Mutation in hereditary extriangulated categories.

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Université de Picardie Jules Verne

October 14th 2021 - FD Seminar

Introduction

- Cluster algebras

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- Cluster algebras: mutation

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- Cluster tilting

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- Two-term silting

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Aim

Those mutations arise because of the presence of some "nice" extriangulated structures.

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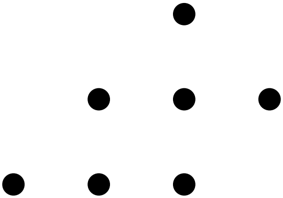
Theorem (Koenig–Yang, Brüstle–Yang)

Fix a co-t-structure $(\mathscr{A}, \mathscr{B})$. Then, there is a mutation theory for intermediate co-t-structures $(\mathscr{A}', \mathscr{B}')$, where the mutation changes precisely one indecomposable isoclass in $(\Sigma\mathscr{A}') \cap \mathscr{B}'$.

I.4 - Non-kissing facets



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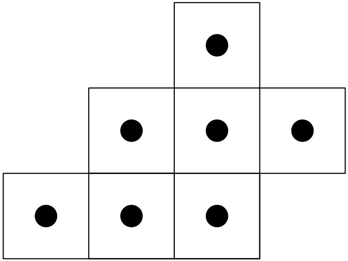

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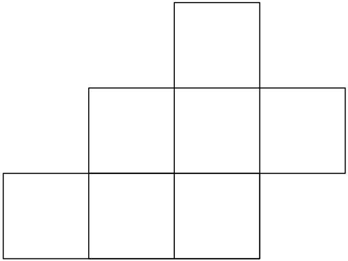

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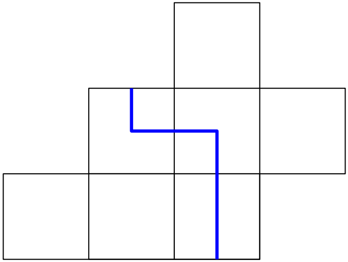

A walk

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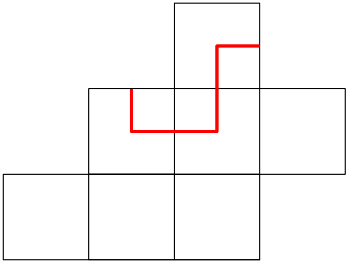

Not a walk

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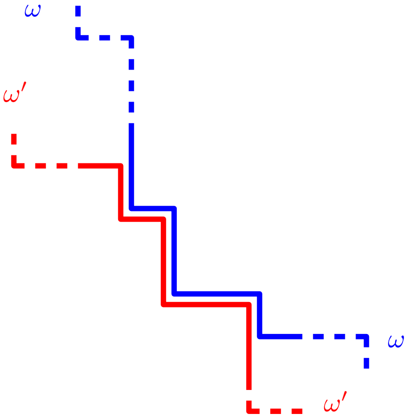

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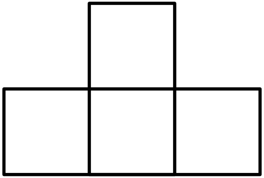

A grid

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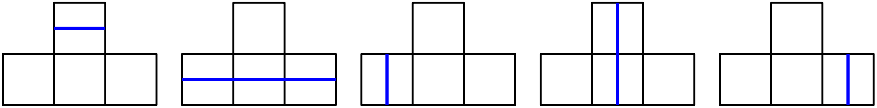

Straight walks

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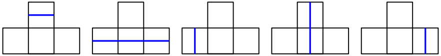

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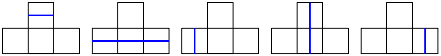

