Symmetric subcategories and good tilting modules

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Main aim: To understand

• derived module categories of the endomorphism algebras of **arbitrary good tilting modules** T

or to establish

• **a general form** of Happel's theorem for not necessarily finitely generated tilting modules

or to describe

• **kernels** of the derived functors $T \otimes_B^{\mathbb{L}} -$

This reports joint works with Hongxing Chen.

Notations

A :	ring (algebra) with 1
A-Mod:	cat. of all left A-modules
$\mathrm{add}(M)$:	summands of f. dir. sums of $M \in A$-Mod
$\mathrm{Add}(M)$:	summands of dir. sums of M
A-Proj:	cat. of all left proj. A-modules
$\mathscr{D}(A)$:	(unbounded) derived cat. of A (or A-Mod)

Definition of tilting modules

Back to BGP, APR, Brenner-Butler, HR, Miyashita

Definition (Angeleri-Huegel + Coelho, 2001)

${}_AT \in A$-*Mod*: *n-tilting module if*

• *pd*$_A(T) \leq n$: $\quad P^\bullet \longrightarrow T \longrightarrow 0$

• $\mathrm{Ext}^i_A(T, T^{(I)}) = 0$ *for all* $i > 0$ *and all sets* I

• $\exists$ *exact seq.*: $0 \to {}_AA \to T_0 \to \cdots \to T_n \to 0$, $T_j \in \mathrm{Add}(T)$

- *good* if $T_i \in \mathrm{add}(T)$.
- *classical* if T: good and f. g. [Brenner-Butler, 1979].

Define $B := \mathrm{End}_A(T)$

Happel's Theorem

Theorem (Happel)

$_AT$: *class.* n-*tilt.* $\Longrightarrow \mathscr{D}(A) \sim \mathscr{D}(B)$

Happel: f. d. algebras
Cline-Parshall-Scott: rings

Note: Classical tilting procedure

- Invariant of derived categories
- No new triangulated categories

Example. I : ideal of ring R

$$\begin{pmatrix} R & I & I & I \\ R & R & I & I \\ R & R & R & I \\ R & R & R & R \end{pmatrix} \overset{der}{\simeq} \begin{pmatrix} R & R/I & R/I & R/I \\ & R/I & R/I & R/I \\ & & R/I & R/I \\ & & & R/I \end{pmatrix}$$

by tilting module of pd ≤ 1.

Significant roles of tilting modules

- Rickard's Morita theory on derived cat.s motivated by Happel Thm. on tilt. mod.s
- Representation theory of Lie algebras and algebraic groups via quasi-hered. alg.s, [Dlab-Ringel, Ringel]
- Representations of algebras: finitistic dimension conjecture [Angeleri-Huegel + Trlifahj]
- Other fields: Adéle rings in number theory [Crawley-Boevey, Ringel, Chen-Xi]

Results on good tilting modules

Theorem (Bazzoni, Bazzoni-Mantese-Tonolo)

${}_AT$: *good* n*-tilt.* $\Longrightarrow$ $\exists$ *recoll. of trian. cat.s:*

$$\mathrm{Ker}(T\otimes^{\mathbb{L}}_B -) \;\begin{matrix}\longleftarrow\\ \longrightarrow\\ \longleftarrow\end{matrix}\; \mathscr{D}(B) \;\begin{matrix}\longleftarrow\\ \xrightarrow{T\otimes^{\mathbb{L}}_B -}\\ \longleftarrow\end{matrix}\; \mathscr{D}(A)$$

- $\mathscr{D}(A) \sim \mathscr{D}(B)/\mathrm{Ker}(T\otimes^{\mathbb{L}}_B -)$
- $\mathrm{Ker}(T\otimes^{\mathbb{L}}_B -) = 0$ iff T class. $\Rightarrow$ Happel's Thm.

Note: Tilting procedure:

- Different trian. cat.s ($\mathscr{D}(A) \not\sim \mathscr{D}(B)$)
- Inf. g. tilting is NOT derived invariant

Definition of recollements

Definition (Beilinson-Bernstein-Deligne, 1982)

$\mathcal{D}$, $\mathcal{D}'$, $\mathcal{D}''$: trian. cat.s, $\mathcal{D}$: recollement of $\mathcal{D}'$ and $\mathcal{D}''$ (or $\exists$ recollement $(\mathcal{D}'', \mathcal{D}, \mathcal{D}')$) if $\exists$ trian. functors i_* and $j^!$:

$$\mathcal{D}'' \xrightarrow{i_*=i_!} \mathcal{D} \xrightarrow{j^!=j^*} \mathcal{D}'$$

