LATTICES AND THICK SUBCATEGORIES

jt with G. Stevenson
(in progress)

k - field

<u>Example 1</u> $\quad Q = \bullet\circlearrowleft$

$\rightsquigarrow \quad kQ = k[x]$

We can classify :

- all finitely generated $k[x]$-modules
- all thick subcategories of
  $$T := D^b(\mathrm{mod}(k[x]))$$

$\mathcal{T}$ - essentially small triangulated category

<u>Def</u>: A subcategory $\mathcal{L} \subseteq \mathcal{T}$ is <u>thick</u> if $\mathcal{L}$ is a triangulated subcategory closed under summands.

$$\mathrm{Thick}(\mathcal{T}) = \{\text{thick subcategories of } \mathcal{T}\}$$

$\mathrm{Thick}(T) = \{\text{thick subcategories of } T\}$ is a lattice under inclusion.

poset L with

- joins: $\forall a, b \in L$

$$\exists\ a \vee b = \min\{c \in L \mid a \le c,\ b \le c\}$$

- meets: $\forall a, b \in L$

$$\exists\ a \wedge b = \max\{c \in L \mid c \le a,\ c \le b\}$$

$$\begin{tikzcd}
 & a \vee b \arrow[dl, no head] \arrow[dr, no head] & \\
a \arrow[dr, no head] & & b \arrow[dl, no head] \\
 & a \wedge b &
\end{tikzcd}$$

For $A, B \in \mathrm{Thick}(T)$:

$$A \wedge B = A \cap B$$

$$A \vee B = \mathrm{thick}(A, B)$$

Back to

<u>Example 1</u> $\quad Q = \bullet\circlearrowleft$

$\rightsquigarrow \quad kQ = k[x] \ , \quad T := D^b(k[x])$

<u>Theorem:</u> [Hopkins-Neeman]

$$\mathrm{Thick}(T) \cong \left\{ \begin{array}{l} \text{specialisation closed} \\ \text{subsets of} \\ \mathrm{Spec}\, k[x] \end{array} \right\}$$

$T := D^b(k[x])$

<u>Theorem</u> : [Hopkins - Neeman]

$$\mathrm{Thick}(T) \cong \left\{ \begin{array}{l} \text{specialisation closed} \\ \text{subsets of} \\ \operatorname{Spec} k[x] \end{array} \right\}$$

In particular : $\exists$ topological space X and a lattice isomorphism

$$\mathrm{Thick}(T) \cong \mathcal{O}(X) = \{ U \subseteq X \mid U \text{ open} \}$$

$\rightsquigarrow$ In Example 1: "Thick subcategories are controlled by a space."

This is atypical in the world of representation theory!

<u>Example 2</u>: Take the universal cover of $\cdot \circlearrowright$:

$$Q = \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots$$

$$\text{mod}\, kQ \cong \text{gr}\, k[x]$$

$\mathbb{Z}$-graded, $|x| = 1$

$$T := D^b(\text{mod}\, kQ)$$

$\rightsquigarrow$ $\text{Thick}(T) = ?$

$Q = \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots$

$T := D^b(kQ)$

<u>Theorem</u>: [G. - Stevenson]

$$\mathrm{Thick}(T) \cong NC(\mathbb{Z} \cup \{-\infty\})$$

non-crossing partitions

L - linearly ordered set

$\mathcal{P} = \{B_i \mid i \in I\}$ partition of $L = \coprod_{i \in I} B_i$.

$\mathcal{P}$ is <u>non-crossing</u> if $x, y \in B_i$; $u, v \in B_j$

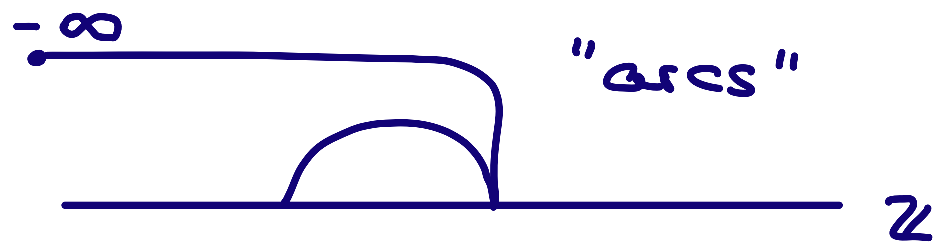
with $x < u < y < v \Rightarrow B_i = B_j$.