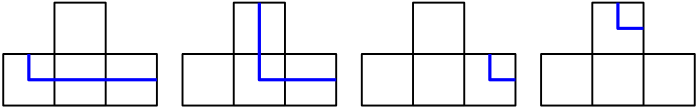

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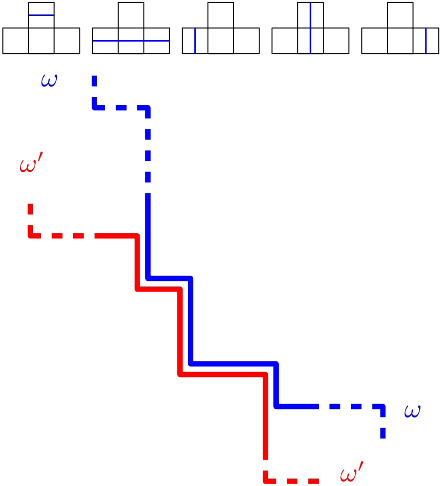

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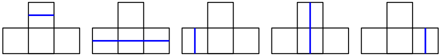

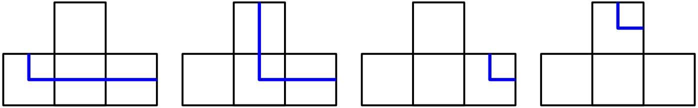

A reduced non-kissing facet

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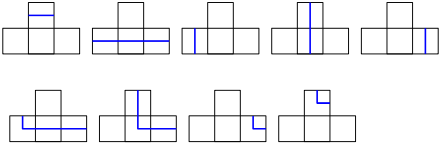

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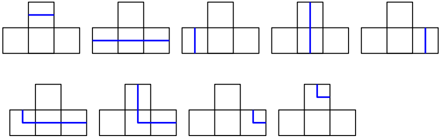

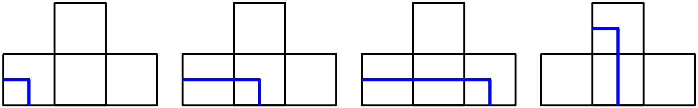

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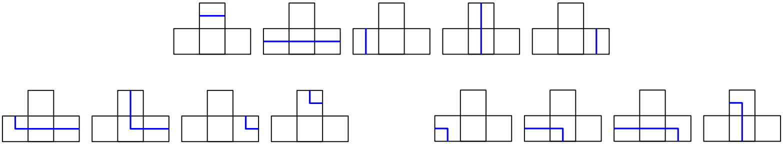

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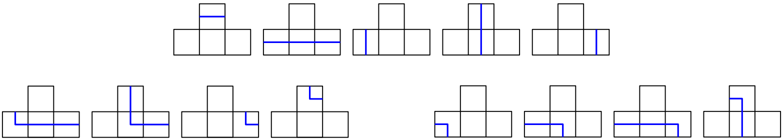

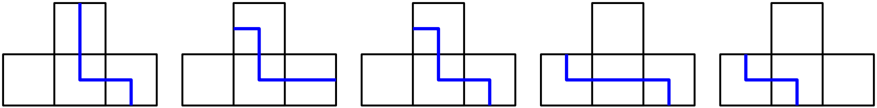

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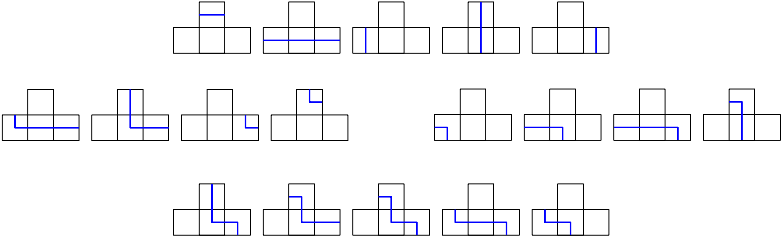