(1) $j^! i_* = 0$,

(2) i_* *has left, right adjoints* $i^*, i^!$;

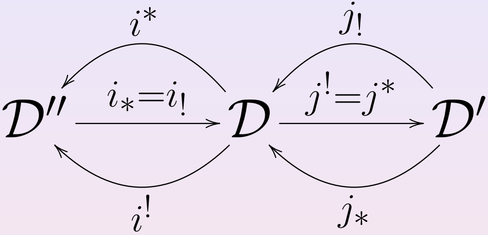
$j^!$ *has left, right adjoints* $j_!, j_*$,

(3) i_*, j^*, $j_!$: *fully faithful, and*

(4) $\forall$ *object* $X \in \mathcal{D}$, $\exists$ *two triangles in* $\mathcal{D}$:

$$i_! i^!(X) \longrightarrow X \longrightarrow j_* j^*(X) \longrightarrow i_! i^!(X)[1]$$

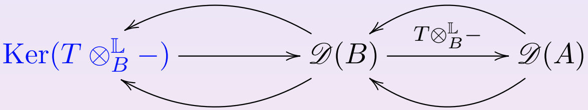
$$j_! j^!(X) \longrightarrow X \longrightarrow i_* i^*(X) \longrightarrow j_! j^!(X)[1].$$



- Derived recollements mean recoll.s of der. categories of rings or exact cat.s

Question for arbitrary good tilting modules

KNOWN:



QUESTION:

How to understand $\mathrm{Ker}(T \otimes_B^{\mathbb{L}} -)$ for good tilt. mod.s?

For n=1 case

Theorem (Chen-X., 2012, Proc. Lond. Math. Soc.)

${}_AT$: *good tilt., proj.dim* ≤ 1, $\Longrightarrow \exists$ *homol. ring epi.* $B \to C$ *and recoll. of der. mod. cat.s:*

$$\mathscr{D}(C) \underset{\longleftarrow}{\overset{\longleftarrow}{\longrightarrow}} \mathscr{D}(B) \underset{\longleftarrow}{\overset{\longleftarrow}{\xrightarrow{j^!}}} \mathscr{D}(A)$$

- $j^! := T \otimes^{\mathbb{L}}_B -, \mathrm{Ker}(j^!) \simeq \mathscr{D}(C)$.
- T: class., $\Rightarrow C = 0$, Happel Theorem.
- C: universal localization of B.

Definition of homological ring epimorphisms

Definition

A ring epimorphism $\lambda : R \to S$ *is called* **homological** *if* $\mathrm{Tor}_j^R(S, S) = 0$ *for* $j > 0$.

Or equivalently, the restriction functor $D(\lambda_*) : \mathscr{D}(S) \to \mathscr{D}(R)$ is fully faithful.

[Geigle-Lenzing: J. Algebra 144(1991)273-343]

In the literature:

- D.Yang 2012:

$$\mathscr{D}(C) \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow}\; \mathscr{D}(B) \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow}\; \mathscr{D}(A)$$

C: dg algebra

- S.Bazzoni and A.Pavarin 2013:

$$\mathscr{D}(E) \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow}\; \mathscr{D}(A) \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow}\; \mathscr{D}(B)$$

E: dg algebra

General question

Does the theorem for $n = 1$ extend to $n \geq 2$?

Definition

A full trian. subcat. $\mathcal{T}$ *of* $\mathscr{D}(B)$ *is called* ***homological*** *if* $\exists$ *homol. ring epi* $\lambda : B \to C$ *s. t.* $\mathscr{D}(C) \sim \mathcal{T}$ *(as trian. cat.s) by restriction.*

Now, the question becomes:

When is $\mathrm{Ker}(T \otimes^{\mathbb{L}}_B -)$ homol. in $\mathscr{D}(B)$?

