Dynamics in triangulated categories

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Dynamical system: (X, f)

(compact, Hausdorff) top. space $f: X \longrightarrow X$ continuous

QUESTION: What is the COMPLEXITY of (X, f)?

We measure it using

$$\boxed{h_{top}(f) = \text{TOPOLOGICAL ENTROPY of } f}$$

Categorification: (Dimitrov-Haiden-Katzarkov-Kontsevich)

$(\mathcal{D}, \Phi)$ $\mathcal{D}$ triangulated category$/\mathbb{K}$ w/ split-generator

$\Phi: \mathcal{D} \longrightarrow \mathcal{D}$ an endofunctor

Def: G split generator:

(*) $\forall F \in \mathcal{D}\ \exists F'\ \exists\ 0 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow \cdots \longrightarrow E_p = F \oplus F'$

$$\begin{tikzcd}[column sep=small]
0 \arrow[rr] & & E_1 \arrow[dl] \arrow[rr] & & E_2 \arrow[dl] \arrow[r] & \cdots \arrow[r] & E_p \arrow[dl] \\
& G[n_1] \arrow[ul, dotted] & & G[n_2] \arrow[ul, dotted] & & G[n_p] \arrow[ul, dotted] &
\end{tikzcd}$$

We use (*) to measure the complexity of F w.r.t. G:

$$\underset{t \in \mathbb{R}}{\delta_t(G, F)} := \begin{cases} 0 & F \cong 0 \\ \inf \left\{ \sum_{i=1}^{p} e^{n_i t} \;\middle|\; \text{exists a seq. as (*)} \right\} & \end{cases}$$

Def (DHKK): $h_t(\Phi): \mathbb{R} \longrightarrow [-\infty, +\infty)$

$$\boxed{h_t(\Phi) = \lim_{n \to +\infty} \frac{1}{n} \log\left(\delta_t(G, \Phi^n G)\right)}$$

Rmk: $h_t(\Phi)$ measures the exponential growth of $\delta_t(G, \Phi^n G)$. The polynomial growth is measured by the polynomial categorical entropy (Fan–Filip–Haiden).

Ex:
- $h_0(id) = 0$ because $\delta_0(G, G) = 1$
- In general, $h_t(id) \neq 0$ if $t \neq 0$.

$R = k[e_\pm]/(e_+ e_- - 1)$ $\quad \deg e_\pm = \pm 2$, $d(e_\pm) = 0$

$R \xrightarrow[e_\pm \cdot]{} R[\pm 2]$ is an isom. of right R dg-mod

$\Rightarrow \delta_t(R, R) = 0 \quad \forall\, t \neq 0$

$$\Rightarrow h_t(id_{D(R)^c}) = \begin{cases} 0 & t = 0 \\ -\infty & t \neq 0 \end{cases}$$

Thm: (DHKK, FAN–FILIP)

R perfect as a bimod.

$\dim_k H^\cdot(R) < \infty$

If $\mathcal{D} \simeq D(R)^c$ with R smooth & proper dga, then:

$$h_t(\Phi) = \lim_{n \to +\infty} \frac{1}{n} \log(\varepsilon_t(G, \Phi^n G))$$

$$\varepsilon_t(G, F) := \sum_{m \in \mathbb{Z}} \dim \operatorname{Hom}_{\mathcal{D}}(G, F[m])\, e^{-mt}$$

$h_t(\Phi)$ is finite, convex and continuous.

Ex:
- $h_t(id) \equiv 0$
- $h_t(\Phi^n) = n\, h_t(\Phi)$
- $h_t([1]) = t$

$\Rightarrow$ $\mathcal{D}$ fract. CY dim m/n

$h_t(S_{\mathcal{D}}) = \frac{m}{n} t$

Ex/Thm: (Kikuta-Takahashi)

$f \in Aut(X)$, $h_0(f^*) = h_{top}(f)$

Def: The CATEGORICAL ENTROPY of Φ is: $h_{cat}(\Phi) := h_0(\Phi)$.