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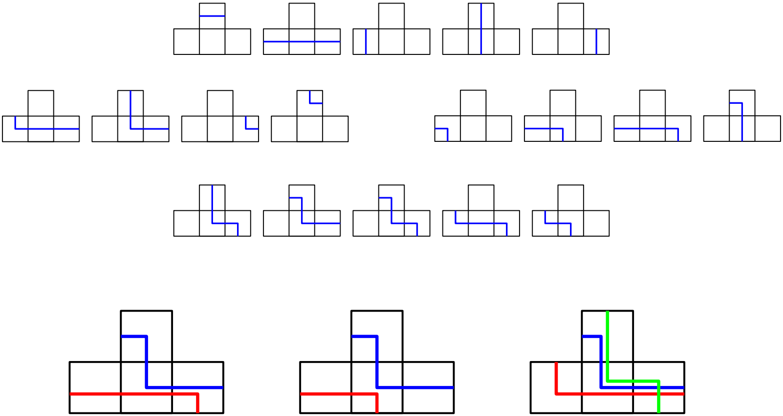

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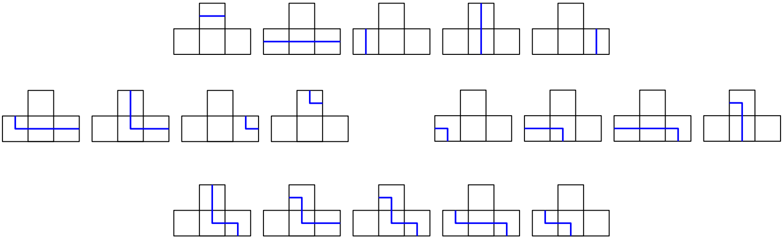

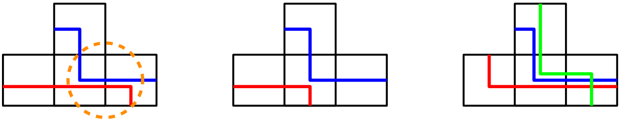

A kiss

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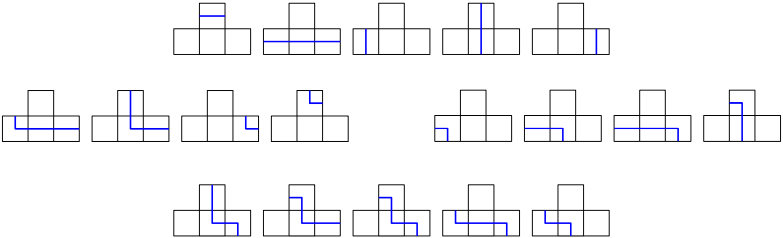

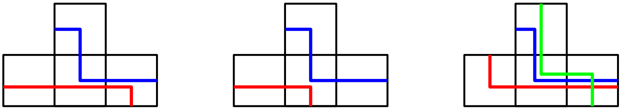

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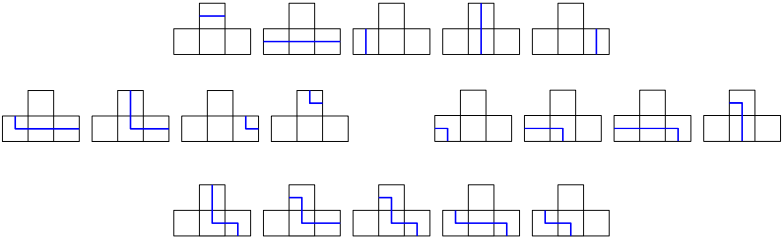

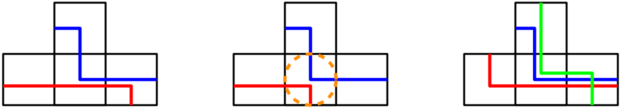

A kiss

A (shy) kiss

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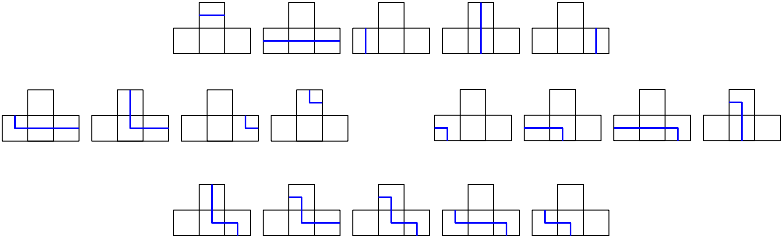