Theorem (Chen-X. J.Math.Soc.Jap. 71 (2019) 515-554)

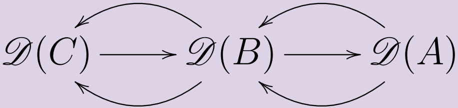
${}_AT$: *good tilt.* $B := \mathrm{End}_A(T)$. *TFAE:*

(1) $\mathrm{Ker}(T \otimes^{\mathbb{L}}_B -)$: *homol. in* $\mathscr{D}(B)$

(2) $H^i(\mathrm{Hom}_A(P^\bullet, A) \otimes_A T) = 0$ *for* $i \geq 2$

Theorem (continued)

In this case, $\exists$ *der. recoll. of der. mod. cat.s of rings*



$P^\bullet$: proj. resol. of T. C: generalized localization of B at T_B

Definition

R: ring,
Σ: a set of complexes of R-modules
$\lambda_\Sigma : R \to R_\Sigma$ hom. of rings is generalized localization of R at Σ if
(1) λ_Σ: Σ-exact: $\forall\, P^\bullet \in \Sigma$, $R_\Sigma \otimes_R P^\bullet$ is exact, and
(2) λ_Σ is univ. Σ-exact.

that is, if $\varphi : R \to S$, Σ-exact hom. of rings, then $\exists$ unique ring hom. $\psi : R_\Sigma \to S$ s.t. $\varphi = \lambda_\Sigma \psi$.

Recall **main aim**:

For arbitrary good tilting module ${}_AT$, to describe $\mathrm{Ker}(T \otimes_B^{\mathbb{L}} -)$ or to establish a counterpart of Happel's Theorem

Definition of n-symmetric subcategories

$\mathcal{A}$: bicompl. abel. cat. (with coprod.s + products).
$\mathcal{E}$: full subcat. of $\mathcal{A}$, $0 \le n \in \mathbb{N}$

Definition

$\mathcal{E}$: n-symmetric subcat. of $\mathcal{A}$ if

- *$\mathcal{E}$: closed under ext.s, prod.s + coprod.s.*
- *For ex. seq.*

$$0 \to X \to M_n \to \cdots \to M_1 \to M_0 \to Y \to 0 \text{ in } \mathcal{A},$$

there hold $X, Y \in \mathcal{E}$ whenever all $M_i \in \mathcal{E}$.

- Example: $\mathscr{E} = \{X \in B\text{-Mod} \mid \mathrm{Tor}_i^B(T_B, X) = 0 \ \forall\, i \ge 0\}$ n-symm. if $n = pd(T_B) < \infty$
- n-sym. subcat.s are ex. cat.s

Symmetric subcategories

$\mathcal{B}$: add. full subcat. of bicompl. abel. cat. $\mathcal{A}$.

- $\mathcal{B}$: n-sym. subcat. of $\mathcal{A} \Longrightarrow \mathcal{B}$ ex., thick subcat., $(n+1)$-sym.
- $\mathcal{B}_i$: m_i-sym. subcat.s of $\mathcal{A} \Longrightarrow \mathcal{B}_1 \cap \mathcal{B}_2$: max$\{m_1, m_2\}$-sym.

- $\mathcal{B}$: ext. closed, Def.(2)$\Longrightarrow$ $\mathcal{B}$: n-wide subcat. of $\mathcal{A}$ in the sense of Matsui-Nam-Takahashi-Tri-Yen.

- $\mathcal{B}$: 0-sym. $\Longleftrightarrow$ $\mathcal{B}$: Serre subcat. $\&$ closed under coprod.s, products $\Longleftrightarrow$ $\mathcal{B}$: localizing subcat. closed under products.

- $\mathcal{B}$: 1-sym. $\Longleftrightarrow$ $\mathcal{B}$: abel. subcat. closed under ext.s, coprod. and products.

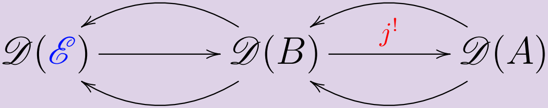
Given an exact category $\mathscr{E}$, define

$\mathscr{D}(\mathscr{E}) = \mathscr{K}(\mathscr{E})/\mathscr{K}_{ac}(\mathscr{E})$: Verdier quotient of $\mathscr{K}(\mathscr{E})$ modulo $\mathscr{K}_{ac}(\mathscr{E})$ of exact complx.s over $\mathscr{E}$

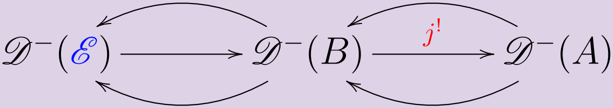
Theorem (Chen-X., 2021)

$_AT$: good tilt. / ring A, $B := \mathrm{End}(_AT)$

$\Longrightarrow$ *$\exists$ n-sym. subcat. $\mathscr{E}$ of B-*Mod *and recoll.*



