From gentle to string algebras: a geometric model

jt. work with Karin Baur

§1. String and gentle algebras – background

Definition • $A = kQ/I$ string algebra if:

(S1) $\forall\, i \in Q_0$, $\exists$ at most two arrows starting at i &
$\exists$ at most two arrows ending at i.

(S2) $\forall\, a \in Q_1$, $\exists$ at most one arrow b st. $ba \notin I$
& $\exists$ at most one arrow c s.t. $ac \notin I$.

(S3) I generated by paths of length ≥ 2.

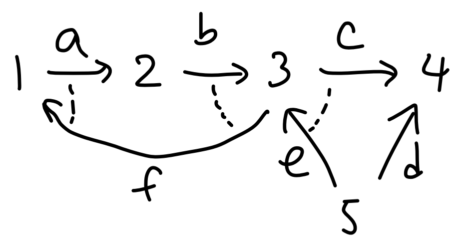
• A gentle algebra if additionally:

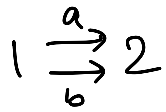
(S2′) $\forall\, a \in Q_1$, $\exists$ at most one arrow b' st. $b'a \in I$
& $\exists$ at most one arrow c' s.t. $ac' \in I$.

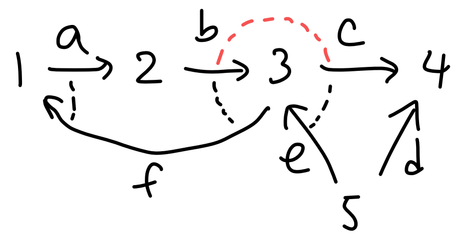
&

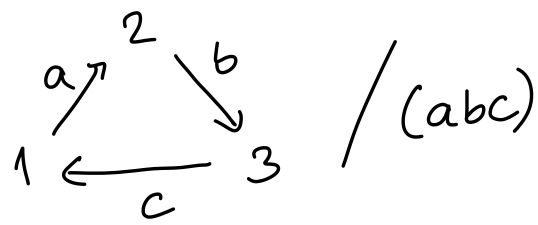
(S3′) I generated by paths of length 2.

Examples









Definitions

* Given $a \in Q_1$, define a formal inverse a^{-1} s.t. $s(a^{-1}) = t(a)$ & $t(a^{-1}) = s(a)$.

* <u>walk</u> = sequence $w = w_1 \cdots w_r$ st. $t(w_i) = s(w_{i+1})$, with $w_i \in Q_1^{\pm 1}$

* <u>string</u> = walk with no subwalks of the form aa^{-1} or $a^{-1}a$ $(a \in Q_1)$
  or subwalks v with $v \in I$ or $v^{-1} \in I$.

* <u>Trivial strings</u> : e_i, $i \in Q_0$.

* <u>Bands</u> : $b = b_1 \cdots b_n$ = cyclic string $(t(b_n) = s(b_1))$ s.t. any power b^m of b is a string but b itself is not a proper power of any string.

Examples

$$\begin{tikzcd} 1 \arrow[r, "a"] & 2 \arrow[r, "b"] & 3 \arrow[r, "c"] \arrow[ll, bend left, "f"] & 4 \\ & & & 5 \arrow[ul, "e"] \arrow[u, "d"'] \end{tikzcd}$$

eb^{-1} string $\qquad \nexists$ bands

ec not string

$eb^{-1}b$ not string

$$\begin{tikzcd} 1 \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] & 2 \end{tikzcd}$$

ab^{-1} band & string

$ab^{-1}ab^{-1}$ string but not a band.

Representation theory of string algebras:

- indec. modules — string modules
  — band modules

Gelfand - Ponomarev
Wald - Waschbüsch

- morphisms Crawley - Boevey; Krause
- AR-sequences Butler - Ringel.

§ Representation theory via surfaces – WHY?

* Description of extensions between modules [Çanakçı – Schroll, Çanakçı – Pauksztello – Schroll]

* τ-tilting theory : [Adachi – Iyama – Reiten]
  cf. Palu – Pilaud – Plamondon
  Brüstle – Douville – Mousavand – Thomas – Yildirim
  He – Zhou – Zhu : classification of support τ-tilting for skew-gentle algs.
  – properties of τ-tilting graph in gentle case. [Fu – Geng – Liu – Zhou]

* Bridgeland/King stability conditions
  [Garcia – Garver] – classification of semistable rep^{ns} when S = disc.

