Prime thick subcategories of derived categories associated with noetherian schemes

Hiroki Matsui (Tokushima University)

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Introduction

Tensor triangular geometry (Balmer)

Study a tensor triangulated category $(\mathcal{T}, \otimes, \mathbf{1})$ via the corresponding topological space

$$\mathbf{Spec}_{\otimes}(\mathcal{T}) := \{\text{prime ideals of } \mathcal{T}\}$$

called the tensor-triangular spectrum (tt-spectrum) of $\mathcal{T}$.

- Very successful!!
  Applied to commutative algebra, algebraic geometry, representation theory of finite groups, stable homotopy theory,...
- Cannot be applied to triangulated category without tensor structure.

Introduction

Aim

1. Introduce "tensor-free" analog of prime ideals and define a topological space $\mathbf{Spec}_{\triangle}(\mathcal{T})$ without using tensor structure.
2. Study these for $\mathrm{D}^{\mathrm{pf}}(X)$, $\mathrm{D}^{\mathrm{b}}(X)$, $\mathrm{D}^{\mathrm{sg}}(X)$ of a noetherian scheme.

- $\mathrm{D}^{\mathrm{pf}}(X) :=$ the derived category of perfect complexes on X
- $\mathrm{D}^{\mathrm{b}}(X) :=$ the derived category of bounded complexes of coherent sheaves on X
- $\mathrm{D}^{\mathrm{sg}}(X) := \mathrm{D}^{\mathrm{b}}(X)/\mathrm{D}^{\mathrm{pf}}(X)$: singularity category or stable derived category

Tensor triangular geometry

First of all, let us recall Balmer's tensor triangular geometry.

Definition

A tensor triangulated category is a triple $(\mathcal{T}, \otimes, \mathbf{1})$ where

- $\mathcal{T}$: a triangulated category
- $\otimes : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$, $(M, N) \mapsto M \otimes N$: an exact bifunctor
- $\mathbf{1} \in \mathcal{T}$: unit object

satisfying

- associativity $(L \otimes M) \otimes N \cong L \otimes (M \otimes N)$
- commutativity $M \otimes N \cong N \otimes M$
- unitality $M \otimes \mathbf{1} \cong M \cong \mathbf{1} \otimes M$.

Note: commutative ring is a triple $(R, \cdot, 1_R)$ satisfying associativity, commutativity, unitality.

Tensor triangular geometry

Example

1. Let X be a noetherian scheme.
   Then $(\mathrm{D}^{\mathrm{pf}}(X), \otimes^{\mathbb{L}}_{\mathcal{O}_X}, \mathcal{O}_X)$ is a tensor triangulated category.
2. Let k be a field and G a finite group. Denote by stmod kG the stable module category of kG.
   Then $(\mathrm{stmod}\, kG, \otimes_k, k)$ is a tensor triangulated category.

Tensor triangular geometry

Balmer's Idea: regard $\mathcal{T}$ as a commutative ring with multiplication $\otimes$

Recall

Let R be a commutative ring.

1. An additive subgroup $I \subseteq R$ is an ideal if
$$a \in R,\ b \in I \Longrightarrow ab \in I$$
2. An ideal $I \subseteq R$ is radical if
$$I = \sqrt{I} = \{a \in R \mid a^n \in I\ (\exists n \geq 1)\} = \bigcap_{I \subseteq \mathfrak{p}:\text{prime}} \mathfrak{p}$$
3. An ideal $\mathfrak{p} \subsetneq R$ is prime if
$$ab \in \mathfrak{p} \Longrightarrow a \in \mathfrak{p} \text{ or } b \in \mathfrak{p}.$$