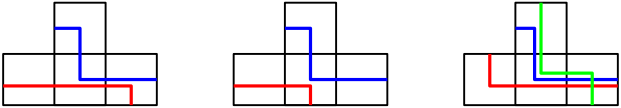

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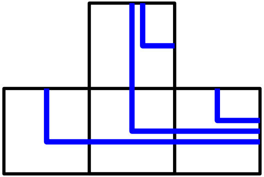

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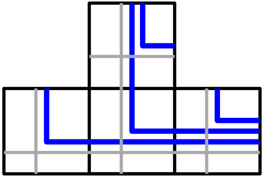

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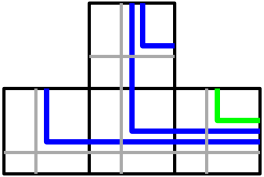

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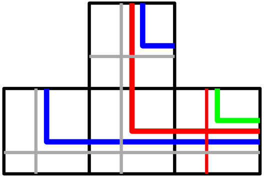

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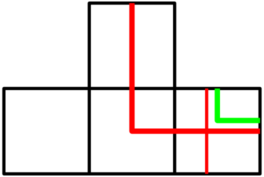

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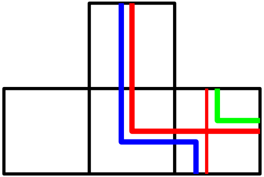

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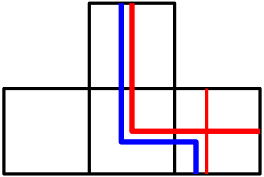

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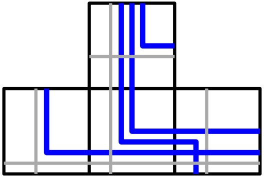

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Definition (Dehy–Keller, P.)

$\mathrm{ind}_T X = (a_i - b_i) \in \mathrm{K}_0^{\mathrm{split}}(addT)$.

II.1 - g-vectors and cluster categories

As in I.1, $\mathscr{C}$ Krull–Schmidt, 2-Calabi–Yau, triangulated category with a basic cluster tilting object $T = T_1 \oplus \cdots \oplus T_n$.

Proposition (Buan–Marsh–Reiten, Keller–Reiten)

For all $X \in \mathscr{C}$, there is a triangle $\oplus T_i^{b_i} \to \oplus T_i^{a_i} \to X \to \Sigma T^{\underline{b}}$.

Definition (Dehy–Keller, P.)

$\mathrm{ind}_T X = (a_i - b_i) \in \mathrm{K}_0^{\mathrm{split}}(addT)$.

Remark (P.)

Given a triangle $X \to Y \to Z \xrightarrow{\varepsilon} \Sigma X$, we have

$$\mathrm{ind}_T Y = \mathrm{ind}_T X + \mathrm{ind}_T Z \Leftrightarrow \varepsilon \in (\Sigma T).$$

II.1 - g-vectors and cluster categories

Idea (Padrol–P.–Pilaud–Plamondon)

When studying g-vectors, endow $\mathscr{C}$ with the subclass Δ_T of triangles of the form $X \to Y \to Z \xrightarrow{(\Sigma T)} \Sigma X$.

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Moreover, we have :

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II.1 - g-vectors and cluster categories

$$\begin{array}{ccccccc} & & & & T & & \\ & & & & \downarrow & & \\ X & \longrightarrow & Y & \longrightarrow & Z & \xrightarrow[(\Sigma T)]{} & \Sigma X. \end{array}$$

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II.1 - g-vectors and cluster categories

$$\begin{tikzcd} & & T \arrow[dl, dashed] \arrow[d] \arrow[dr, "0"] & \\ X \arrow[r] & Y \arrow[r] & Z \arrow[r, "(\Sigma T)"'] & \Sigma X. \end{tikzcd}$$

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- $T \to 0 \to \Sigma T \xrightarrow{1} \Sigma T \in \Delta_T$.