* Link to Fukaya catgs in symplectic geometry [Lekili – Polishchuk]

* $\exists$ geometric model of $D^b(A)$, A gentle [Opper – Plamondon – Schroll ⇝ used to study derived equivalences [Amiot – Plamondon – Schroll; Opper].
  [Broomhead] DDCs ; Brauer graph algs – [Opper – Zvonareva]).

§2. Representation theory of gentle algs via surfaces

Theorem 1 (Baur - CS)

① Let A be a finite dim'l alg. TFAE

see also [OPS]

* A is gentle
* A is a tiling algebra associated to (S, M, P)

finite set of marked pts on ∂S

unpunctured surface with ∂S

partial triangulation

* A is the endomorphism algebra of a partial cluster-tilting object of a generalised cluster catg associated to some unpunctured surface

Particular cases: • P triangulation → Jacobian algs. ([Assem-Brüstle-Charbonneau-Plamondon]) [Brüstle-Zhang]

• P cut of a triangulation → surface algs [David-Roesler-Schiffler]

Theorem 1 (Baur-CS)

② A gentle algebra, (S,M,P) corresponding tiling

* indecomposable modules
  - string modules $\overset{1-1}{\longleftrightarrow}$ equivalence classes of permissible arcs in S
  - band modules $\underset{1-1}{\longleftrightarrow}$ homotopy classes of certain permissible closed curves (intersection number with $P \geq 2$).

* $M = M(\gamma)$ string module, γ corresponding arc

irreducible morphisms:

$$\begin{tikzcd} & M(\gamma_s) \\ M(\gamma) \arrow[ur] \arrow[dr] & \\ & M(\gamma_e) \end{tikzcd}$$

γ_s, γ_e obtained from γ by pivot elementary moves on their endpoints

* AR-sequences are of the form:

$$0 \to M(\gamma) \to M(\gamma_s) \oplus M(\gamma_e) \to M(\gamma_{s,e}) \to 0.$$

$$\tau^{-1}\gamma = \gamma_{s,e} = (\gamma_s)_e = (\gamma_e)_s.$$

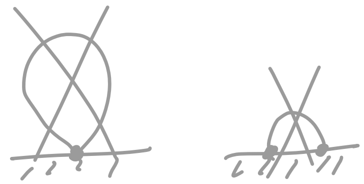
S unpunctured oriented, connected surface with boundary ∂S

M finite set of marked pts on ∂S.

(S, M) marked surface.

<u>Arc</u> in (S, M): curve $\gamma : [0,1] \longrightarrow S$ s.t.

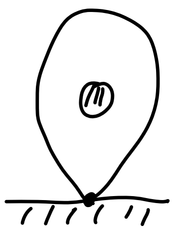
* $\gamma(0), \gamma(1) \in M$
* $\gamma \cap M$ only at its endpts
* γ does not cut a monogon or digon



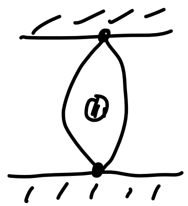
P <u>partial triangulation</u> = collection of arcs that do not intersect themselves or each other in the interior of S.

(S, M, P) tiling s.t. P divides S into the following regions / tiles:

* m-gons $(m \geqslant 3)$: edges are arcs in P or boundary segments & $\nexists$ unmarked body components in its interior.
* 1-gon

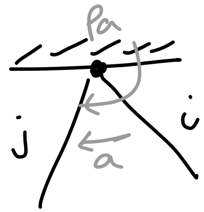


* 2-gon

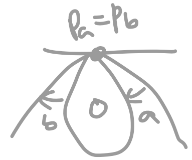


Definition A_P tiling algebra associated to (S, M, P) $\quad A = kQ_P / I_P$

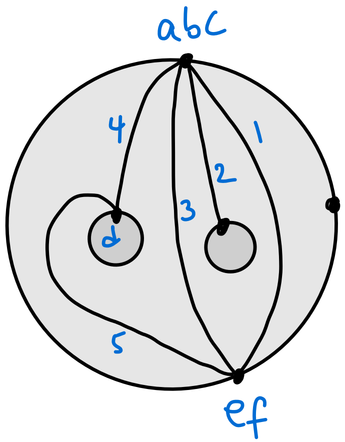
* $(Q_P)_0 \overset{1-1}{\longleftrightarrow}$ arcs in P
* $\exists$ arrow $i \xrightarrow{a} j$ if