$\mathbf{Spec}\, R := \{\text{prime ideals of } R\}$

Tensor triangular geometry

$(\mathcal{T}, \otimes, \mathbf{1})$: tensor triangulated category

Definition

1. A thick subcategory $\mathcal{I} \subseteq \mathcal{T}$ is an ideal if
$$M \in \mathcal{T},\ N \in \mathcal{I} \Longrightarrow M \otimes N \in \mathcal{I}.$$
2. An ideal $\mathcal{I} \subseteq \mathcal{T}$ is radical if
$$\mathcal{I} = \sqrt{\mathcal{I}} := \{M \in \mathcal{T} \mid M^{\otimes n} \in \mathcal{I}\ (\exists n \geq 1)\} = \bigcap_{\mathcal{I} \subseteq \mathcal{P}:\text{prime}} \mathcal{P}$$
$\mathbf{Rad}_{\otimes}(\mathcal{T}) := \{\text{radical ideals of } \mathcal{T}\}$
3. An ideal $\mathcal{P} \subsetneq \mathcal{T}$ is prime if
$$M \otimes N \in \mathcal{P} \Longrightarrow M \in \mathcal{P} \text{ or } N \in \mathcal{P}.$$
$\mathbf{Spec}_{\otimes}(\mathcal{T}) := \{\text{prime ideals of } \mathcal{T}\}$

Tensor triangular geometry

Example

1. For a noetherian scheme X,

$$\mathbf{Supp}^{-1}(W) := \{M \in \mathrm{D}^{\mathrm{pf}}(X) \mid \mathbf{Supp}(M) \subseteq W\}$$

   is a radical ideal of $\mathrm{D}^{\mathrm{pf}}(X)$ for each specialization-closed subset $W \subseteq X$.

2. Let k be a field and G a finite group. Then

$$\mathbf{V}_G^{-1}(W) := \{M \in \operatorname{stmod} kG \mid \mathbf{V}_G(M) \subseteq W\}$$

   is a radical ideal of $\operatorname{stmod} kG$ for each specialization-closed subset $W \subseteq \operatorname{Proj} \mathrm{H}^*(G; k)$. Here, $\mathbf{V}_G(M)$ denotes the support variety of M.

These are prime if and only if W is of the form

$$W = \{y \mid x \notin \overline{\{y\}}\}$$

for some point x of the above topological spaces

Tensor triangular geometry

Recall that the Zariski topology on $\mathbf{Spec}\, R$ is given by closed basis $\{\mathbf{V}(a)\}_{a \in R}$ where

$$\mathbf{V}(a) := \{\mathfrak{p} \in \mathbf{Spec}\, R \mid a \in \mathfrak{p}\}.$$

Definition

The topology of $\mathbf{Spec}_{\otimes}(\mathcal{T})$ is given by closed basis $\{\mathbf{Supp}_{\otimes}(\mathcal{T})\}_{M \in \mathcal{T}}$ where

$$\mathbf{Supp}_{\otimes}(\mathcal{T}) := \{\mathcal{P} \in \mathbf{Spec}_{\otimes}(\mathcal{T}) \mid M \notin \mathcal{P}\}.$$

This topological space $\mathbf{Spec}_{\otimes}(\mathcal{T})$ is called the tensor-triangular spectrum (tt-spectrum) of $\mathcal{T}$.

- Kock-Pitsch (2017)
  This topology is the Hochster dual of the Zariski topology

Tensor triangular geometry

$(\mathcal{T}, \otimes, \mathbf{1})$: tensor triangulated category

Classification theorem (Balmer(2005))

$\exists$ a bijection

$$\mathbf{Rad}_{\otimes}(\mathcal{T}) \underset{\mathbf{Supp}_{\otimes}^{-1}}{\overset{\mathbf{Supp}_{\otimes}}{\rightleftarrows}} \mathbf{Thom}(\mathbf{Spec}_{\otimes}(\mathcal{T})) = \{\mathbf{Supp}_{\otimes}(\mathcal{I}) \mid \mathcal{I} \in \mathbf{Rad}_{\otimes}(\mathcal{T})\}$$

Thomason subset = union of complements of quasi-compact open subsets
= subsets of the form $\mathbf{Supp}_{\otimes}(\mathcal{I})$ for $\mathcal{I} \in \mathbf{Rad}_{\otimes}(\mathcal{T})$
= specialization-closed subset if noeth. top. space

This theorem unifies several known results, e.g.,

- Hopkins (1987), Neeman (1992), Thomason (1997): Classificaion of radical ideals of $\mathrm{D}^{\mathrm{pf}}(X)$ via $\mathbf{Spcl}(X)$.
- Benson-Carlson-Rickard (1997), Benson-Iyengar-Krause (2011): Classificaion of radical ideals of stmod kG via $\mathbf{Spcl}(\operatorname{Proj} \mathrm{H}^*(G; k))$.

