An explicit dg enhancement of singularity category

(joint work with X.-W. Chen)

① Background

② The dg enhancement — singular Yoneda dg category

③ Applications

① Background

A a (left) noetherian algebra

Def (Buchweitz 86, Orlov 03)

$D_{sg}(A) \triangleq D^b(A\text{-mod})/\operatorname{per}(A)$ the singularity category of A.

Rem If $\operatorname{gldim} A < \infty$ then $D_{sg}(A) = 0$.

Thm (Krause 05)
There is a triangle equivalence (up to direct summands)

$$D_{sg}(A) \xrightarrow{\sim} K_{ac}(A\text{-Inj})^c$$

- $K_{ac}(A\text{-Inj})$ is the homotopy category consisting of acyclic complexes of injective A-modules.
- $K_{ac}(A\text{-Inj})^c \subset K_{ac}(A\text{-Inj})$ compact objects (i.e. $\mathrm{Hom}(X,-)$ commutes with coproducts).

Thm (Smith 12, Chen-Yang 15)
Let Q be a finite quiver. Let $A = kQ/J^2 \cong kQ_0 \oplus kQ_1$. Then

$$D_{sg}(A) \xrightarrow{\sim} \mathrm{Per}(L(Q))$$

"universal localisation of kQ"

where $L(Q)$ is the (graded) Leavitt path algebra

$$\underbrace{k\overline{Q}}_{\text{the double quiver of } Q} \Big/ \left(\begin{array}{ll} \alpha\beta^* = \delta_{\alpha,\beta}\, e_{t(\alpha)} & \forall\, \alpha, \beta \in Q_1 \\ \sum\limits_{\{\alpha \in Q_1 \mid s(\alpha) = i\}} \alpha^*\alpha = e_i & \forall\, i \in Q_0 \end{array} \right)$$

$|\alpha| = -1 \quad |\alpha^*| = 1$

We will give an explicit realisation of the above triangle equivalences.

② The dg enhancement — singular Yoneda dg category

Recall A dg enhancement of $D_{sg}(A)$ is a pretriangulated dg category $\mathcal{C}$ such that $H^0(\mathcal{C}) \simeq D_{sg}(A)$ as triangulated categories.

Rem By Keller and Drinfeld, the dg quotient $D^b_{dg}(A\text{-mod})/\mathrm{per}_{dg}(A)$ is a dg enhancement of $D_{sg}(A)$.

Recall The normalised bar resolution $\mathrm{Bar}(A) \triangleq A \otimes T s\bar{A} \otimes A$ $\quad \bar{A} \triangleq A/k\cdot 1$

Note that $\mathrm{Bar}(A) \otimes_A X$ is a dg projective resolution of X.

Def The Yoneda dg category $\mathcal{Y}$ of A

- Objects: the same as those in $D^b(A\text{-mod})$
- Morphisms:
$$\mathcal{Y}(X,Y) \triangleq \mathrm{Hom}_A(\mathrm{Bar}(A) \otimes_A X, Y) \simeq \prod_{i \geq 0} \mathrm{Hom}(s\bar{A}^{\otimes i} \otimes X, Y)$$
- Composition:
$$\mathcal{Y}(Y,Z) \times \mathcal{Y}(X,Y) \xrightarrow{\odot} \mathcal{Y}(X,Z)$$
$\quad f: s\bar{A}^{\otimes m} \otimes Y \to Z$
$\quad g: s\bar{A}^{\otimes n} \otimes X \to Y$

$$f \odot g(s\bar{a}_1 \otimes \cdots \otimes s\bar{a}_{m+n} \otimes x) \triangleq (-1)^{m|g|} f(s\bar{a}_1 \otimes \cdots \otimes s\bar{a}_m \otimes g(s\bar{a}_{m+1} \otimes \cdots \otimes s\bar{a}_{m+n} \otimes x))$$

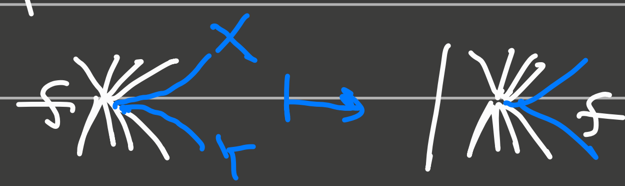
Prop $\mathcal{Y}$ is a dg enhancement of $D^b(A\text{-mod})$. $\quad H^*(\mathcal{Y}(X,X)) \simeq \mathrm{Ext}^*_A(X,X)$