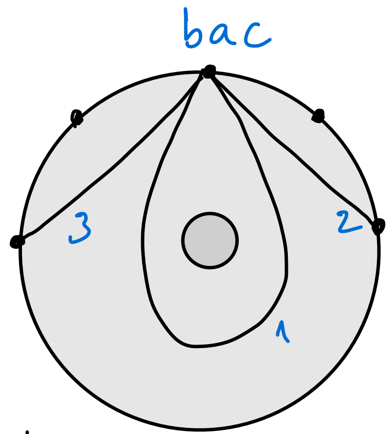


* I_P generated by :
  * paths ab st $p_a \neq p_b$
  * paths ab st $t(a) = s(b)$ is a loop arc



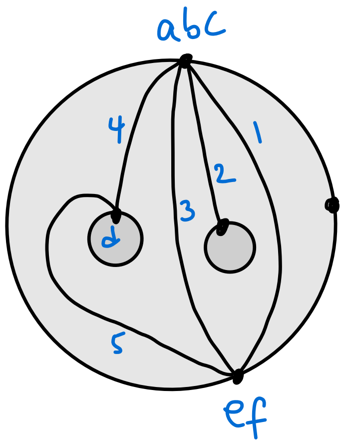
Egs



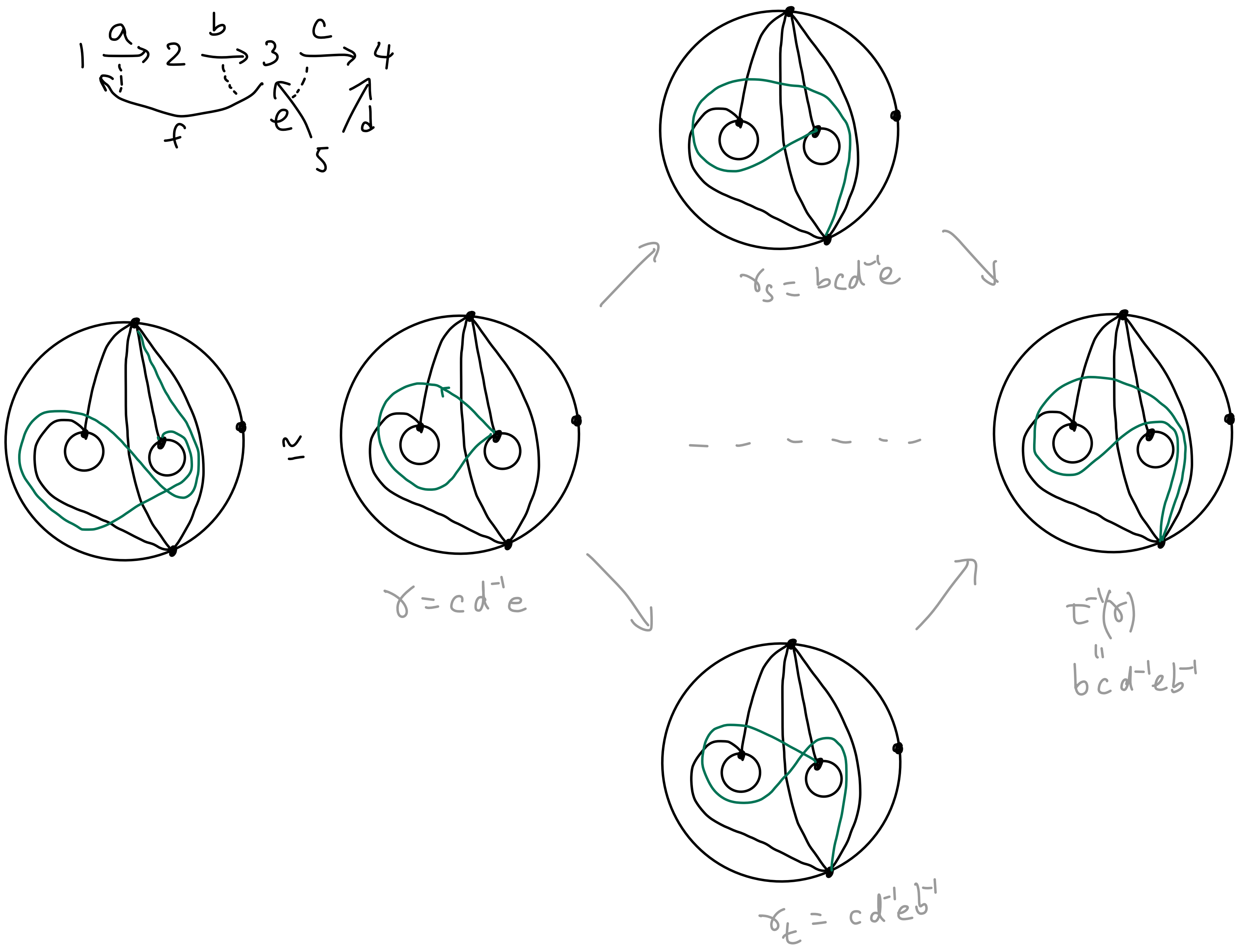


$/(a^2, bc)$

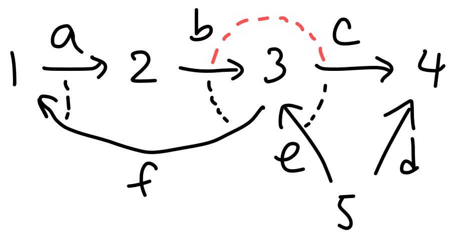
Example : permissible, equivalence of arcs, pivot elementary move, AR-translate

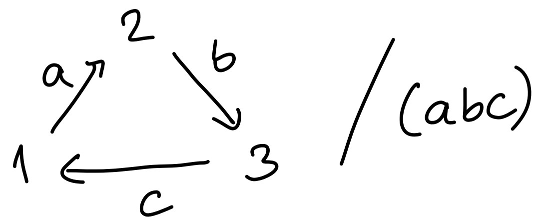


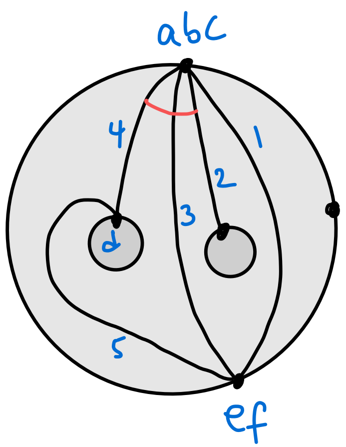
$$\begin{tikzcd} 1 \arrow[r, "a"] & 2 \arrow[r, "b"] & 3 \arrow[r, "c"] \arrow[ll, bend left, "f"] & 4 \\ & & & 5 \arrow[ul, "e"] \arrow[u, "d"] \end{tikzcd}$$



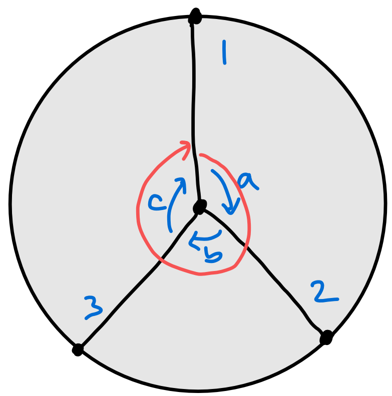
§3. Representation theory of string algebras via surfaces







Labelled
tiling algebra



Theorem 2 (Baur-CS)

① A string algebra $\iff$ A is a labelled tiling algebra associated to

(S, M, P, L) ← labelled tiling

partial triangulation

surface with punctures

marked pts on ∂S & interior

labels at marked pts s.t. each puncture has at least one label.