Moreover, this recoll. induces



- $\mathscr{E} := \{X \in B\text{-Mod} \mid T \otimes^{\mathbb{L}}_B X = 0\}$: n-sym. subcat. with $n = \mathrm{pd}(T_B)$
- $j^! := T \otimes^{\mathbb{L}}_B -$
- $\mathscr{D}^-(\mathscr{E})$: der. cat. of bounded-above complx.s over $\mathscr{E}$

Comment:

This might be regarded as Happle's Thm for good tilt. mod.s since the 3 categories

$$\mathscr{E}, B\text{-Mod}, A\text{-Mod}$$

are the same kind of categories, namely subcategories of modules over rings

Corollary

TFAE for good tilt. A-mod. T:

(1) $\mathrm{Ker}(T\otimes^{\mathbb{L}}_B -)$: *homol. in* $\mathscr{D}(B)$

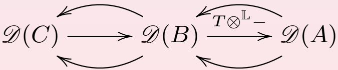
(2) $\mathscr{E}$: *abel. subcat*

(3) $H^m(\mathrm{Hom}_A(P^\bullet, A)\otimes_B T) = 0$ *for all* $m \geq 2$, $P^\bullet$: *proj. resol. of* ${}_AT$.

(4) $(\mathscr{E}, \mathscr{E}^{\perp})$: *der. decom. of* B-Mod

$$\mathscr{E}^{\perp} := \{Y \in B\text{-Mod} \mid \mathrm{Ext}^n_B(X, Y) = 0, \forall\ X \in \mathscr{E}, n \geq 0\}.$$

Recall: T is homol. if $\exists$ homol. ring epi. $B \to C$ of rings s.t.



Definition (Chen-X. Pacific J. Math. 312 (2021))

$\mathcal{A}$: *abel. cat.* $\mathcal{B}, \mathcal{C}$: *full subcat.s of* $\mathcal{A}$.

$(\mathcal{B}, \mathcal{C})$: **der. decomposition** *of* $\mathcal{A}$ *if*

- $\mathcal{B}, \mathcal{C}$: *abel. subcat. of* $\mathcal{A}$, *inclusions induce f. fait. functors* $\mathscr{D}^b(\mathcal{B}) \to \mathscr{D}^b(\mathcal{A})$ *and* $\mathscr{D}^b(\mathcal{C}) \to \mathscr{D}^b(\mathcal{A})$, *resp.*
- $\mathrm{Hom}_{\mathscr{D}^b(\mathcal{A})}(B, C[n]) = 0$ *for* $B \in \mathcal{B}$, $C \in \mathcal{C}$ *and* $n \in \mathbb{Z}$
- *For* $M^\bullet \in \mathscr{D}^b(\mathcal{A})$, $\exists$ *triangle*

$$B_{M^\bullet} \to M^\bullet \to C^{M^\bullet} \to B_{M^\bullet}[1]$$

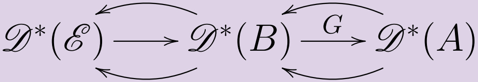
in $\mathscr{D}^b(\mathcal{A})$ *s.t.* $B_{M^\bullet} \in \mathscr{D}^b(\mathcal{B})$, $C^{M^\bullet} \in \mathscr{D}^b(\mathcal{C})$.

Corollary

A: left coherent ring, $\quad {}_AT$*: good tilt.,*

$B := \mathrm{End}_A(T) \Longrightarrow$

$\exists$ *recoll. of der. cat.s*



for $* \in \{b, +, -, \emptyset\}$

$\mathscr{E}$: sym. subcat. of B-Mod, $\quad G = T \otimes^{\mathbb{L}}_B -$

Left coher. ring if f. g. left ideals are f. presented

Ideas of proof of the main result

- $i : \mathscr{E} \longrightarrow B\text{-Mod}, \ D(i): \ \mathscr{D}(\mathscr{E}) \longrightarrow \mathscr{D}(B)$
- There is decomposition

$$\mathscr{D}(\mathscr{E}) \xrightarrow{\overline{D(i)}} \mathrm{Ker}(G) \xrightarrow{\kappa} \mathscr{D}(B), \qquad D(i) = \kappa \circ \overline{D(i)}$$

- $\overline{D(i)}$: trian. equiv.

