The definable subcategory induced by a large canonical module

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§1. Notation and terminology

A : a ring

$\mathrm{Mod}\,A$: the category of right A-modules

$\mathrm{mod}\,A$: ______ finitely presented right A-modules

- A monomorphism $M \longrightarrow N$ in $\mathrm{Mod}\,A$ is pure if the induced morphism $M \otimes_A L \longrightarrow N \otimes_A L$ is monic for all $L \in \mathrm{Mod}\,A^{\mathrm{op}}$.

- A full subcategory $\mathcal{X} \subseteq \mathrm{Mod}\,A$ is definable if it is closed under direct limits, direct products, and pure submodules.

- $M \in \mathrm{Mod}\,A$ is pure-injective if any pure monomorphism $M \longrightarrow N$ splits

$\mathcal{A}$: an abelian category, $\mathcal{X} \subseteq \mathcal{A}$: a full subcategory closed under isomorphisms

- A morphism $f : X \longrightarrow M$ in $\mathcal{A}$ is an $\mathcal{X}$-precover if $X \in \mathcal{X}$ and $\mathrm{Hom}_{\mathcal{A}}(X', f)$ is surjective for all $X' \in \mathcal{X}$.

- A morphism $f : X \longrightarrow M$ in $\mathcal{A}$ is an $\mathcal{X}$-cover if it is an $\mathcal{X}$-precover and any $h \in \mathrm{End}_{\mathcal{A}}(X)$ with $f \circ h = f$ is an isomorphism.

- A morphism $g : M \longrightarrow X$ in $\mathcal{A}$ is an $\mathcal{X}$-(pre)envelope if g is an $\mathcal{X}$-(pre)cover in $\mathcal{A}^{op}$.

$\mathcal{C} \subseteq \mathcal{A}$: a full subcategory

$\mathcal{C}^{\perp_1} := \{ M \in \mathcal{A} \mid \mathrm{Ext}^1_A(C, M) = 0, {}^\forall C \in \mathcal{C} \}$ $\quad$ ${}^{\perp_1}\mathcal{C} := \{ M \in \mathcal{A} \mid \mathrm{Ext}^1_A(M, C) = 0, {}^\forall C \in \mathcal{C} \}$

$${}^{\perp_{>0}}\mathcal{C} := \{ M \in \mathcal{A} \mid \mathrm{Ext}^i_A(M, C) = 0, {}^\forall C \in \mathcal{C}, {}^\forall i > 0 \}$$

$\mathcal{X}, \mathcal{Y} \subseteq \mathcal{A}$: full subcategories

- $(\mathcal{X}, \mathcal{Y})$ is a cotorsion pair if $\mathcal{X}^{\perp_1} = \mathcal{Y}$ and ${}^{\perp_1}\mathcal{Y} = \mathcal{X}$
- A cotorsion pair $(\mathcal{X}, \mathcal{Y})$ is complete if for ${}^\forall M \in \mathcal{A}$,

$${}^\exists 0 \to \underset{\substack{\cap\!\!\!\!\!\!\!\;\;\; \\ \mathcal{Y}}}{Y} \to \underset{\substack{\cap \\ \mathcal{X}}}{X} \to M \to 0 : \text{ex.}, \quad {}^\exists 0 \to M \to \underset{\substack{\cap \\ \mathcal{Y}}}{Y'} \to \underset{\substack{\cap \\ \mathcal{X}}}{X'} \to 0 : \text{ex.}$$

- A cotorsion pair $(\mathcal{X},\mathcal{Y})$ is **perfect** if every $M\in\mathcal{A}$ has an $\mathcal{X}$-cover $X\to M$ and a $\mathcal{Y}$-envelope $M\to Y$.

(If $\mathcal{A}$ has enough projectives and enough injectives, then any perfect cotorsion pair in $\mathcal{A}$ is complete)

- A cotorsion pair $(\mathcal{X},\mathcal{Y})$ is hereditary if $\mathrm{Ext}^i_{\mathcal{A}}(X,Y)=0$ for all $X\in\mathcal{X}$, $Y\in\mathcal{Y}$, and $i>0$.