② A string algebra, (S, M, P, L) corresponding labelled tiling

- indecomposable modules
  - string modules $\overset{1-1}{\longleftrightarrow}$ equivalence classes of permissible arcs in S
  - band modules $\underset{1-1}{\longleftrightarrow}$ homotopy classes of certain permissible closed curves

new condition on permissible: does not cross labels

* $M = M(\gamma)$ string module, γ corresponding arc

irreducible morphisms:

$$
\begin{array}{ccc}
 & & M(\gamma_s) \\
 & \nearrow & \\
M(\gamma) & & \\
 & \searrow & \\
 & & M(\gamma_e)
\end{array}
$$

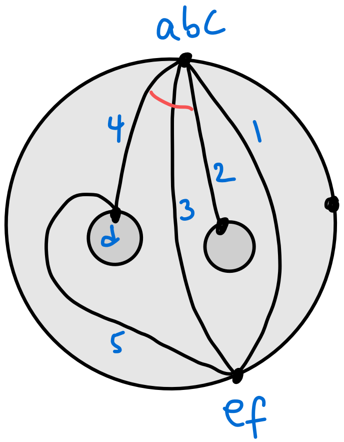
γ_s, γ_e obtained from γ by pivot elementary moves on their endpoints
$\downarrow$
depend on labels

* AR-sequences are of the form:

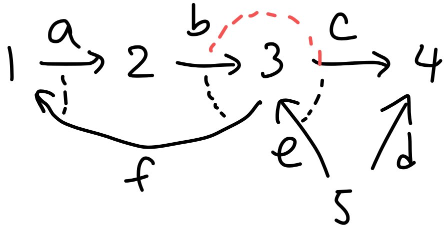
$$0 \to M(\gamma) \to M(\gamma_s) \oplus M(\gamma_e) \to M(\gamma_{s,e}) \to 0.$$

$$\underbrace{\gamma_{s,e} = (\gamma_s)_e = (\gamma_e)_s.}_{\text{true if } \gamma \text{ not injective}}$$