Tensor triangular geometry

Uniqueness theorem (Balmer)

If $\exists$ a noetherian topological space X that classifies radical ideals of $\mathcal{T}$ i.e., $\exists$ a bijection

$$\mathbf{Rad}_{\otimes}(\mathcal{T}) \rightleftarrows \mathbf{Spcl}(X),$$

then $\mathbf{Spec}_{\otimes}(\mathcal{T}) \cong X$.

Theorem (Balmer)

1. For a noetherian scheme X,
$$\mathbf{Spec}_{\otimes}(\mathrm{D}^{\mathrm{pf}}(X)) \cong X$$
2. For a field k and a finite group G,
$$\mathbf{Spec}_{\otimes}(\mathrm{stmod}\, kG) \cong \mathrm{Proj}\, \mathrm{H}^*(G; k)$$

- X (resp. $\mathrm{Proj}\, \mathrm{H}^*(G; k)$) is reconstructed from $\mathrm{D}^{\mathrm{pf}}(X)$ (resp. $\mathrm{stmod}\, kG$) using tensor structure.

Tensor triangular geometry

$(\mathcal{T}, \otimes, \mathbf{1})$: tensor triangulated category

Observation

$\mathcal{P} \in \mathbf{Rad}_{\otimes}(\mathcal{T})$ is prime ideal iff
$\exists!$ a radical ideal $\mathcal{I}$ with $\mathcal{P} \subsetneq \mathcal{I}$ s.t. $\nexists$ radical ideals $\mathcal{J}$ with $\mathcal{P} \subsetneq \mathcal{J} \subsetneq \mathcal{I}$.
In other words, $\mathcal{P}$ is prime iff it has a unique cover in the lattice $\mathbf{Rad}_{\otimes}(\mathcal{T})$.

- "if part": Assume such $\mathcal{I}$ exists. If $\mathcal{P}$ is not prime, then $\mathcal{P} \subsetneq \mathcal{Q}$ for each prime ideal $\mathcal{Q}$ that contains $\mathcal{P}$. Therefore

$$\mathcal{P} \subsetneq \mathcal{I} \subseteq \bigcap_{\mathcal{P} \subseteq \mathcal{Q} \in \mathbf{Spec}_{\otimes}(\mathcal{T})} \mathcal{Q} = \sqrt{\mathcal{P}} = \mathcal{P}.$$

  "only if part": Use Balmer's classification and need some argument on topological space $\mathbf{Spec}_{\otimes}(\mathcal{T})$. □
- Prime ideals of $\mathcal{T}$ is determined by the lattice structure on $\mathbf{Rad}_{\otimes}(\mathcal{T})$.
- Our strategy is to replace $\mathbf{Rad}_{\otimes}(\mathcal{T})$ with the lattice $\mathbf{Th}(\mathcal{T})$ of thick subcategories.

Prime thick subcategories and Spectrum

$\mathcal{T}$: triangulated category

Definition

$\mathcal{P} \in \mathbf{Th}(\mathcal{T})$ is a prime thick subcategory if $\mathcal{P}$ has a unique cover in $\mathbf{Th}(\mathcal{T})$.
$\mathbf{Spec}_{\triangle}(\mathcal{T}) := \{\text{prime thick subategories of } \mathcal{T}\}$

- We will see later that for a radical ideal $\mathcal{P}$ of $\mathrm{D}^{\mathrm{pf}}(X)$,

$$\mathcal{P}: \text{ prime ideal } \iff \mathcal{P}: \text{ prime thick subcategory}$$

- We can consider that prime thick subcategories are "tensor-free" analog of prime ideals.