$Q = \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots$

$T := D^b(kQ)$

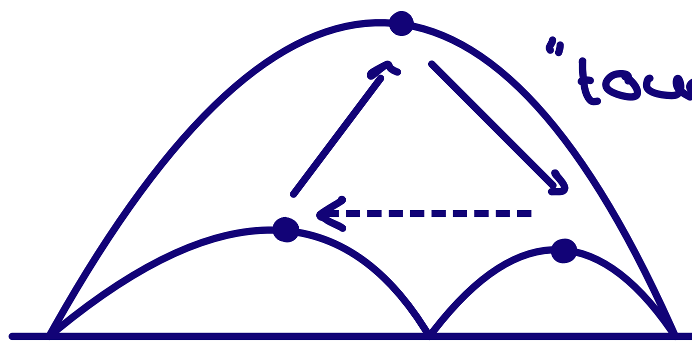
<u>Idea</u> :

Σ-orbits of indec. in T $\rightsquigarrow$



"arcs"

Δs in T $\rightsquigarrow$



"touching arcs"

Thick(T) $\rightsquigarrow$ "saturated sets of arcs"

$\updownarrow$

nc partitions

The lattice

$$\mathrm{Thick}(D^b(k\, \bullet \to \bullet \to \bullet \to \cdots)) \cong NC(\mathbb{Z} \amalg \{-\infty\})$$

is of a very different flavour than the lattice

$$\mathrm{Thick}(D^b(k\, \circlearrowleft)) \cong \mathcal{O}(X) \quad .$$

In particular, it is not of the form $\mathcal{O}(X)$ for any space X.

$\hookrightarrow$ How can we see that?

Let's analyse $\mathcal{O}(X)$ for X a space.
This is a lattice under $\subseteq$ with
$\wedge = \cap$ and $\vee = \cup$.

If $U, V, W \in \mathcal{O}(X)$ then

$$U \cap (V \cup W) = (U \cap V) \cup (U \cap W)$$

<u>Def</u>: Let L be a lattice. We say that L is <u>distributive</u> if $\forall\, l, m, n \in L$:

$$l \wedge (m \vee n) = (l \wedge m) \vee (l \wedge n).$$

Key observation:

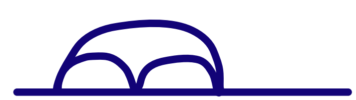
$\mathrm{Thick}(D^b(kQ))$ is <u>not</u> distributive.

Consider the non-split short exact sequence

$$0 \longrightarrow S_1 \longrightarrow M \longrightarrow S_2 \longrightarrow 0$$

in $\mathrm{mod}(kQ)$.

$$Q = \; 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4 \longrightarrow 5 \longrightarrow \cdots$$



$$0 \to S_1 \to H \to S_2 \to 0$$

H, S_1, S_2 : these are all exceptional

$A = \mathrm{thick}(H)$, $B_1 = \mathrm{thick}(S_1)$, $B_2 = \mathrm{thick}(S_2)$

$$A \wedge (B_1 \vee B_2) = A \cap \mathrm{thick}(S_1, S_2) = A$$
$$\#$$
$$(A \wedge B_1) \vee (A \wedge B_2) = 0 \vee 0 = 0$$

$\Rightarrow$ $\mathrm{Thick}(D^b(kQ)) \not\cong \mathcal{O}(X)$ for any space X.

① For a lattice L to satisfy

$$L \cong \mathcal{O}(X)$$

we need L to be <u>distributive.</u>

<u>But</u>: This is <u>not</u> enough.

We need an infinite analogue:

U, $\{V_i \mid i \in I\}$ open subspaces of X

$$\Rightarrow \quad U \cap \left(\bigcup_{i \in I} V_i \right) = \bigcup_{i \in I} (U \cap V_i).$$

Def: A lattice L is a **frame** if for all ℓ, $\{m_i \mid i \in I\}$ in L we have

$$\ell \wedge \left(\bigvee_{i\in I} m_i\right) = \bigvee_{i \in I} (\ell \wedge m_i).$$

② For a lattice L to satisfy

$$L \cong \mathcal{O}(X)$$

we need L to be **a frame**.

But: This is **still** not enough.