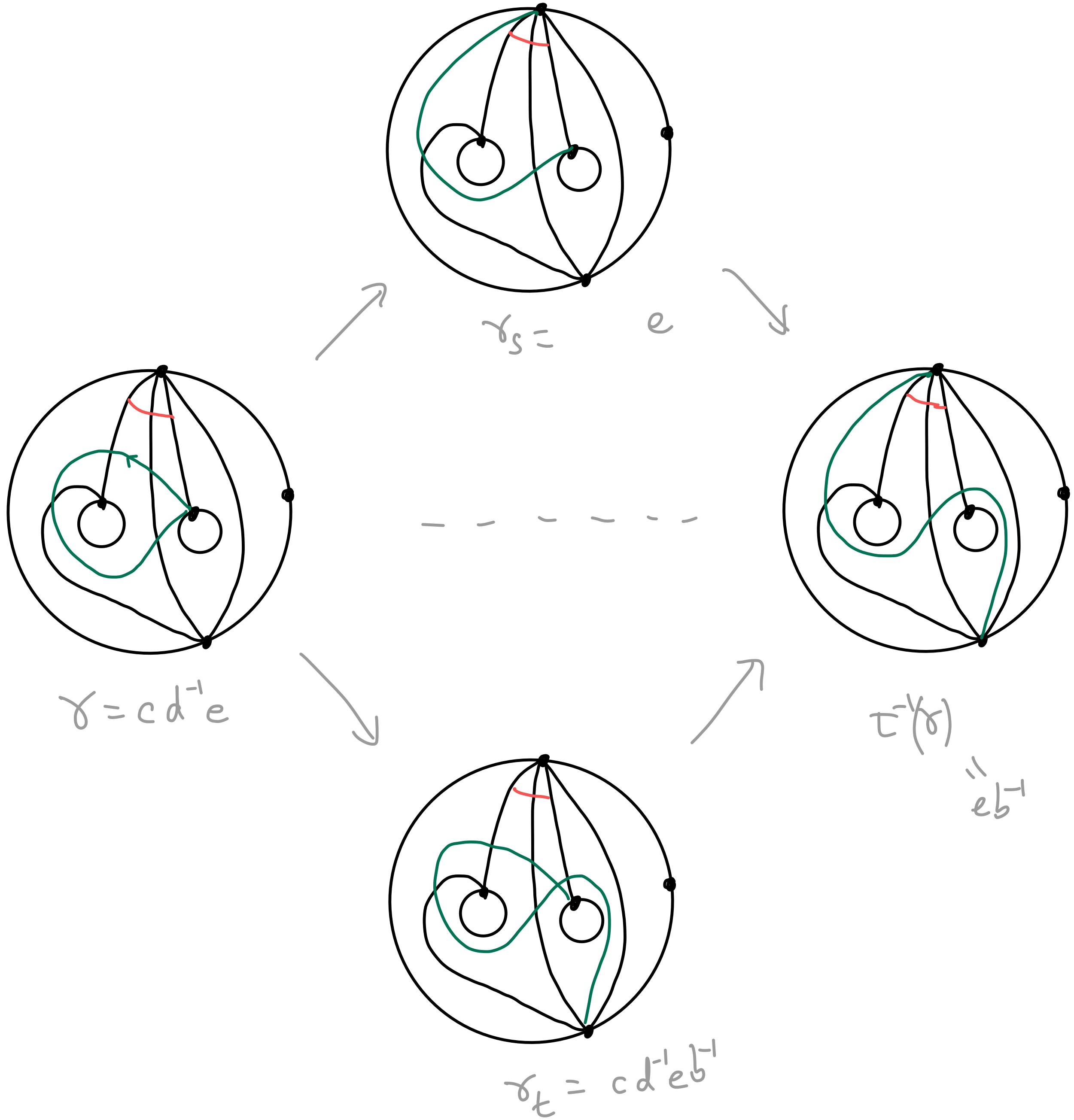
Example



abc
4
1
2
3
d
5
ef



1
a
2
b
3
c
4
f
e
5
d



$\gamma_s =$ e
$\gamma = c d^{-1} e$
$\tau^{-1}(\gamma)$
$= e b^{-1}$
$\gamma_t = c d^{-1} e b^{-1}$

§4. An application: τ-tilting theory

Def^{n}. A finite dim^{l} alg, $M \in \operatorname{mod} A$.

$|M|$ = # non-isomorphic indec. direct summands of M.

* M <u>τ-rigid</u> if $\operatorname{Hom}(M, \tau M) = 0$.
* M <u>τ-tilting</u> if M τ-rigid & $|M| = |A|$.
* M support τ-tilting if $\exists$ idempotent $e \in A$ s.t. M is τ-tilting $(A/\langle e \rangle)$-module.
* <u>support τ-tilting pair</u> : (P, M) s.t.
  * $\operatorname{Hom}(P, M) = 0$
  * M τ-rigid
  * $|M| + |P| = |A|$.

<u>Fact</u> [AIR]:
M support τ-tilting $\Leftrightarrow$ $\exists$ proj. P s.t. (P, M) is support τ-tilting pair.

[He-Zhou-Zhu] If A is gentle,

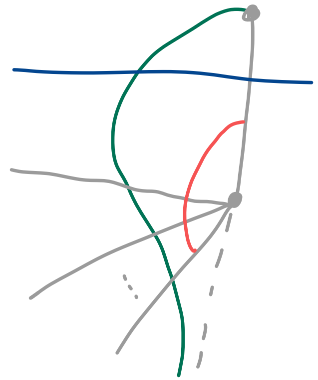
$$\left\{\begin{matrix}\text{support } \tau\text{-tilting} \\ \text{pairs}\end{matrix}\right\} \xleftrightarrow{1-1} \left\{\begin{matrix}\text{maximal collections of noncrossing} \\ \text{generalised permissible arcs}\end{matrix}\right\}$$

may include arcs in P

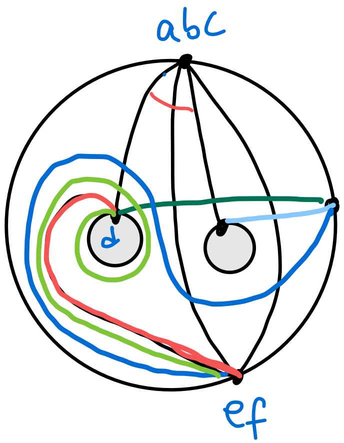
↳ right choice of representative (clockwise most)

Conjecture (Baur-CS) A string alg.

$$\left\{ \begin{array}{c} \text{support } \tau\text{-tilting} \\ \text{pairs} \end{array} \right\} \xleftrightarrow{1-1} \left\{ \begin{array}{c} \text{maximal collections of permissible} \\ \text{arcs for which each crossing} \\ \text{satisfies} \end{array} \right\}$$



Examples



$(P_5,\ 4 \oplus 1 \oplus f^{-1}c \oplus ab)$ support τ-tilting over string algebra (but not the gentle alg.)