Trees and chicken feet

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G finite group

__Def:__ A __transfer system__ $\mathcal{J}$ on G consists of pairs of subgroups (H, L), $H \leq L$ s.th.

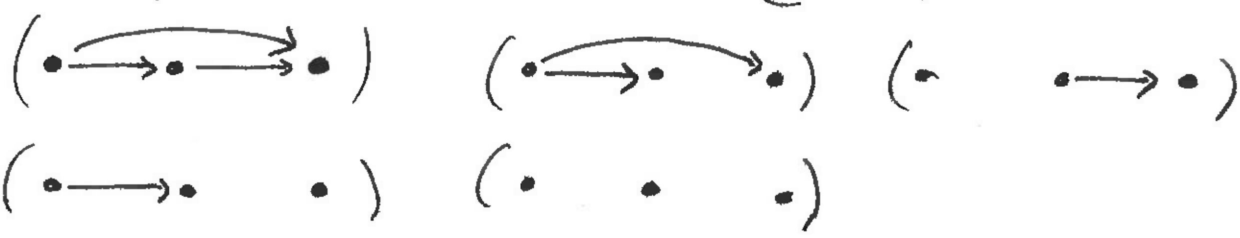
- composition: $(H,L), (L,M) \in \mathcal{J} \Rightarrow (H,M) \in \mathcal{J}$
- restriction: $(H,L) \in \mathcal{J}$, $M \leq G \Rightarrow (H \cap M, L \cap M) \in \mathcal{J}$.

(all up to conjugacy) $\rightsquigarrow$ display as graph $\underset{H}{\bullet} \longrightarrow \underset{L}{\bullet}$

__Examples__ $G = C_{p^n}$: $\underset{e}{\bullet} \quad \underset{C_p}{\bullet} \quad \underset{C_{p^2}}{\bullet} \quad \cdots\cdots \quad \underset{C_{p^n}}{\bullet}$ ($n+1$ dots)

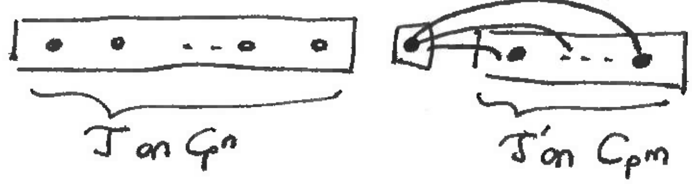
__n=1:__ two options $(\bullet \quad \bullet)$ and $(\bullet \longrightarrow \bullet)$

__n=2:__



__WHY?__ $\left(\begin{matrix}\text{transfer systems}\\ \text{on } G\end{matrix}\right) \longleftrightarrow \left(\begin{matrix}\text{types of equivariant}\\ \text{homotopy commutativity}\end{matrix}\right)$

How many transfer systems for C_{p^n}?



$= \mathcal{J} \odot \mathcal{J}'$ on $C_{p^{n+m+2}}$

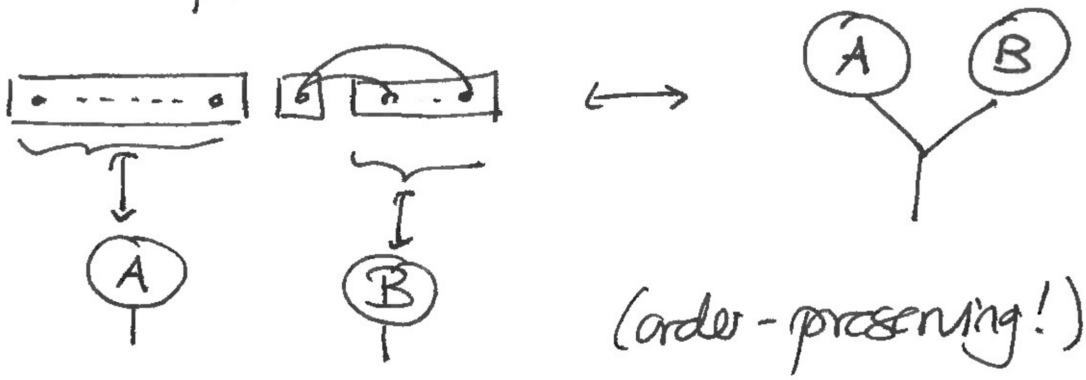
$\rightsquigarrow$ classification via position of pivot

$\Rightarrow$ __Thm__ $|\mathrm{Tr}(C_{p^n})| = \sum_{i=0}^{n} |\mathrm{Tr}(C_{p^{i-1}})| \cdot |\mathrm{Tr}(C_{p^{n-i-1}})|$

__Corollary__ $|\mathrm{Tr}(C_{p^n})| = \mathrm{Cat}(n+1)$

$=$ # trees, binary, with $n+2$ leaves

This is in fact more structured:



(order-preserving!)