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$\Rightarrow$ extriangles $X \rightarrowtail Y \twoheadrightarrow Z \dashrightarrow$

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$\mathscr{C}$ is reduced 0-Auslander if 0 is the only projective-injective.

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extended coheart : $\mathscr{C} = \mathscr{S} * \Sigma\mathscr{S} = (\Sigma^2\mathscr{A}) \cap \mathscr{B}$

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$\mathscr{C}$ reduced 0-Auslander extriangulated, Hom-finite, Krull–Schmidt with a projective generator P. Then

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- *Conditions (i) to (iv) are equivalent.*

III.2 - Mutation

Theorem

$\mathscr{C}$ reduced 0-Auslander extriangulated, Krull–Schmidt with a basic tilting object $R = \overline{R} \oplus X$ where X indecomposable.

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$$X \rightarrowtail E \twoheadrightarrow Y \dashrightarrow \quad \textit{or} \quad Y \rightarrowtail E' \twoheadrightarrow X \dashrightarrow$$

with E or E' in $\operatorname{add} \overline{R}$.

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Corollary

Applied to $(\mathscr{C}, \Delta_T)$, recovers cluster tilting mutation.

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Corollary

Applied to $\mathcal{K}^{[-1,0]}(\operatorname{proj} \Lambda)$, recovers 2-term silting mutation.

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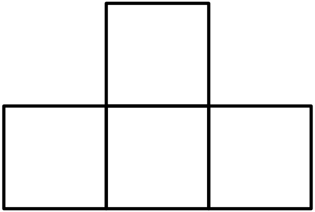
Corollary

Applied to extended cohearts and combined with a theorem by Adachi–Tsukamoto, recovers mutation of intermediate co-t-structures.

III.3 - Flips are mutations



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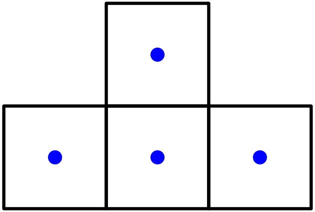

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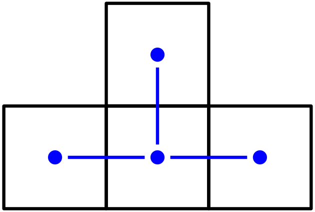

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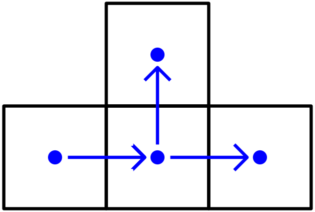

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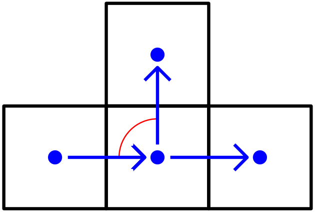

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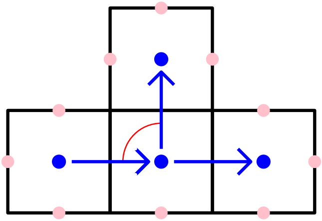

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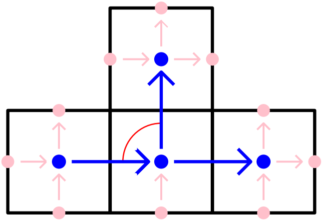

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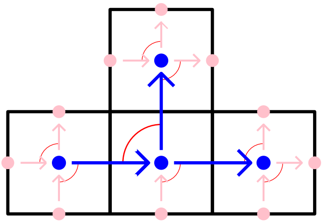

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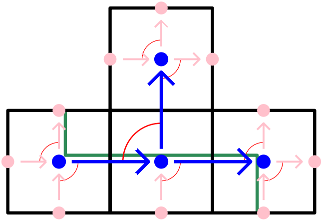

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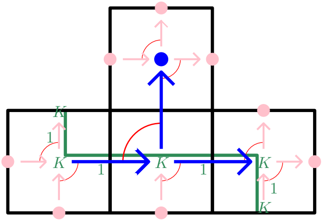

III.3 - Flips are mutations

Let (Q, I) be the (gentle) blossming bound quiver associated with a grid.