Prime thick subcategories and Spectrum

Example

Let R be a commutative noetherian ring. Then

$$\mathcal{S}^{\mathsf{pf}}(\mathfrak{p}) := \{M \in \mathsf{D}^{\mathsf{pf}}(R) \mid M_{\mathfrak{p}} \cong 0 \text{ in } \mathsf{D}^{\mathsf{pf}}(R_{\mathfrak{p}})\}$$

is a prime thick subcategory of $\mathsf{D}^{\mathsf{pf}}(R)$ for any $\mathfrak{p} \in \mathbf{Spec}\, R$.

$(\because)$
Use the lattice isomorphism

$$\mathbf{Th}(\mathsf{D}^{\mathsf{pf}}(R)) \cong \mathbf{Spcl}(\mathbf{Spec}\, R), \quad \mathcal{X} \mapsto \mathbf{Supp}(\mathcal{X}) := \bigcup_{M \in \mathcal{X}} \mathbf{Supp}(M)$$

by Hopkins-Neeman. By this correspondence, $\mathcal{S}^{\mathsf{pf}}(\mathfrak{p})$ corresponds to $W_0 = \{\mathfrak{q} \in \mathbf{Spec}\, R \mid \mathfrak{p} \notin \overline{\{\mathfrak{q}\}}\}$. Since $W_0 \cup \{\mathfrak{p}\}$ is a unique cover W_0, $\mathcal{S}^{\mathsf{pf}}(\mathfrak{p})$ has a unique cover by the above correspondence.

Prime thick subcategories and the Spectrum

Example

Let R be a hypersurface local ring (i.e., $R \cong S/(x)$ for some RLR S). Then

$$\mathcal{S}^{\mathrm{sg}}(\mathfrak{p}) := \{M \in \mathrm{D}^{\mathrm{sg}}(R) \mid M_{\mathfrak{p}} \cong 0 \text{ in } \mathrm{D}^{\mathrm{sg}}(R_{\mathfrak{p}})\}$$

is a prime thick subcategory of $\mathrm{D}^{\mathrm{sg}}(R)$ for any $\mathfrak{p} \in \mathbf{Sing}\, R$.

$(\because)$
Here, use the lattice isomorphism

$$\mathbf{Th}(\mathrm{D}^{\mathrm{sg}}(R)) \cong \mathbf{Spcl}(\mathbf{Sing}\, R)$$

by Takahashi (2010). Then we do the same argument as above: $\mathcal{S}^{\mathrm{sg}}(\mathfrak{p})$ corresponds to $W_0 = \{\mathfrak{q} \in \mathbf{Sing}\, R \mid \mathfrak{p} \notin \overline{\{\mathfrak{q}\}}\}$.

Prime thick subcategories and the Spectrum

Example

Let $(R, \mathfrak{m})$ be a complete intersection local ring (i.e., $R \cong S/(\underline{x})$ for some RLR S and a regular sequence $\underline{x}$). Then

$$\mathcal{S}^{\mathrm{b}}(\mathfrak{m}) := \{M \in \mathrm{D}^{\mathrm{b}}(R) \mid M_{\mathfrak{m}} \cong 0 \text{ in } \mathrm{D}^{\mathrm{b}}(R)\} = \mathbf{0}$$

is a prime thick subcategory of $\mathrm{D}^{\mathrm{b}}(R)$.

$(\because)$
It follows from Dwyer-Greenlees-Iyengar (2006) and Hopkins-Neeman, non-zero thick subcategory contains

$$\mathcal{X} := \{M \in \mathrm{D}^{\mathrm{pf}}(R) \mid \mathbf{Supp}(M) \subseteq \{\mathfrak{m}\}\}.$$

Therefore, $\mathcal{X}$ is a unique cover of $\mathbf{0}$.

Prime thick subcategories and the Spectrum

Definition

Define a topology on $\mathbf{Spec}_\triangle(\mathcal{T})$ by closed basis $\{\mathbf{Supp}_\triangle(M)\}_{M\in\mathcal{T}}$ where

$$\mathbf{Supp}_\triangle(M) := \{\mathcal{P} \in \mathbf{Spec}_\triangle(\mathcal{T}) \mid M \notin \mathcal{P}\}.$$

We call $\mathbf{Spec}_\triangle(\mathcal{T})$ the triangular spectrum of $\mathcal{T}$.