(When $\mathcal{A}$ has enough projectives, a cotorsion pair $(\mathcal{X},\mathcal{Y})$ is hereditary iff $\mathcal{X}$ is resolving, i.e., (i) $\mathrm{Proj}\,\mathcal{A}\subseteq\mathcal{X}$, (ii) $\mathcal{X}$ is cl und. ext., and
(iii) $0\to X_3\to \underset{\mathcal{X}}{X_2}\to \underset{\mathcal{X}}{X_1}\to 0$ ex $\Rightarrow X_3\in\mathcal{X}$.)

§2. Basics from commutative ring theory

$(R, \mathfrak{m}, k)$: a comm. noeth. local ring

$M \in \operatorname{Mod} R$ $\qquad \operatorname{depth}_R M := \inf\{ i \mid \operatorname{Ext}_R^i(k, M) \neq 0 \}$ $\qquad (\inf \emptyset = \infty)$

- $\underline{x} = x_1, \ldots, x_i$ $(x_j \in R)$ is M-regular if (1) $M/(x_1, \ldots, x_{j-1})M \xhookrightarrow{x_j} M/(x_1, \ldots, x_{j-1})M$ $1 \leq {}^\forall j \leq i$ and

($\underline{x}$ is weak M-regular if (1) is satisfied) (2) $(x_1, \ldots, x_i)M \neq M$.

<u>Fact</u> $0 \neq M \in \operatorname{mod} R \Rightarrow \operatorname{depth}_R M = \max\{ i \mid {}^\exists \underline{x} = x_1, \ldots, x_i : \text{M-regular} \}$

$$\begin{array}{rcl} \Gamma_{\mathfrak{m}} : \operatorname{Mod} R & \longrightarrow & \operatorname{Mod} R \\ \rotatebox{90}{$\in$} \; & & \\ M & \longmapsto & \{ x \in M \mid {}^\exists n > 0 \text{ s.t. } \mathfrak{m}^n \cdot x = 0 \} \end{array}$$

$R\Gamma_{\mathfrak{m}} : D(R) \longrightarrow D(R)$ $\qquad D(R) := D(\mathrm{Mod}\, R)$

$H^i_{\mathfrak{m}} := H^i R\Gamma_{\mathfrak{m}}$

i th local cohomology functor

$X \in D(R)$ $\qquad \inf X := \inf\{ i \mid H^i X \neq 0 \}$

corrected

<u>Fact</u> $\mathrm{depth}_R M = \inf R\Gamma_{\mathfrak{m}} M \leq \dim R$, ${}^{\forall} M \in \mathrm{Mod}\, R$.

- $M \in \mathrm{mod}\, R$ is **maximal CM** if $\mathrm{depth}_R M \geq \dim R$.

(Cohen-Macaulay)

- R is a CM local ring if it is maximal CM as an R-module.

- A canonical module of R is a maximal CM module ω_R s.t. $\text{inj.dim}_R\, \omega_R < \infty$ and $\operatorname{Ext}^d_R(k, \omega_R) \cong k$, where $d := \dim R$.

<u>Fact</u> $M \in \operatorname{mod} R$ is a canonical module if and only if M is a dualizing complex for R, i.e., $\mathbf{R}\mathrm{Hom}_R(-, M)$ induces an equivalence $D^b_{fg}(R)^{op} \xrightarrow{\sim} D^b_{fg}(R)$

M: dualizing, $\inf M = 0$, $M \longrightarrow I$: minimal inj. resol. $\Rightarrow I^i \cong \bigoplus\limits_{i = d - \dim R/\mathfrak{p}} E_R(R/\mathfrak{p}) \quad {}^{\forall} i \geq 0$

<u>Rem</u> ${}^{\exists}$A canonical module of R $\underset{\not\Leftarrow}{\Rightarrow}$ R : CM.
If a canonical module exists, it is unique up to isomorphism

R : comm. noeth. ring

- $M \in \operatorname{mod} R$ is maximal CM if $\operatorname{depth}_{R_{\mathfrak{m}}} M_{\mathfrak{m}} \geq \dim R_{\mathfrak{m}}$ for all maximal ideals $\mathfrak{m}$ of R.
- R is a CM ring if it is maximal CM as an R-module.
- $\omega \in \operatorname{mod} R$ is a canonical module of R if $\omega_{\mathfrak{m}}$ is a canonical module of $R_{\mathfrak{m}}$ for all maximal ideals $\mathfrak{m}$ of R.