Definition

The category of walks $\mathscr{W}$ is the full, additive subcategory of $\operatorname{mod} \mathbb{K}Q/I$ whose indecomposable objects are the indecomposable representations associated with walks in the grid.

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For a walk ω, write M_ω for the associated indecomposable representation.

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Theorem

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- *$\mathscr{W}$ is 0-Auslander.*
- *A walk ω kisses a walk γ if and only if $\operatorname{Ext}^1_{\mathscr{W}}(M_\gamma, M_\omega) \neq 0$.*

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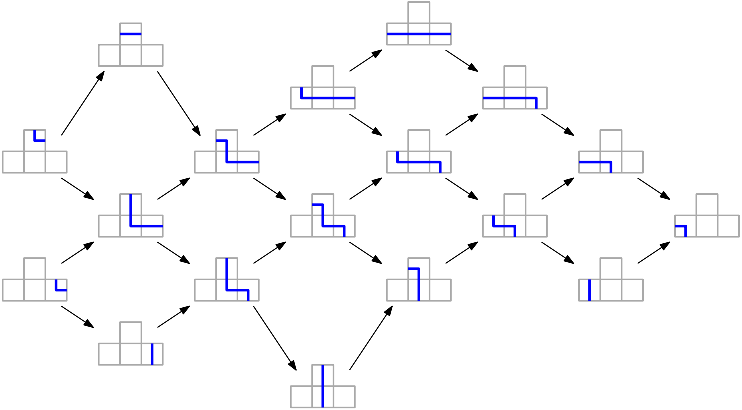
III.3 - Flips are mutations

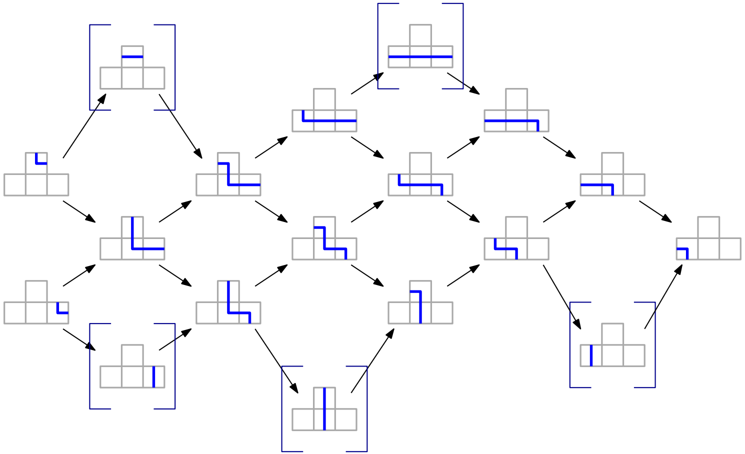
Corollary

- Non-kissing facets correspond to tilting objects in $\mathscr{W}$.
- Their flips correspond to mutation in $\mathscr{W}$.