Def $\Omega^p_{nc}(X) \triangleq s\bar{A}^{\otimes p} \otimes X$ graded noncommutative differential p-forms
Cuntz–Quillen 95

Rem $\Omega^p_{nc}(X)$ carries a left dg A-module structure:

$$a_0 \triangleright (s\bar{a}_1 \otimes \cdots \otimes s\bar{a}_p \otimes x) \triangleq s\overline{a_0 a_1} \otimes \cdots \otimes s\bar{a}_p \otimes x + \sum_{i=1}^{p-1} (-1)^i s\bar{a}_0 \otimes \cdots \otimes s\overline{a_i a_{i+1}} \otimes \cdots \otimes s\bar{a}_p \otimes x$$
$$+ (-1)^p s\bar{a}_0 \otimes \cdots \otimes s\bar{a}_{p-1} \otimes a_p x$$

Def The singular Yoneda dg category $\mathcal{SY}$ of A



- objects: the same as in $D^b(A\text{-mod})$
- morphisms:

$\mathcal{SY}(X,Y) \triangleq$ the colimit of the following complexes

$$\mathcal{Y}(X,Y) \hookrightarrow \mathcal{Y}(X, \Omega^1_{nc}(Y)) \hookrightarrow \cdots \hookrightarrow \mathcal{Y}(X, \Omega^p_{nc}(Y)) \hookrightarrow \mathcal{Y}(X, \Omega^{p+1}_{nc}(Y)) \hookrightarrow \cdots$$
$$f \longmapsto \theta_{\Omega^p_{nc}(Y)} \odot f \doteq \mathrm{id}_{s\bar{A}} \otimes f$$

where $\theta_{\Omega^p_{nc}(Y)} \in \mathcal{Y}(\Omega^p_{nc}(Y), \Omega^{p+1}_{nc}(Y)) \triangleq \prod_{i \geq 0} \mathrm{Hom}(s\bar{A}^{\otimes i} \otimes s\bar{A}^{\otimes p} \otimes Y, s\bar{A}^{\otimes p+1} \otimes Y)$ is given by the identity map $s\bar{A}^{\otimes p+1} \otimes Y \to s\bar{A}^{\otimes p+1} \otimes Y$

Thm (Chen–W. 21) $\mathcal{SY}$ is a dg enhancement of $D_{sg}(A)$.

Rem $\mathcal{SY}$ is a "dg localisation" of $\mathcal{Y}$. ($\mathcal{Y} \to \mathcal{SY}$ satisfies a universal property).

③ Applications

1) An explicit realisation of Krause's equivalence $D_{sg}(A) \xrightarrow{\sim} K_{ac}(A\text{-Inj})^c$

Rem Since $A \cong 0$ in $D_{sg}(A)$, the complex $\mathcal{SY}(A,X)$ is acyclic for $X \in D_{sg}(A)$. But $\mathcal{SY}(A,X) \in K_{ac}(A\text{-Inj})$.

Thm (Chen–W. 22) Krause's triangle equivalence is naturally isomorphic to
$$\mathcal{SY}(A,-) : D_{sg}(A) \xrightarrow{\sim} K_{ac}(A\text{-Inj})^c$$

2) A generalisation of Smith & Chen–Yang's equivalence $D_{sg}(kQ/J^2) \cong \mathrm{Per}(L(Q))$

Let $A = kQ/I$ be a finite dimensional k-algebra. Denote $E \triangleq kQ_0$.

Rem Replacing $\otimes$ by $\otimes_E$ in the definitions of $\Omega^p_{nc}(X)$ and $\mathrm{Bar}(A)$, we may define the E-relative singular Yoneda dg category $\mathcal{SY}_E$

Prop (Chen–W. 21) Let $A = kQ/J^2$. Then there is an isomorphism of dg algebras
$$\mathcal{SY}_E(E,E) \cong L(Q)^{op}$$
As a result, Smith & Chen–Yang's equivalence is isomorphic to
$$D_{sg}(A) \xrightarrow{\mathcal{SY}_E(E,-)} \mathrm{Per}(\mathcal{SY}_E(E,E)^{op}) \xrightarrow{\sim} \mathrm{Per}(L(Q))$$

Que How about general algebras $A = kQ/I$?