$\operatorname{CM} R := \{\text{maximal CM } R\text{-modules}\}$

<u>Fact</u> $^{\exists}\omega$: a canonical module of R

$\Rightarrow \operatorname{CM} R = \{ M \in \operatorname{mod} R \mid \operatorname{Ext}^i_R(M, \omega) = 0 \ {}^{\forall} i > 0 \}$

Rem $R : CM \iff R_{\mathfrak{p}} : CM \ \forall \mathfrak{p} \in \operatorname{Spec} R$

$R_{\mathfrak{p}} : CM \iff \widehat{R_{\mathfrak{p}}} : CM \iff \widehat{R_{\mathfrak{p}}}$ has a canonical module $\omega_{\widehat{R_{\mathfrak{p}}}}$.

$$\left(\widehat{R_{\mathfrak{p}}} := \Lambda^{\mathfrak{p}} R_{\mathfrak{p}} \qquad \Lambda^{\mathfrak{p}} := \varprojlim_{n \geq 1} (- \otimes_R R/\mathfrak{p}^n) \right)$$

Fact $(R, \mathfrak{m}, k) : CM \qquad d = \dim R$

$$\omega_{\widehat{R}} \cong \operatorname{Hom}_R (H^d_{\mathfrak{m}}(R), E_R(k))$$

local duality

§3 Large CM modules and large canonical modules

R: a comm. noeth. ring

<u>Def</u> We call $M \in \operatorname{Mod} R$ large CM if $\operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} \geq \dim R_\mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Spec} R$.

$\operatorname{LCM} R := \{\text{large CM } R\text{-modules}\}$

<u>Prop</u> $\operatorname{LCM} R$ is a definable subcategory of $\operatorname{Mod} R$.

<u>Rem</u> $(\operatorname{mod} R) \cap (\operatorname{LCM} R) = \operatorname{CM} R$.

Def R : CM

We call $C_R := \prod_{\mathfrak{p} \in \operatorname{Spec} R} \omega_{\widehat{R_{\mathfrak{p}}}}$ a large canonical module of R.

Lem R · CM

(1) $\operatorname{LCM} R = \{ M \in \operatorname{Mod} R \mid \operatorname{Ext}_R^i(M, C_R) = 0,\ i > 0 \}$.

(2) C_R is pure-injective.

(3) $C_R \in \operatorname{LCM} R$.

Def R · CM

An R-algebra A is called an R-order if A is maximal CM as an R-module.

$$\begin{tikzcd} R \arrow[rr, "\varphi"] \arrow[dr] & & A \\ & Z(A) \arrow[ur] & \end{tikzcd} \qquad \operatorname{Mod} A \xrightarrow{\varphi_*} \operatorname{Mod} R$$

Cor A: an R-order, $\mathrm{LCM}\,A := \{ M \in \mathrm{Mod}\,A \mid \varphi_*(M) \in \mathrm{LCM}\,R \}$

$C_A := \mathrm{Hom}_R(A, C_R)$

(1) $\mathrm{LCM}\,A = \{ M \in \mathrm{Mod}\,R \mid \mathrm{Ext}^i_A(M, C_A) = 0,\ i > 0 \}$

(2) C_A is pure-injective.

(3) $C_A \in \mathrm{LCM}\,A$.

We call C_A a large canonical module of the R-order A

$\mathrm{LCM}\,A$ is also a definable subcategory. $(\mathrm{Flat}\,A \subseteq \mathrm{LCM}\,A)$

§4 Large CM approximations

Thm1 (N) R: a CM ring, A: an R-order

$(\mathrm{LCM}\,A, (\mathrm{LCM}\,A)^{\perp_1})$ is a perfect hereditary cotorsion pair in $\mathrm{Mod}\,A$
(complete)

Proof. $\mathcal{C} := \{C_A\}$. Then $\mathrm{LCM}\,A = {}^{\perp_{>0}}\mathcal{C}$ by the corollary above.
Apply the theorem below. $\square$

Thm (Eklof and Trlifaj, 2000 ; Göbel and Trlifaj, 2012)
A: a ring $\mathcal{C}$: a class of pure-injective modules

$\Rightarrow ({}^{\perp_{>0}}\mathcal{C}, ({}^{\perp_{>0}}\mathcal{C})^{\perp_1})$ is a perfect hereditary cotorsion pair