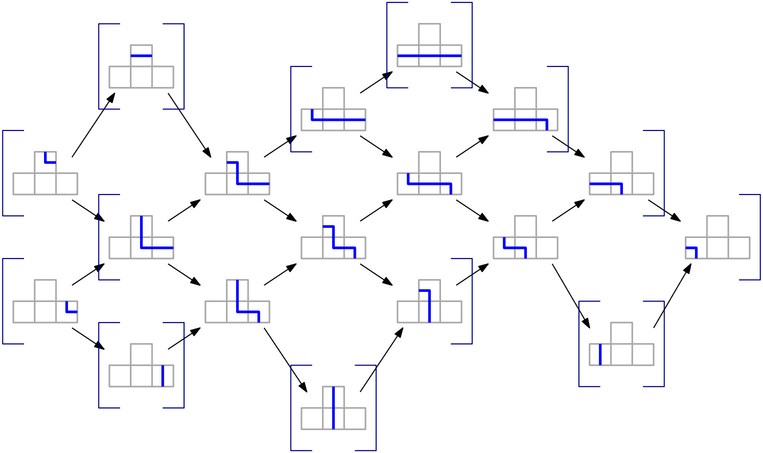
III.3 - Flips are mutations



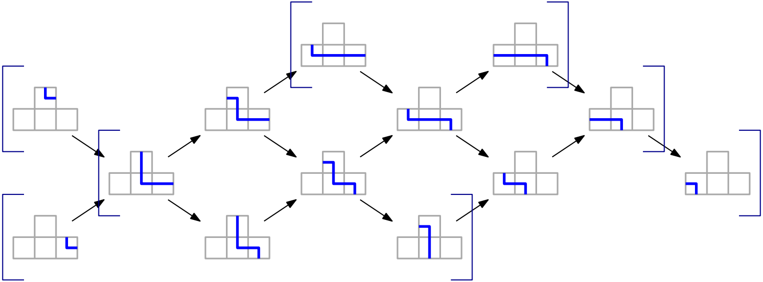
III.3 - Flips are mutations



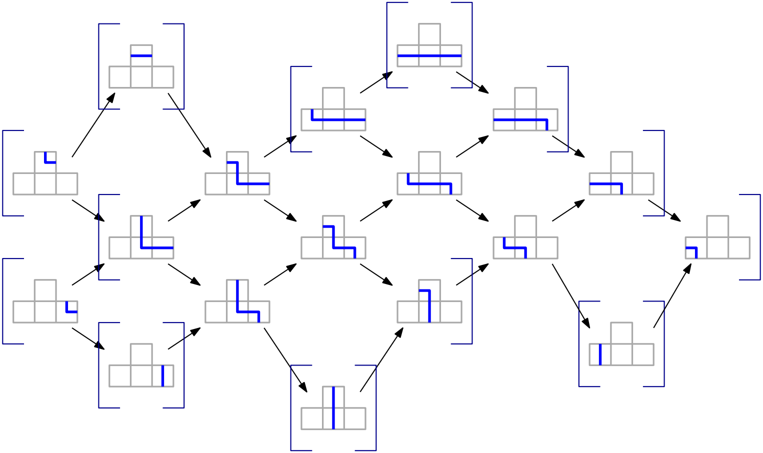
III.3 - Flips are mutations



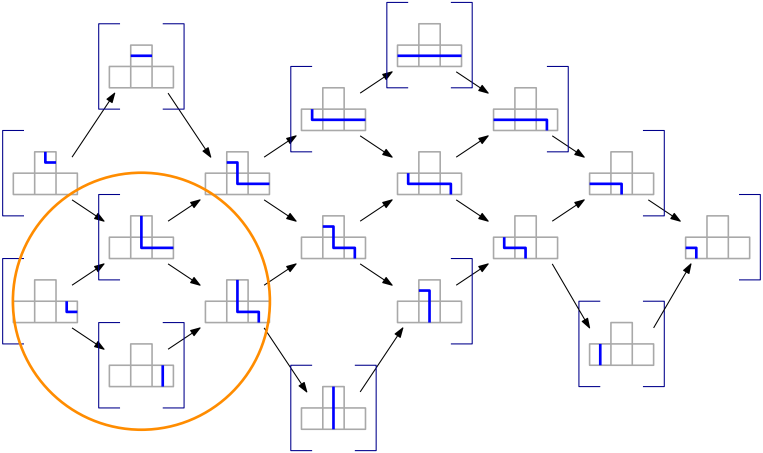
III.3 - Flips are mutations



III.3 - Flips are mutations



III.3 - Flips are mutations



III.3 - Flips are mutations

Thank you for your attention!