Idea • (Schaps 1988) $A = kQ/I$ is a deformation of $\tilde{A} = k\tilde{Q}/J^2$.
That is, $\tilde{A} \cong A$ as E-E-bimodules $(\tilde{Q}_0 = Q_0)$
and $(\tilde{A}, \mu) \cong A$ as algebras where $\mu \in C^2(\tilde{A}, \tilde{A})$ is a Maurer–Cartan element of $(C^*(\tilde{A},\tilde{A}), \delta, [\cdot,\cdot])$ $\quad \delta\mu + \frac{1}{2}[\mu,\mu] = 0$

• (Chen–Li–W. 21) there is a morphism of dg Lie algebras

$$\begin{aligned} \phi : C^*(\tilde{A},\tilde{A}) &\longrightarrow C^*(L(\hat{Q}), L(\hat{Q})) \cong \prod_{i\geq 0} \mathrm{Hom}(L(\hat{Q})^{\otimes i}, L(\hat{Q})) \\ \mu &\longmapsto d : L(\hat{Q}) \to L(\hat{Q}) \end{aligned}$$

Thm (Chen–W. 21) Let $A = kQ/I$. Then there is an isomorphism of dg algebras

$$\mathrm{Sy}_E(E,E) \cong (L(\hat{Q}), d)^{op}$$

As a result, we have a triangle equivalence

$$D_{sg}(A) \xrightarrow{\mathrm{Sy}_E(E,-)} \mathrm{Per}(\mathrm{Sy}_E(E,E)^{op}) \xrightarrow{\sim} \mathrm{Per}(L(\hat{Q}), d)$$

Thm (Keller–Y. Wang 21) The dg algebra $(L(\hat{Q}), d)$ is a derived localisation (in the sense of Braun–Chuang–Lazarev) of the Koszul dual of A.

Rmk • The above construction works for A_∞-algebras A, which can be used to describe the generalised cluster categories (Chen–Keller–W. in progress).

• $$\begin{array}{ccc} D_{sg}(\hat{A}) & \xrightarrow[\sim]{\text{Smith, Chen-Yang}} & \mathrm{Per}(L(\tilde{Q})) \\ \mu \,\wr & \overset{\text{M.C. formalism}}{\longleftrightarrow} & \wr\, d \\ D_{sg}(A) & \xrightarrow{\sim} & \mathrm{Per}(L(\tilde{Q}), d) \end{array}$$

A purely deformation-theoretic proof ?

Ex $$A = kQ/I = k\left(\cdot \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \cdot \right) / (\alpha\beta\alpha, \beta\alpha\beta)$$

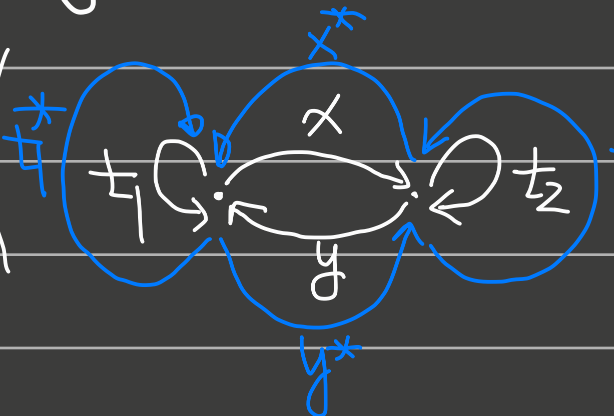
• A is a deformation of $\hat{A} = k\tilde{Q}/J^2$ where $\tilde{Q} = t_1 \circlearrowleft \cdot \underset{y}{\overset{x}{\rightleftarrows}} \cdot \circlearrowright t_2$

• The Maurer–Cartan element $\mu : k\tilde{Q}_1 \otimes k\tilde{Q}_1 \to \hat{A}$

$$x \otimes y \longmapsto t_2$$
$$y \otimes x \longmapsto t_1$$

• The dg Leavitt path algebra

$$L(\hat{Q}) = k\left(\cdots \right) \Big/ \begin{array}{l} yy^* = e_1 = t_1 t_1^* \\ xx^* = e_2 = t_2 t_2^* \\ x^*x + t_1^* t_1 = e_1 \\ y^*y + t_2^* t_2 = e_2 \\ \alpha\beta^* = 0 \text{ if } \alpha \neq \beta \in Q_1 \end{array}$$



$$dx = y^* t_1, \quad dy = x^* t_2, \quad dt_1^* = x^* y^*, \quad dt_2^* = y^* x^*,$$
$$dt_1 = dt_2 = dx^* = dy^* = 0$$

Thank you !