<u>Def</u> $(R,\mathfrak{m})$: comm. noeth local ring , $d := \dim R$ $\quad$ ($x_i \in \mathfrak{m}$, $\dim R/(x_1,\dots,x_d) = 0$)

An R-module M is big CM if there exists a system of parameters $\underline{x} = x_1,\dots,x_d$ of R s.t. $\underline{x}$ is M-regular. (Hochster, 1975)

________________ balanced big CM if every system of parameters of R is M-regular. (Sharp, 1981)

________________ weak balanced big CM if every system of parameters of R is weak M-regular. (Holm, 2017)

$$\begin{array}{ccc} 0 & & \\ \cap & & \\ \mathrm{bbCM}\,R := \{\text{balanced big CM } R\text{-modules}\} & \subsetneq & \mathrm{wbbCM}\,R := \{\text{weak balanced big CM } R\text{-modules}\} \\ \cup\!\!\!\!\times & & \cup\!\!\!\!\times \\ \mathrm{CM}\,R \setminus \{0\} & & \mathrm{CM}\,R \end{array}$$

Rem In general, $\mathrm{wbbCM}\,R \subseteq \mathrm{LCM}\,R$.

The equality holds if R is CM, but this is not a necessary condition.

By this remark and Thm 1, we have the perfect hereditary cotorsion pair

$$(\mathrm{wbbCM}\,R, (\mathrm{wbbCM}\,R)^{\perp_1}) = (\mathrm{LCM}\,R, (\mathrm{LCM}\,R)^{\perp_1})$$

whenever R is a CM local ring. So Thm 1 recovers and generalizes:

Thm (Holm, 2017) R: a CM local ring with a canonical module

$(\mathrm{wbbCM}\,R, (\mathrm{wbbCM}\,R)^{\perp_1})$ is a perect hereditary cotorsion pair.

Def R : CM local, $R \xrightarrow{\varphi} A$: an R-order

$$\mathrm{bbCM}\,A := \{ M \in \mathrm{Mod}\,A \mid \varphi_*(M) \in \mathrm{bbCM}\,R \}$$

A balanced big CM cover (precover) of $M \in \mathrm{Mod}\,A$ is a $(\mathrm{bbCM}\,A)$-cover (precover).

Def R : CM, A . an R-order

A large CM cover (precover) of $M \in \mathrm{Mod}\,A$ is an $(\mathrm{LCM}\,A)$-cover (precover).

A large CM approximation of $M \in \mathrm{Mod}\,A$ is an exact sequence

$$0 \to \underset{(\mathrm{LCM}\,A)^{\perp_1}}{\underset{\cap\!\!\!\cap}{N}} \longrightarrow \underset{(\mathrm{LCM}\,A)}{\underset{\cap\!\!\!\cap}{L}} \longrightarrow M \longrightarrow 0 .$$

$\underline{\text{Cor 2}}$ $(R, \mathfrak{m})$: CM , A : R-order , $0 \neq N \in \operatorname{Mod} A$

Let $f: M \to N$ be an LCM cover of N, which exists by Thm 1.

(1) $N \neq \mathfrak{m} N \Rightarrow f$ is a bbCM cover.

(2) $N \cong \widehat{N} \Rightarrow$ ________________ and $M \cong \widehat{M}$.

(3) $N \cong H_0^{\mathfrak{m}}(N) \Rightarrow$ ________________ and $M \cong H_0^{\mathfrak{m}}(M)$.

$H_0^{\mathfrak{m}}(-) := H^0 L\Lambda^{\mathfrak{m}}$

0th local homology functor

$\underline{\text{Rem}}$ (2) generalizes a result of Simon (2009) where $A = R$.

(1) ____________________ Holm (2017) where $A = R$ and $\exists \omega_R$.

§5. A Govorov–Lazard type result for large CM modules

<u>Thm3</u> (N) R : a CM ring with a canonical module ω, $\dim R < \infty$

$R \xrightarrow{\varphi} A$: an R-order, $\mathrm{CM}\,A := \{ M \in \mathrm{mod}\,R \mid \varphi_*(M) \in \mathrm{CM}\,R \} = \{$maximal CM A-modules$\}$

A right A-module is large CM iff it is a direct limit of maximal CM A-modules.
Consequently, $\mathrm{LCM}\,A$ is the smallest definable subcategory contaning $\mathrm{CM}\,A$.