Model category structures on $[n] = \{\underset{0}{\bullet} \to \underset{1}{\bullet} \to \cdots \to \underset{n}{\bullet}\}$

model category: weak equivalences W $\xrightarrow{\sim}$
fibrations F $\twoheadrightarrow$
cofibrations C $\hookrightarrow$
+ axioms

think: w.eq. = weak htpy equivalence / isos in htpy category

- retracts
- 2-out-of-3 for $\xrightarrow{\sim}$
- factorisation axioms
- lifting axioms

$AF = W \cap F$
$AC = W \cap C$

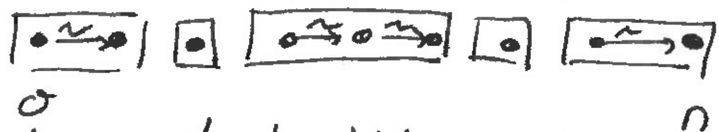
$AF \leftrightarrow C$, $AC \leftrightarrow F$

How many model structures on $[n]$?

other cats: $A \to C \times A \to A$

- no retracts
- weak equiv's are "decomposable": $g \circ f$ w.eq. $\Rightarrow$ f and g are w.eq.

$\Rightarrow$ weak equivalences are determined by partitions



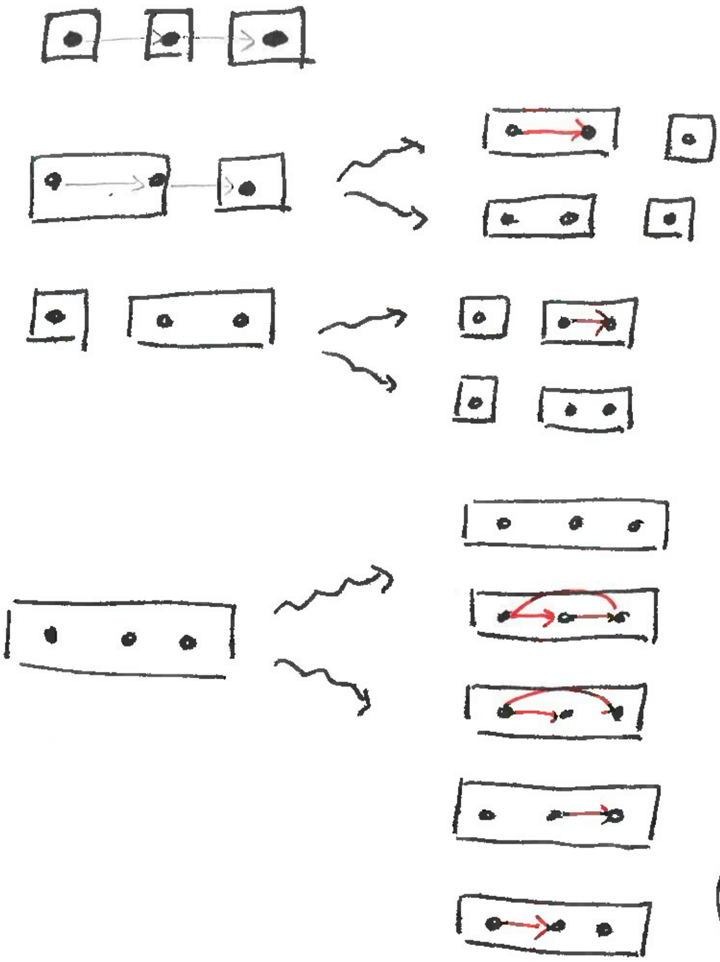
- in general, a model str. is determined by W and AF
  ($AF \leftrightarrow C \rightsquigarrow AC \leftrightarrow F$)

$\Rightarrow$ model str. is determined by the AF on each "block"

- Restricted to each block, the AF form a transfer system
(conversely, every such choice determines a model str.)

Example $n=2$:

AF in red



total: 10 model str. on $[2]$

[BOOR]

Thm # model structures on $[n]$

$$= \sum_{p \text{ partition}} \prod_{i=1}^{k} Cat(a_{i+1}-a_i) = \binom{2n+1}{n}$$

$$p = [0,a_1] \amalg [a_1+1, a_2] \amalg \ldots [a_{k+1}+1, n]$$

$$\binom{2n+1}{n} = \# \text{ monotonic functions } [n] \longrightarrow [n] = End([n])$$

Questions bijection? (fixed points $\longleftrightarrow$ bifibrant objects)
identify things in $End([n])$ with homotopic phenomena

chicken foot: C_{pq}