Definition

A thick subcategory $\mathcal{X} \in \mathbf{Th}(\mathcal{T})$ is radical if

$$\mathcal{X} = \sqrt{\mathcal{X}} := \bigcap_{\mathcal{X}\subseteq\mathcal{P}\in\mathbf{Spec}_\triangle(\mathcal{T})} \mathcal{P}.$$

$\mathbf{Rad}_\triangle(\mathcal{T}) := \{$radical thick subcategories of $\mathcal{T}\}$

Prime thick subcategories and the Spectrum

Theorem A

1. $\exists$ a bijection
$$\mathbf{Rad}_\triangle(\mathcal{T}) \underset{\mathbf{Supp}_\triangle^{-1}}{\overset{\mathbf{Supp}_\triangle}{\rightleftarrows}} \{\mathbf{Supp}_\triangle(\mathcal{X}) \mid \mathcal{X} \in \mathbf{Rad}_\triangle(\mathcal{T})\}$$
2. If $\exists$ a noetherian top. sp. X that classifies thick subcategories of $\mathcal{T}$ i.e., $\exists$ a bijection
$$\mathbf{Th}(\mathcal{T}) \rightleftarrows \mathbf{Spcl}(X),$$
then $\mathbf{Spec}_\triangle(\mathcal{T}) \cong X$.

problem

- $\mathbf{Supp}_\triangle(\mathcal{X})$ may not be a Thomason subset. Can we give a topological characterization of RHS in (1)?
- If (2), can we replace $\mathbf{Th}(\mathcal{T})$ with $\mathbf{Rad}_\triangle(\mathcal{T})$?

Prime thick subcategories and the Spectrum

Corollary

1. Let X be a quasi-affine noetherian scheme (i.e., $\mathcal{O}_X$ is ample). Then
$$\mathbf{Spec}_\triangle(\mathrm{D}^{\mathrm{pf}}(X)) \cong X$$
2. Let X be a quasi-affine noetherian scheme which is locally hypersurface (i.e., $\mathcal{O}_{X,x}$ is hypersurface for $\forall x \in X$). Then
$$\mathbf{Spec}_\triangle(\mathrm{D}^{\mathrm{sg}}(X)) \cong \mathbf{Sing}(X)$$
3. Let k be a field and G a finite p-group. Then
$$\mathbf{Spec}_\triangle(\mathrm{stmod}\, kG) \cong \mathrm{Proj}\,\mathrm{H}^*(G; k)$$

- Reconstruction without tensor structure.
- By (1), $\dim X$ is an invariant of triangulated category $\mathrm{D}^{\mathrm{pf}}(X)$.
- By (2), the p-rank
$r_p(G) = \inf\{n \geq 0 \mid (\mathbb{Z}/pZZ)^{\oplus n} \subseteq G\} \overset{Quillen}{=} \dim \mathrm{Proj}\,\mathrm{H}^*(G; k)$
is an invariant of triangulated category $\mathrm{stmod}\, kG$.

Prime thick subcategories and the Spectrum

Example

Let $\mathbb{P}^1$ be a projective line over a field k. Then it was shown by Krause-Stevenson (2019) that there is a lattice isomorphism

$$\mathbf{Th}(\mathrm{D}^{\mathrm{pf}}(\mathbb{P}^1)) \cong \mathbf{Spcl}(\mathbb{P}^1) \sqcup \mathbb{Z}.$$

Restricting this to prime thick subcategories, we get a homeomorphism

$$\begin{aligned}\mathbf{Spec}_{\triangle}(\mathrm{D}^{\mathrm{pf}}(\mathbb{P}^1)) &= \mathbf{Spec}_{\otimes}(\mathrm{D}^{\mathrm{pf}}(\mathbb{P}^1)) \sqcup \{\mathrm{thick}(\mathcal{O}_{\mathbb{P}^1}(i)) \mid i \in \mathbb{Z}\} \\ &\cong \mathbb{P}^1 \sqcup \mathbb{Z}\end{aligned}$$

