Bridgeland stability conditions with massless objects

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January, 2022

Plan

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Background

Set up

- $\mathcal{T}$ triangulated category
- Λ finite rank free quotient of $K(\mathcal{T})$
- $\mathrm{Aut}_{\Lambda}(\mathcal{T})$ subgroup of autoequivalences descending to Λ

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Stability spaces are conjecturally contractible (when non-empty) and to simplify the exposition I assume they are connected!

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Approach

Allow stability conditions with massless objects!

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4. for each t there are $s_j \in \mathcal{P}(\varphi_j)$ with $\varphi_0 > \cdots > \varphi_n$ and a filtration

$$\begin{array}{ccccccccc} 0 & \longrightarrow & t_0 & \longrightarrow & \cdots & \longrightarrow & t_{n-1} & \longrightarrow & t_n & = & t \\ & \nwarrow & \downarrow & & & & & \nwarrow & \downarrow & & \\ & & s_0 & & & & & & s_n & & \end{array}$$

Stability conditions and spaces

Support property

A stability condition $(\mathcal{P}, Z)$ is a pre-stability condition such that

$$\inf\left\{\frac{m(t)}{||t||} : t \text{ semistable}\right\} > 0$$

where $||\cdot||$ is (any) norm on $\Lambda \otimes \mathbb{R}$. This implies $\mathcal{P}(I)$ is a quasi-abelian length category whenever $|I| < 1$, in particular $\mathcal{P}$ is a locally-finite slicing.

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Stability spaces

The Bridgeland stability space

$$\mathrm{Stab}_{\Lambda}(\mathcal{D}) \subset \mathrm{Slice}(\mathcal{T}) \times \mathrm{Hom}(\Lambda, \mathbb{C})$$

is the subset of stability conditions with the subspace topology.

Degenerate stability conditions

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Remarks

1. The slices $\mathcal{P}(\varphi)$ are quasi-abelian but not necessarily abelian.
2. The slicing $\mathcal{P}$ is not necessarily locally-finite.
3. Any $(\mathcal{P}, Z)$ in $\overline{\mathrm{Stab}_\Lambda(\mathcal{T})}$ is a degenerate pre-stability condition.

Massless subcategories and quotient stability conditions

Proposition

Let $(\mathcal{P}, Z)$ *be a degenerate stability condition. Then*

1. *The massless subcategory* $\mathcal{M} = \{t \in \mathcal{T} \colon m(t) = 0\}$ *is thick*

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3. $\mathcal{P}$ *restricts to a slicing* $\mathcal{P}_{\mathcal{M}}$ *of* $\mathcal{M}$
4. $\mathcal{P}$ *descends to a slicing* $\mathcal{P}_{\mathcal{T}/\mathcal{M}}$ *of* $\mathcal{T}/\mathcal{M}$.

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Corollary

Let $\Lambda_{\mathcal{M}}$ be the saturation of the image of $K(\mathcal{M}) \to K(\mathcal{T}) \to \Lambda$. Then

$$(P_{\mathcal{T}/\mathcal{M}}, Z) \in \mathrm{Stab}_{\Lambda/\Lambda_{\mathcal{M}}}(\mathcal{T}/\mathcal{M})$$

Roughly, a degenerate stability condition consists of a massless part, a slicing on $\mathcal{M}$, and a massive part, a stability condition on $\mathcal{T}/\mathcal{M}$.

Glueing slicings

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Suppose $\mathcal{M} \subset \mathcal{T}$ *is thick and* $(\mathcal{Q}, \mathcal{R}) \in \mathrm{Slice}\,(\mathcal{M}) \times \mathrm{Slice}\,(\mathcal{T}/\mathcal{M})$.

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- Local-finiteness is required in order to construct the Harder–Narasimham filtrations for the glued slicing $\mathcal{P}$.
- This is the key to lifting deformations of the charge Z to deformations of degenerate stability conditions.

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The space of degenerate stability conditions

Theorem

1. *There is a real manifold with boundary*

$$\mathrm{DStab}_\Lambda(\mathcal{T}) \subset \overline{\mathrm{Stab}_\Lambda(\mathcal{T})}$$

with a decomposition

$$\mathrm{DStab}_\Lambda(\mathcal{T}) \cong \mathrm{Stab}_\Lambda(\mathcal{T}) \cup \bigcup_{\mathcal{M} \in M} \mathbb{R} \times \mathrm{Stab}_{\Lambda/\Lambda_\mathcal{M}}(\mathcal{T}/\mathcal{M})$$

where M is the set of massless subcategories $\mathcal{M}$ with $\mathrm{rk}\,\Lambda_\mathcal{M} = 1$.

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where M is the set of massless subcategories $\mathcal{M}$ with $\mathrm{rk}\,\Lambda_\mathcal{M} = 1$.

2. *The boundary component where objects in $\mathcal{M}$ are massless has a deleted neighbourhood isomorphic to*

$$\mathrm{Stab}_{\Lambda_\mathcal{M}}(\mathcal{M}) \times \mathrm{Stab}_{\Lambda/\Lambda_\mathcal{M}}(\mathcal{T}/\mathcal{M}) \cong \mathbb{C} \times \mathrm{Stab}_{\Lambda/\Lambda_\mathcal{M}}(\mathcal{T}/\mathcal{M}).$$

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The charge map extends continuously to $\mathcal{Z} \colon \mathrm{DStab}_{\Lambda}(\mathcal{T}) \to \mathrm{Hom}(\Lambda, \mathbb{C})$, but no longer has discrete fibres over the boundary components.