Example

Recall:

Definition

R: n-Gorenstein ring if R is comm. noether. of inj.dim$({}_RR) = n$

A : **2-Gorenstein local domain**,

$\mathfrak{m}$: max. ideal of A, Q: its fraction field,

Minimal inj. resol. of A by a result of Bass:

$$0 \to A \xrightarrow{\lambda} Q \xrightarrow{\alpha} \bigoplus_{\mathfrak{p}\in\mathscr{H}_1} E(A/\mathfrak{p}) \xrightarrow{\beta} E(A/\mathfrak{m}) \to 0$$

$E(M)$: inj. envelope of M

$$\mathscr{H}_1 := \{\mathfrak{p} \lhd A \mid \mathfrak{p} \text{ prime ideal with height } 1\}$$

- Known:

$T' := Q \oplus \bigoplus_{\mathfrak{p}\in\mathscr{H}_1} E(A/\mathfrak{p}) \oplus E(A/\mathfrak{m})$: 2-tilt.

- Modify this construction: $\emptyset \neq \mathscr{S} \subseteq \mathscr{H}_1$

$$T_2 := E(A/\mathfrak{m})$$
$$T_1 := \bigoplus_{\mathfrak{p}\in\mathscr{S}} E(A/\mathfrak{p})$$
$$T_0 := \alpha^{-1}(T_1 \cap \mathrm{Ker}(\beta))$$
$$T := T_0 \oplus T_1 \oplus T_2$$

$$0 \longrightarrow A \xrightarrow{f_0} T_0 \xrightarrow{f_1} T_1 \xrightarrow{f_2} T_2$$

f_0: the inclusion; f_1: induced by α;
f_2: restr. of β

Proposition

(1) $\mathscr{S}$ contains a principal ideal, $\Longrightarrow$ T: 2-tilt.
(2) A: complete, $\mathscr{S}$ consists of f. m. principal ideals of A, $\Longrightarrow$

$$\operatorname{End}_A(T) \simeq \begin{pmatrix} T_0 & T_0 \otimes_A B_1 & T_0 \otimes_A C \\ 0 & B_1 & B_1 \\ 0 & 0 & A \end{pmatrix}$$

$B_1 := \operatorname{End}_A(T_1)$, $T_0 = \operatorname{End}_A(T_0)$ *and*
$Q = \operatorname{End}_A(Q)$

$\mathrm{End}_A(T)$-Mod is identified with category $\mathscr{C}(A,T)$:

Objects:

Complexes $X^\bullet : 0 \to X^{-2} \to X^{-1} \to X^0 \to 0$ in $\mathscr{C}(A)$,

$$X^{-1} \in B_1\text{-Mod},\ X^0 \in T_0\text{-Mod},$$

where B_1-modules and T_0-modules regarded as A-modules via given ring homomorphisms θ_{T_1} and f_0, respectively.

Morphism:

Chain map $f^\bullet := (f^{-2}, f^{-1}, f^0) : X^\bullet \to Y^\bullet$ in $\mathscr{C}(A)$, $f^{-1} \in \mathrm{Hom}_{B_1}(X^{-1}, Y^{-1})$,
$f^0 \in \mathrm{Hom}_{T_0}(X^0, Y^0)$.

$\mathscr{C}_{\mathrm{ac}}(A,T)$: full exact subcat. of $\mathscr{C}(A,T)$ consisting of all exact complexes.

Example

A: complete, $\mathscr{S}$ consists of f. m. prin. ideals of A

$\Longrightarrow$

(1) 2-*sym. subcat.* $\mathscr{E}$ *by* T *is equ. to* $\mathscr{C}_{\mathrm{ac}}(A,T)$.

(2) *Recoll.*

$$\mathscr{D}^*(\mathscr{C}_{\mathrm{ac}}(A,T)) \longrightarrow \mathscr{D}^*(\mathrm{End}_A(T)) \xrightarrow{G} \mathscr{D}^*(A)$$

$* \in \{-,\emptyset\}$, $G := T \otimes^{\mathbb{L}}_B -$

Questions:

A: ring, or algebra/field

(1) Given n, parameterize n-symm. subcat.s of $A\text{-Mod}$.

(2) Which n-sym. subcat.s of A-Mod can be realised by n-tilt. modules?

that is, under which cond.s on n-sym. subcat. $\mathscr{E}$ of $A\text{-Mod}$ is there an n-til. mod. T_A s. t. $\mathscr{E} \simeq \{Y \in A\text{-Mod} \mid \mathrm{Tor}_i^A(T_A, Y) = 0, \forall\, i \geq 0\}$ as ex. cat.s?

(3) Find methods to construct homological tilting modules, or cotilting modules.

Symmetric subcategories and good tilting modules

Thank you !

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URL: http://math0.bnu.edu.cn/~ccxi/
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