Without classification, it is quite difficult to determine tensor (triangular) spectra in general.
$\rightarrow$ We will determine a certain subset of triangular spectrum

Applications to derived categories of a noetherian scheme

X: noetherian scheme

Observation

By Balmer's theorem $\mathbf{Spec}_\otimes(\mathrm{D}^{\mathrm{pf}}(X)) \cong X$, every prime ideal of $\mathrm{D}^{\mathrm{pf}}(X)$ is of the form

$$\mathcal{S}^{\mathrm{pf}}(x) := \{M \in \mathrm{D}^{\mathrm{pf}}(X) \mid M_x \cong 0 \text{ in } \mathrm{D}^{\mathrm{pf}}(\mathcal{O}_{X,x})\}$$

for some $x \in X$.

Question

For $* \in \{\mathrm{pf}, \mathrm{b}, \mathrm{sg}\}$ and $x \in X$, is

$$\mathcal{S}^*(x) := \{M \in \mathrm{D}^*(X) \mid M_x \cong 0 \text{ in } \mathrm{D}^*(\mathcal{O}_{X,x})\}$$

a prime thick subcategory?

Applications to derived categories of a noetherian scheme

Recall (Examples)

Let R be a commutative noetherian ring.

1. For any $\mathfrak{p} \in \mathbf{Spec}\, R$,

$$\mathcal{S}^{\mathrm{pf}}(\mathfrak{p}) = \{M \in \mathrm{D}^{\mathrm{pf}}(R) \mid M_{\mathfrak{p}} \cong 0 \text{ in } \mathrm{D}^{\mathrm{pf}}(R_{\mathfrak{p}})\}$$

   is a prime thick subcategory.

2. If $(R, \mathfrak{m})$ is a complete intersection local ring, then

$$\mathcal{S}^{\mathrm{b}}(\mathfrak{m}) = \mathbf{0}$$

   is a prime thick subcategory.

3. If $(R, \mathfrak{m})$ is a hypersurface local ring, then

$$\mathcal{S}^{\mathrm{sg}}(\mathfrak{p}) = \{M \in \mathrm{D}^{\mathrm{sg}}(R) \mid M_{\mathfrak{p}} \cong 0 \text{ in } \mathrm{D}^{\mathrm{sg}}(R_{\mathfrak{p}})\}$$

   is a prime thick subcategory for each $\mathfrak{p} \in \mathbf{Sing}\, R$.

Applications to derived categories of a noetherian scheme

Theorem B

Let X be a separated noetherian scheme.

1. For any $x \in X$, $\mathcal{S}^{\mathsf{pf}}(x)$ is a prime thick subcategory of $\mathsf{D}^{\mathsf{pf}}(X)$.
2. For any $x \in X$, $\mathcal{S}^{\mathsf{b}}(x)$ is a prime thick subcategory of $\mathsf{D}^{\mathsf{b}}(X)$ iff $\mathcal{O}_{X,x}$ is complete intersection.
3. For any $x \in \mathbf{Sing}(X)$, if $\mathcal{O}_{X,x}$ is hypersurface, then $\mathcal{S}^{\mathsf{sg}}(x)$ is a prime thick subcategory. The converse holds if $\mathcal{O}_{X,x}$ is complete intersection.

- The converse of (3) does not holds in general without complete intersection assumption.
- (3) is a paraphrase of Takahashi (2021) for a Zariski spectrum of a noetherian local ring $(R, \mathfrak{m})$ and $x = \mathfrak{m}$.
- This theorem gives a categorical characterization of hypersurface points and complete intersection points.

Applications to derived categories of a noetherian scheme

Corollary

For a radical ideal $\mathcal{P} \subseteq \mathrm{D}^{\mathrm{pf}}(X)$,

$$\mathcal{P} \in \mathbf{Spec}_{\otimes}(\mathrm{D}^{\mathrm{pf}}(X)) \Longleftrightarrow \mathcal{P} \in \mathbf{Spec}_{\triangle}(\mathrm{D}^{\mathrm{pf}}(X))$$

$\Longleftarrow$ holds for a general tensor triangulated category.