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Definition

Forgetting the phases of massless objects we obtain the space of quotient stability conditions

$$\mathrm{QStab}_{\Lambda}(\mathcal{T}) \cong \mathrm{Stab}_{\Lambda}(\mathcal{T}) \cup \bigcup_{\mathcal{M} \in M} \mathrm{Stab}_{\Lambda/\Lambda_{\mathcal{M}}}(\mathcal{T}/\mathcal{M})$$

whose charge map is a local homeomorphism on each stratum. We recover $\mathrm{DStab}_{\Lambda}(\mathcal{T})$ by performing a real blowup along each boundary stratum.

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The actions of $\mathrm{Aut}_\Lambda(\mathcal{T})$ and G extend to $\mathrm{DStab}_\Lambda(\mathcal{T})$ and $\mathrm{QStab}_\Lambda(\mathcal{T})$.

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Alternative approaches

Metric completion (Bolognese '19)

Under certain conditions, Bolognese constructs a metric completion of $\mathrm{Stab}_{\Lambda}(\mathcal{T})$ whose boundary points correspond to stability conditions on quotients of $\mathcal{T}$ by thick subcategories. This should be closely related to $\mathrm{QStab}_{\Lambda}(\mathcal{T})$, but it is difficult to compare our notion of support with her notion of 'limiting support'.

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Thurston compactification (Bapat, Deopurkar, Licata '20)

Bapat, Deopurkar and Licata constuct a 'Thurston compactification' of $\mathrm{Stab}_\Lambda(\mathcal{T})/\mathbb{C}$ for $\mathcal{T} = \mathcal{D}(\Gamma_2 Q)$ by embedding it into projective space using the mass functionals. They conjecture that the closure of the image is a compact manifold with boundary and interior $\mathrm{Stab}_\Lambda(\mathcal{T})/\mathbb{C}$. This holds in the A_2 case and our partial compactification embeds in it.

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Two-dimensional stability spaces

Let $\operatorname{rk}\Lambda = 2$. Then $\operatorname{Stab}_\Lambda(\mathcal{T})/\mathbb{C}$ is a non-compact Riemann surface, and

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- *s massless stable $\iff$ s simple in heart $\mathcal{H}$ with $\mathcal{H}_s^\flat$, $\mathcal{H}$, $\mathcal{H}_s^\sharp$ algebraic.*

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2. $\mathrm{Stab}(Q)$ where Q is a 2-vertex quiver with oriented loops [Dimitrov, Haiden, Katzarkov and Kontsevich '14]

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Suppose there is $\sigma \in \mathrm{Stab}_\Lambda(\mathcal{T})$ with non-dense phases. Equivalently $\mathcal{T}$ has an algebraic heart. Assume $\Lambda = K(\mathcal{T}) \cong \mathbb{Z}^2$.

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- *masses of the two simples vanish at ends of the wall*

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Proposition (Wall-and-chamber structure)

In $\mathrm{Stab}_\Lambda(\mathcal{T})/\mathbb{C}$

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- *walls correspond to 'non-trivial' algebraic hearts (up to shift)*
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- *no two walls intersect*
- *each chamber is isomorphic to* $\mathbb{D}$.

Speiser graphs

- The ‘dual graph’ $\Gamma_{\mathcal{T}}$ of G-orbit structure is the Speiser graph of

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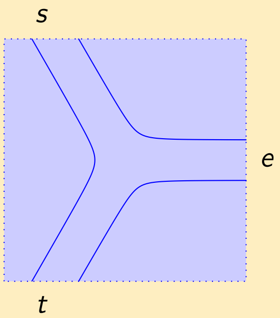
- It is recurrent if vertices of $\Gamma_{\mathcal{T}}$ embed in $\mathbb{R}^2$ with bounded below pairwise distances and bounded above edge lengths [Doyle, Snell '84].

The simplest interesting example...

$\mathcal{T} = \mathcal{D}(A_2)$ [King '00s; Qiu '11]



Random walk recurrent so $\mathrm{Stab}(A_2)/\mathbb{C} \cong \mathbb{C}$; Serre functor rotates.

… and its 2-Calabi–Yau cousin

$\mathcal{T} = \mathcal{D}(\Gamma_2 A_2)$ [Thomas '06; Bridgeland '09; Qiu '11; Bridgeland, Qiu, Sutherland '20]

Spherical twists about simples s and t of the standard heart generate subgroup Br_3 of automorphisms. There is a free Br_3 orbit of chambers.

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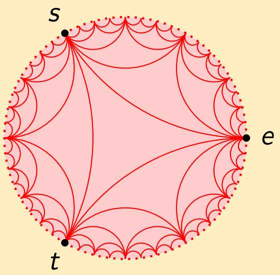

Walk transient so $\mathrm{Stab}(\Gamma_2 A_2)/\mathbb{C} \cong \mathbb{D}$; twists act by ideal rotations.

A discrete derived category

$\mathcal{T} = \mathcal{D}(Q_{1,2,0})$ [W '18; Broomhead, Pauksztello, Ploog '16]

The bounded derived category of the quiver with relations

$$Q_{1,2,0}: \quad \bullet \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \bullet \qquad \alpha\beta = 0$$

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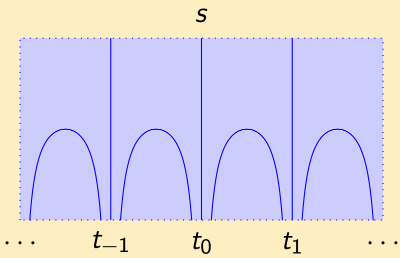

Walk recurrent so $\mathrm{Stab}(Q_{1,2,0})/\mathbb{C} \cong \mathbb{C}$; twist acts by translation.

Coherent sheaves on $\mathbb{CP}^1$

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