<u>Def:</u> A frame L is called <u>spatial</u> if there exists a space X such that

$$L \cong \mathcal{O}(X).$$

Duh.

One can describe this in terms of points of a lattice.

To summarize:

Spatial frame $\longleftrightarrow$ governed by a space

$\Downarrow$

frame

$\Downarrow$

distributive lattice

Question: When is Thick(T) a spatial frame for an essentially small triangulated category T ?

Thm: [G. - Stevenson]

$$\mathrm{Thick}(T) \text{ is a spatial frame}$$
$$\Updownarrow$$
$$\mathrm{Thick}(T) \text{ is distributive.}$$

<u>Corollary:</u> If for all $L, M, N \in \mathrm{Thick}(T)$:

$$L \cap \mathrm{thick}(M, N) = \mathrm{thick}(L \cap M, L \cap N)$$

then there exists an up to isomorphism unique sober space X such that

$$\mathrm{Thick}(T) \cong \mathcal{O}(X).$$

Note: This does not help us with things like

$$L = \mathrm{Thick}\left(D^b\left(k \cdot \overset{\textstyle\rightarrow}{\underset{\textstyle\rightarrow}{\rightarrow}} \cdot\right)\right)$$

which is not distributive.

Trailer: We can universally "approximate" L by a space.

$\rightsquigarrow$ upcoming preprint

Coherent frame

$\Downarrow$

Spatial frame

$\Downarrow$

frame

$\Downarrow$

distributive lattice

$\Downarrow$

modular lattice

Back to

<u>Example 1</u> $\quad Q = \bullet\circlearrowleft$

$\rightsquigarrow \quad kQ = k[x] \,, \quad T := D^b(k[x])$

<u>Theorem:</u> [Hopkins-Neeman]

$$\mathrm{Thick}(T) \cong \left\{ \begin{array}{l} \text{specialisation closed} \\ \text{subsets of} \\ \operatorname{Spec} k[x] \end{array} \right\}$$

$$\cong G(X)$$

this is <u>almost</u> Spec $k[x]$

If R is a commutative ring then $\operatorname{Spec} R$ is a very nice space.

C1 It is quasi-compact

C2 Every irreducible closed subset has a unique generic point] sober

C3 It has a basis of quasi-compact open subsets closed under finite intersections

A topological space satisfying C1 - C3 is called <u>coherent</u> (or <u>spectral</u>).

Thm: [Hochster]

If X is coherent then there exists a commutative ring R such that

$$X \cong \operatorname{Spec} R .$$

Q: If $\operatorname{Thick}(T) \cong \mathcal{O}(X)$, i.e. if $\operatorname{Thick}(T)$ is distributive,

- how nice is the space X?
- when is X coherent?

Let T be an essentially small triangulated category s.t. $\mathrm{Thick}(T)$ is distributive.

Let X be s.t. $\mathrm{Thick}(T) \cong \mathcal{O}(X)$.

<u>Lemma:</u> - Every irreducible closed subset of X has a unique generic point

- X has a basis of quasi-compact open subsets

$\rightsquigarrow$ promising ...

Lemma: C2: Every irreducible closed subset of X has a unique generic point
C3 part I: X has a basis of quasi-compact open subsets

For X to be coherent we'd additionally need ① C1: X is quasi-compact.

② C3 part II: the intersection of two quasi-compat open subsets is quasi-compact

Do we always have ① ? ② ?

(1) C1 : X is quasi-compact.

Do we always have (1) ? <u>No</u>.

<u>Example</u> : $T = D^b_{tors}(\mathrm{mod}\, k[x])$

$$= \{ X \in D^b(\mathrm{mod}\, k[x]) \mid H^* X \text{ is f.d.} \}$$

$$\mathrm{Thick}(T) \longleftrightarrow \underbrace{\mathrm{Thick}(D^b(\mathrm{mod}\, k[x]))}_{\text{distributive}}$$

$\Rightarrow$ Thick(T) distributive

$\Rightarrow$ Thick(T) is a spatial frame

T consists of tubes labelled by closed pts of $\mathbb{A}^1$.

$$\rightsquigarrow \mathrm{Thick}(T) \cong \bigoplus_{\alpha \in \mathbb{A}^1 \setminus \{\eta\}} \mathrm{Thick}(k(\alpha))$$

T is <u>not</u> finitely generated, i.e. there exists no $g \in T$ such that

$$\mathrm{thick}(g) = T.$$

$\Rightarrow$ X is <u>not</u> quasi-compact.