Corollary

There are immersion of topological spaces:

1. $X \hookrightarrow \mathbf{Spec}_{\triangle}(\mathrm{D}^{\mathrm{pf}}(X))$
2. $\mathrm{CI}(X) \hookrightarrow \mathbf{Spec}_{\triangle}(\mathrm{D}^{\mathrm{b}}(X))$,
   where $\mathrm{CI}(X) := \{x \in X \mid \mathcal{O}_{X,x} : \text{ complete intersection}\}$.
3. $\mathrm{HS}(X) \hookrightarrow \mathbf{Spec}_{\triangle}(\mathrm{D}^{\mathrm{sg}}(X))$,
   where $\mathrm{HS}(X) := \{x \in X \mid \mathcal{O}_{X,x} : \text{ singular hypersurface}\}$.

Restrict the RHS of (1) to radical ideals
$\Rightarrow X \cong \mathbf{Spec}_{\triangle}(\mathrm{D}^{\mathrm{pf}}(X) \cap \mathbf{Rad}_{\otimes}(\mathrm{D}^{\mathrm{pf}}(X)) = \mathbf{Spec}_{\otimes}(\mathrm{D}^{\mathrm{pf}}(X))$

Sketch of the proof of Theorem B

(1) Reduce to affine case $X = \mathbf{Spec}\, R$, $x = \mathfrak{p}$:

Proposition

For an affine open $U \subseteq X$ with $Z := X \setminus U$

$$\mathrm{D}^*(X)/\mathrm{D}^*_Z(X) \cong \mathrm{D}^*(U)$$

Here, $\mathrm{D}^*_Z(X) = \bigcap_{x \in U} \mathcal{S}^*(x)$.

- $* = \mathrm{pf}$: Thomason-Trobaugh (1990), Balmer(2002)
- $* = \mathrm{b}$: Keller (1999), Schlichiting (2008)
- $* = \mathrm{sg}$: Orlov (2004, 2011), Chen (2010) (Under (ELF) condition)

For $* = \mathrm{sg}$, we can remove (ELF) assumption using Krause's stable derived category $S(X)$.

- $S(X)$ is compactly generated triang. cat. such that $S(X)^c \cong \mathrm{D}^{\mathrm{sg}}(X)$.
- $S(X)/S_Z(X) \cong S(U)$

Sketch of the proof of Theorem B

(2) Reduce to local case $(R, \mathfrak{m})$, $x = \mathfrak{m}$, and $\mathcal{S}^*(\mathfrak{m}) = \mathbf{0}$:

Proposition

For a commutative noetherian ring R and $\mathfrak{p} \in \mathbf{Spec}\, R$

$$\mathrm{D}^*(R)/\mathcal{S}^*(\mathfrak{p}) \cong \mathrm{D}^*(R_\mathfrak{p})$$

- Hom commutes with localization $\Rightarrow \mathrm{D}^*(R)/\mathcal{S}^*(\mathfrak{p}) \to \mathrm{D}^*(R_\mathfrak{p})$ is fully faithful.
- Check essentially surjective directly.

(3) Check whether $\mathbf{0}$ is prime or not. We use:

- $* =$ pf: Hopkins-Neeman (1992)
- $* =$ b : Dwyer-Greenlees-Iyengar (2006), Pollitz (2019)
- $* =$ sg: Takahashi (2021)

Further problem

For a noetherian scheme X its Fourier-Mukai partner is a noetherian scheme Y such that $\mathrm{D}^{\mathrm{pf}}(X) \cong \mathrm{D}^{\mathrm{pf}}(Y)$ as tringlated categories.

$$\Longrightarrow Y \subseteq \mathbf{Spec}_{\otimes}(\mathrm{D}^{\mathrm{pf}}(Y)) \cong \mathbf{Spec}_{\otimes}(\mathrm{D}^{\mathrm{pf}}(X))$$

Question

For a noetherian scheme X, when does the inclusion

$$\bigcup \{Y \mid Y \text{ is a FM partner of } X\} \subseteq \mathbf{Spec}_{\otimes}(\mathrm{D}^{\mathrm{pf}}(X))$$

become equality?

Thank you for your kind attention!