Thm: (Gromov-Yomdin)

X smooth, projective variety

$f: X \to X$ continuous, invertible

If f is holomorphic, then

$$h_{top}(f) = \log\left(\rho\left(f^*|_{\bigoplus_p H^{p,p}(X)}\right)\right)$$

Spectral radius:

$\varphi: V \to V$, $\rho(\varphi) := \max\{|\lambda| : \lambda \text{ eigenvalue of } \varphi\}$

Thm: (Kikuta-Shiraishi-Takahashi)

$K_{num}(\mathcal{D}) := K(\mathcal{D})$ / radical of Euler pairing

Assume $K_{num}(\mathcal{D})$ of finite rank. Then:

$$h_{cat}(\Phi) \geq \log(\rho([\Phi]))$$

$[\Phi]: K_{num}(\mathcal{D}) \otimes \mathbb{R} \longrightarrow K_{num}(\mathcal{D}) \otimes \mathbb{R}$

CONJECTURE: $h_{cat}(\Phi) = \log(\rho([\Phi])) \quad \forall \Phi$

When is this true? (Anti) Fano [KST], curves [Kikuta]
abelian surfaces [Yoshioka]

The conjecture is false:

- Fan: Hypersurfaces in $\mathbb{P}^d$ of degree $d+1$, d odd
- Ouchi: K3 surfaces
- Matteo: surfaces w/ -2- curves
- B-Kim: we use compositions of spherical twists to give a systematic way to construct counterexamples (e.g. hypersurfaces in $\mathbb{P}^n \times \mathbb{P}^m$; Milnor fibre A_2-sing. in dim $2d > 2$).

QUESTION: What is the categorical way to say holomorphic?

Recall: Stability conditions

full, additive

Def: A stability condition σ on $\mathcal{D}$ is $\sigma = (Z, \{\mathcal{P}(\varphi)\}_{\varphi \in \mathbb{R}})$

where:
- $Z : K(\mathcal{D}) \longrightarrow \mathbb{C}$
- $\mathcal{P}(\varphi + 1) = \mathcal{P}(\varphi)[1]$
- $\mathrm{Hom}(E_1, E_2) = 0$ if $E_i \in \mathcal{P}(\varphi_i)$, $\varphi_1 > \varphi_2$
- (HN) $\forall\, 0 \neq E \in \mathcal{D}$ $\exists\, \varphi_1 > \varphi_2 > \dots > \varphi_m$ $\exists$

$$\begin{array}{ccccccccc} 0 = E_0 & \xrightarrow{\alpha_0} & E_1 & \xrightarrow{\alpha_1} & E_2 & \longrightarrow \cdots \longrightarrow & E_m = E \\ & \nwarrow \swarrow & & \nwarrow \swarrow & & & \nwarrow \swarrow \\ & A_1 & & A_2 & & & A_m \end{array}$$

w/ $A_i \in \mathcal{P}(\varphi_i)$ Harder-Narasimhan factors of E

s.t.
- $Z(E) = \underset{\mathbb{R}_{>0}}{\underset{\cap}{m(E)}}\, e^{i\pi\varphi}$ $\forall\, 0 \neq E \in \mathcal{P}(\varphi)$

Rmk: We have $\mathrm{Aut}(\mathcal{D}) \circlearrowright \mathrm{Stab}(\mathcal{D}) \circlearrowleft \widetilde{GL}_2^+(\mathbb{R})$:

$$\Phi \cdot (Z, \mathcal{P}) = (Z \circ [\Phi]^{-1}, \{\Phi(\mathcal{P}(\varphi))\}_{\varphi \in \mathbb{R}})$$

$$(Z, \mathcal{P}) \cdot g = (M_g^{-1} Z, \{\mathcal{P}(f(\varphi))\}_{\varphi \in \mathbb{R}})$$

(M_g, f) s.t. $M_g \in GL_2^+(\mathbb{R})$, $f : \mathbb{R} \to \mathbb{R}$

strictly increasing

$f(x+1) = f(x) + 1$

&

$M_g e^{i\pi\alpha} \in \mathbb{R}_{>0}\, e^{i\pi f(\alpha)}$

Philosophy of mirror symmetry & Bridgeland-Joyce:

(X, ω) a symplectic, CY manifold, $f : X \to X$

f is holomorphic $\Leftrightarrow$ preserves a holomorphic structure

Take Ω the induced volume form

$$\Rightarrow f^*\Omega = \exp(-i\pi\lambda)\,\Omega \qquad \lambda \in \mathbb{R}$$

$\Rightarrow L$ Lagrangian $\subseteq X$

$$\int_L f^*\Omega = \int_L \Omega \cdot \exp(-i\pi\lambda)$$

$$f_* \cdot \sigma_L = \sigma_\Omega \cdot \lambda$$

$\rightsquigarrow$ Action of $\mathbb{C}$

Action of $\mathrm{Aut}(X)$

Stability condition

Def: A triple $(\Phi, \sigma, g) \in \mathrm{Aut}\,\mathcal{D} \times \mathrm{Stab}(\mathcal{D}) \times \widetilde{GL}_2^+(\mathbb{R})$

is COMPATIBLE if $\Phi \cdot \sigma = \sigma \cdot g$

Rmk: • This property categorifies one of the properties of being holomorphic, but it is more general: Dehn twists on elliptic curves satisfy it.

