,\Lambda/\mathfrak{m}\Lambda & \\ \swarrow & & \searrow \\ \text{Fac}\,P_1 & & \text{Fac}\,P_2 \\ \downarrow & & \downarrow \\ \text{add}\,S_1 & & \text{add}\,S_2 \\ \searrow & & \swarrow \\ & 0 & \end{array} \right\} \xrightarrow{\mathrm{r}_{\mathfrak{m},0}} \left\{ \begin{array}{c} \text{mod}\,\Lambda_0 \\ \downarrow \\ 0 \end{array} \right\} = \text{tors}(\Lambda_0)$$

Example

- Since $P_i = (\Lambda/\mathfrak{m}\Lambda)e_i$, we have

$$\mathrm{r}_{\mathfrak{m},0}(\mathsf{Fac}\, P_i) = \mathrm{r}_{\mathfrak{m},0}(\mathsf{Fac}\, \Lambda e_i \cap \mathsf{mod}\, \Lambda/\mathfrak{m}\Lambda) \overset{\text{Prop}}{=} \mathsf{Fac}(\Lambda e_i)_0 = \mathsf{mod}\, \Lambda_0.$$

- $\mathsf{tors}(\Lambda/\mathfrak{m}\Lambda), \mathsf{Fac}\, P_1, \mathsf{Fac}\, P_2$ go to $\mathsf{mod}\, \Lambda_0$ by $\mathrm{r}_{\mathfrak{m},0}$.
- $\mathsf{tors}\, \Lambda$ has the following Hasse quiver



Theorem for $R \otimes_k A$

Let k be a field, A a finite dimensional k-algebra. A simple A-module S is k-simple if $\mathrm{End}_A(S) \simeq k$.
For example, $k = \overline{k}$ or $A = kQ/I$ (I is an admissible ideal), then all simple modules are k-simple.

Theorem 4 (Iyama-Kimura)

Let A a finite dimensional k-algebra, and R a commutative Noetherian ring which contains k. Assume that

- all simple A-modules are k-simple, and
- tors A is a finite set.

Then we have

$$\mathsf{tors}(R \otimes_k A) \simeq \mathrm{Hom}_{\mathsf{poset}}(\mathsf{Spec}\, R, \mathsf{tors}\, A)$$

Example

Let k be a field. Let Q be a Dynkin quiver and $\mathsf{Cam}(Q)$ the Cambrian lattice of Q. By [Ingalls-Thomas, Reading] there is an isomorphism of posets,

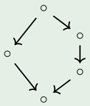
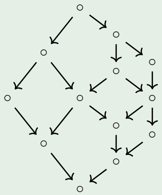
$$\mathsf{tors}(kQ) \simeq \mathsf{Cam}(Q).$$

Therefore for a commutative Noetherian ring R containing the field k, we have

$$\mathsf{tors}\, RQ \simeq \mathrm{Hom}_{\mathrm{poset}}(\mathsf{Spec}\, R, \mathsf{Cam}(Q)).$$

Example

Let $R = k[[x]]$ and $\mathfrak{m} = (x)$. Let $Q = (1 \to 2)$.

$\mathsf{Spec}\, R = \left\{ \begin{array}{c} \mathfrak{m} \\ \downarrow \\ 0 \end{array} \right\}$, $\mathsf{tors}(kQ) = \Big\{$  $\Big\}$, $\mathsf{tors}(RQ) = \Big\{$  $\Big\}$

Problems

Question 1

For which Λ, $\operatorname{Im}\Phi_{\mathrm{t}} = \{$compatible elements of $\mathbb{T}_R(\Lambda)\}$?

$\Rightarrow$ So far we do not know any Λ such that $\operatorname{Im}\Phi_{\mathrm{t}} \neq \{$compatible$\}$.

Question 2

Assume that $(R, \mathfrak{m})$ is a local ring. When does the following equality hold?

$$\mathsf{tors}(\Lambda/\mathfrak{m}\Lambda) = \{\mathsf{Fac}(M/\mathfrak{m}M) \mid M \text{ is a silting } \Lambda\text{-module}\}$$

$\Rightarrow$ We have an analog of the result of [Demonet-Iyama-Jasso] (τ-tilting finiteness).

τ-tilting finite algebra

- f-tors Λ : the set of functorially finite torsion classes of mod Λ

Proposition (Adachi-Iyama-Reiten '14)

Let A be a finite dimensional algebra. Then

$$\text{2-silt}\,A \longrightarrow \text{f-tors}\,A, \quad X \mapsto \operatorname{Fac} H^0(X)$$

is a bijection.

Theorem (Demonet-Iyama-Jasso '19)

Let A be a finite dimensional algebra. TFAE

(i) $\text{tors}\,A = \text{f-tors}\,A$

(ii) $\text{tors}\,A$ is a finite set.

(iii) $\text{2-silt}\,A$ is a finite set.

Finite type

Theorem 5 (Iyama-Kimura)

Assume that $(R, \mathfrak{m})$ is a local ring. Then $(i) \Rightarrow (ii) \Rightarrow (iii)$ hold.

- (i) $\mathsf{tors}(\Lambda/\mathfrak{m}\Lambda) = \{\mathsf{Fac}(H^0(X)/\mathfrak{m}H^0(X)) \mid X \in \mathsf{2\text{-}silt}\,\Lambda\}$
- (ii) $\mathsf{tors}(\Lambda/\mathfrak{m}\Lambda)$ is a finite set.
- (iii) $\mathsf{2\text{-}silt}\,\Lambda$ is a finite set.

If Λ is semi-perfect, then $(iii) \Rightarrow (i)$ holds.

Remark

In general, the right hand side of (i) is strictly smaller than $\mathsf{f\text{-}tors}(\Lambda/\mathfrak{m}\Lambda)$

Reduction of 2-term silting complexes

Assume that $(R, \mathfrak{m})$ is a local ring. For a 2-term complex $X = (X^{-1} \to X^0)$, let $\overline{X} = (X^{-1}/\mathfrak{m}X^{-1} \to X^0/\mathfrak{m}X^0)$.

Proposition (Kimura, Gnedin, Eisele)

The assignment $X \mapsto \overline{X}$ gives an injective map

$$\text{2-silt}\,\Lambda \longrightarrow \text{2-silt}(\Lambda/\mathfrak{m}\Lambda).$$

If Λ is semi-perfect, then this map is a bijection.

Λ is semi-perfect if it admits a decomposition ${}_\Lambda\Lambda = P_1 \oplus \cdots \oplus P_r$ such that each P_i has a local endomorphism ring.
For example,

- If R is complete local, then Λ is semi-perfect.
- If R is local, then RQ is semi-perfect for a finite acyclic quiver Q.