So: $\mathrm{Thick}(T) \cong \mathcal{O}(X) \not\Rightarrow X$ coherent.

② C3 part II: the intersection of two quasi-compact open subsets is quasi-compact.

Do we always have ② ?

We don't know.
Probably NO.

Coherent frame $\iff$ governed by a coherent space

$\Downarrow$

Spatial frame

$\Downarrow$

frame

$\Downarrow$

distributive lattice

$\Downarrow$

modular lattice

Motivating example

R — ring

M — R-module

$\mathrm{Sub}(M)$ lattice of submodules of M.

$\wedge = \cap$, $\vee = +$

usually not distr.

Example: $M = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$

$$\langle(1,1)\rangle \cap (\langle(1,0)\rangle + \langle(0,1)\rangle) = \langle(1,1)\rangle$$

$$(\langle(1,1)\rangle \cap \langle(1,0)\rangle) + (\langle(1,1)\rangle \cap \langle(0,1)\rangle) = 0 \quad \neq$$

<u>But</u>: $\mathrm{Sub}(M)$ is always <u>modular</u>.

<u>Def</u>: A lattice L is <u>modular</u> if $\forall\, l, m, n \in L$ with $l \le n$:

$$l \vee (m \wedge n) = (l \vee m) \wedge n.$$

[$A, B, C \in \mathrm{Sub}(M)$, $A \le C$

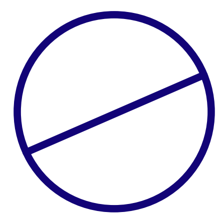
$\Rightarrow A + (B \cap C) = (A + B) \cap C$.]

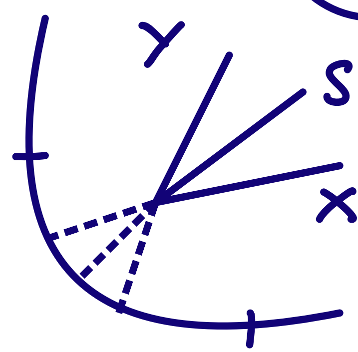
L distributive $\Rightarrow$ L modular.

Question: Is $\mathrm{Thick}(T)$ always modular?

No.

Example: $\mathcal{C}(\mathcal{Z})$ discrete cluster category [Igusa-Todorov] $\mathcal{Z} \subseteq S^1$ discrete with n accumulation points.

indec. objects  arcs

morphisms: 

$$\begin{tikzcd} & S \arrow[dr] & \\ X \arrow[ur] \arrow[rr] & \arrow[u, phantom, "\circlearrowright"] & Y \end{tikzcd}$$

$\mathcal{C}(Z)$ has a Δed structure encoded in the combinatorial picture.

<u>Thm:</u> [G.- Zvonareva]

$$\mathrm{Thick}(\mathcal{C}(Z)) \cong NNC([n])$$

$NNC([n])$ = nc partitions of subposets of $\{1, \dots, n\}$.

$$P_1 = \{B_i \mid i \in I\} \leq P_2 = \{B_j' \mid j \in J\}$$

$$:\Leftrightarrow \forall i \in I \; \exists j \in J : \; B_i \subseteq B_j' .$$

Example:

- $NNC([2]) = $
$$\begin{tikzcd}[column sep=small] & \{\{1,2\}\} \arrow[d, dash] & \\ & \{\{1\},\{2\}\} \arrow[dl, dash] \arrow[dr, dash] & \\ \{1\} \arrow[dr, dash] & & \{2\} \arrow[dl, dash] \\ & \emptyset & \end{tikzcd}$$

- $NNC([4])$ is <u>not</u> modular

$$l = \{\{1,2\}\} \leq n = \{\{1,2\},\{3,4\}\}$$
$$m = \{\{2,3\},\{1,4\}\}$$

$$l \vee (m \wedge n) = l \vee \{\{1\},\{2\},\{3\},\{4\}\} = \{\{1,2\},\{3\},\{4\}\}$$
$$\nparallel$$
$$(l \vee m) \wedge n = \{\{1,2,3,4\}\} \wedge n = \{\{1,2\},\{3,4\}\}$$