• Similar equations have been studied:

(DHKK) 1. Pseudo-Anosov : $g = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}^{\pm 1}$ $|\lambda| > 1$

(TODA) 2. Gepner points : $g \in \mathbb{C}$

Def: $0 \neq E \in \mathcal{D}$, the mass of E w.r.t. σ is:

$$m_{\sigma,t}(E) = \sum_{i=1}^{n} |Z_\sigma(A_i)| \cdot e^{\varphi_i t} \qquad t \in \mathbb{R}$$

Def / Thm: (Ikeda) The mass-growth of $\Phi \in \mathrm{Aut}(\mathcal{D})$ is

$$h_{\sigma,t}(\Phi) := \sup_{0 \neq E \in \mathcal{D}} \left\{ \limsup_{n} \frac{\log(m_{\sigma,t}(\Phi^n E))}{n} \right\} \leq h_t(\Phi)$$

Def: We say that σ has spanning image if $\mathrm{Span}_{\mathbb{R}}(\mathrm{Im}\, Z_\sigma) = \mathbb{C}$.

Thm: (B-Kim)

(Φ, σ, g) a compatible triple s.t. σ has spanning image. Then:

$$h_{\sigma,0}(\Phi) = \log(\rho(M_g)) = \log(\rho([\Phi]))$$

If furthermore $\mathcal{D} \simeq D(R)^c$ w/ R smooth & proper, then:

$$h_{\sigma,t}(\Phi) = \log(\rho([\Phi])) + \underbrace{J(\Phi)}\cdot t$$

$$\overline{J}(\Phi) := \lim_{m \to +\infty} \frac{\varepsilon^+(G, \Phi^m G)}{m}$$

G split-generator

$\varepsilon^+(E,F) = \max\{k : \mathrm{Hom}(E, F[-k]) \neq 0\}$

and :

$$h_{\sigma,t}(\Phi) \equiv h_t(\Phi) \iff h_{cat}(\Phi) = \log(\rho([\Phi]))$$

Applications:

1. Curves:

If C is a smooth, projective curve, then:

$\mathrm{Stab}(C) \simeq \mathbb{C}^2$ (Okada) if $g(C) = 0$

$\mathrm{Stab}(C) \simeq \sigma_0 \cdot \widetilde{GL}_2^+(\mathbb{R})$ if $g(C) > 0$ (Bridgeland, Macrì)

$$\sigma_0 = (-\deg(-) + i\,\mathrm{rk}(-), \mathrm{Coh}(C))$$

$\Rightarrow$ One can prove that $\forall\, \Phi \in \mathrm{Aut}(D^b(C))$ $\exists$ $g \in \widetilde{GL}_2^+(\mathbb{R})$ s.t. (Φ, σ_0, g) is compatible

Thm: (B-Kim)

$\forall\, \Phi \in \mathrm{Aut}(D^b(C))$ $\forall\, \sigma \in \mathrm{Stab}(D^b(C))$ we have

$$h_t(\Phi) = h_{\sigma,t}(\Phi) = \log(\rho([\Phi])) + \overline{J}(\Phi)\, t$$

2. $D^b_{(1)}(X)$:

X irreducible, smooth, proj. variety of $\dim X \geq 2$.

Def/Thm: (Meinhardt - Partsch)

$$D^b_{(1)}(X) := D^b(X) / \{E \in D^b(X) : \operatorname{codim} \operatorname{Supp} E \geq 2\}$$

$\forall\, \Phi \in \mathrm{Aut}(D^b_{(1)}(X))$ $\exists$ $f \in \mathrm{Bir}(X)$ iso. in codim. 1, $L \in \mathrm{Pic}(X)$, $n \in \mathbb{Z}$ s.t.

$$\Phi \simeq f^*(L \otimes -)[n]$$

Thm: (B-Kim)

Assume X has Picard rank 1.

$\Phi = f^*(L \otimes -)[n]$ $\quad$ L numerically trivial

Then, $\forall$ $\sigma \in \mathrm{Stab}(\mathcal{D}^b_{(1)}(X))$ we have:

$$h_{\sigma,0}(\Phi) = \log(\rho(f^*|_{N^1(X)}))$$