<u>Rem</u> If $A = R$ and R is local, Thm3 is essentially due to Holm (2017).

$$\varinjlim \mathrm{proj}\,A = \mathrm{Flat}\,A \subseteq \varinjlim \mathrm{CM}\,A = \mathrm{LCM}\,A \subseteq \varinjlim \mathrm{mod}\,A = \mathrm{Mod}\,A$$

The equality holds iff R is artinian.

Def R : CM, A : an R-order

A is **non-singular** if $\mathrm{gl.dim}\, A_{\mathfrak{m}} = \dim R_{\mathfrak{m}}$ for all maximal ideals $\mathfrak{m}$ of R.

A is **Gorenstein** if C_A is flat (and cotorsion).

Rem A is an Gorenstein R-algebra in the sense of Goto–Nishida (2002) iff A is a Gorenstein R-order in our sense.
When R has a canonical module, A is a Gorenstein R-order in the sense of Iyama–Wemyss (2014) iff A is a Gorenstein R-order in our sense.

Caution "Gorenstein" in our sense can be stronger than "Iwanaga–Gorenstein" if A is non-commutative.

Thm 4 (N) R : a CM ring with a canonical module ω, $\dim R < \infty$

A : an R-order

$$A : \text{non-singular} \iff (\mathrm{LCM}\,A, (\mathrm{LCM}\,A)^{\perp_1}) = (\mathrm{Flat}\,A, \mathrm{Cot}\,A)$$

$$\Downarrow \text{known}$$

$$A : \text{Gorenstein} \iff (\mathrm{LCM}\,A, (\mathrm{LCM}\,A)^{\perp_1}) = (\mathrm{GFlat}\,A, (\mathrm{GFlat}\,A)^{\perp_1})$$

Rem If $A = R$ and R is local, Thm 4 is essentially due to Holm (2017).

Sketch proof of Thm 3 and Thm 4

Let $\omega_A := \mathrm{Hom}_R(A, \omega_R)$. Reduce the proof to the trivial extension $A \ltimes \omega_A$, which is a Gorenstein R-order.

If A is Gorenstein, $\mathrm{Prod}(C_A) = \mathrm{Prod}\left(\prod_{\mathfrak{p} \in \mathrm{Spec} R} \widehat{A_{\mathfrak{p}}}\right) \overset{\uparrow}{=} \mathrm{Prod}\left(\prod_{P \in \mathrm{Spec} A} \mathrm{Hom}_R(I_{A^{op}}(P), E_R(R/P \cap R))\right)$,

by Kanda-N (2021)

$\left(\mathrm{Spec} A \overset{\exists}{\xrightarrow{\sim}} \{\text{indec. inj } A^{op}\text{-modules}\}/\cong, \quad P \longmapsto I_{A^{op}}(P) \right)$

so that ${}^{\perp>0}\{C_A\} = \mathrm{GFlat}\, A$.

The rest argument is similar to Holm's apprproch.

$\square$

$\underline{\text{Ex}}$ $(R, \mathfrak{m})$: regular, $\dim R = 1$

(1) $A = \begin{bmatrix} R & R \\ R & R \end{bmatrix}$ is non-singular

(2) $A = \begin{bmatrix} R & \mathfrak{m} \\ R & R \end{bmatrix}$ is Gorenstein

(3) $A = \begin{bmatrix} R & 0 \\ R & R \end{bmatrix}$ is not Gorenstein

$Q := R_{(0)}$

$\mathrm{Flat}\, A \not\ni [Q \;\; R] \in \mathrm{bbCM}\, A \subset \mathrm{LCM}\, A$

(pure-inj. if $R = \widehat{R}$)

§6. Large canonical modules as cotilting objects

<u>Thm</u> (Hrbek-N-Šťovíček, 2022) R : CM

C_R is a cotilting object in $D(R)$ in the sense of Psaroudakis-Vitória (2018).

If $\dim R < \infty$, then C_R is a cotilting module in the sense of Angeleri Hügel-Coelho (2001).

<u>Rem</u> In the above theorem, C_R and $D(R)$ can be replaced by C_A and $D(A)$ for an R-order

We have $\mathrm{inj.dim}_R C_R = \dim R$ and $\mathrm{inj.dim}_A C_A = \dim R/\mathrm{ann}_R A$.