Double covers of quiver Heisenberg algebras as higher preprojective algebras

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Quivers and path algebras

- K: field, $K = \overline{K}$, $char\, K = 0$.
- $D = \mathrm{Hom}_K(-, K)$.
- Q: finite connected acyclic quiver with at least one arrow.
- KQ: path algebra.
- Note $\dim KQ < \infty$ and $\mathrm{gl.dim}\, KQ = 1$.

Theorem (Gabriel)

KQ is representation finite if and only if Q is Dynkin.

<u>Ex</u> $Q: 1 \xrightarrow{a} 2 \xrightarrow{b} 3$ $\qquad ab \in KQe_3 = P(3)$

KQ-mod

$$\begin{tikzcd}
 & & \begin{smallmatrix}3\\2\\1\end{smallmatrix} \arrow[dr] & & \\
 & \begin{smallmatrix}2\\1\end{smallmatrix} \arrow[ur] \arrow[dr] & & \begin{smallmatrix}3\\2\end{smallmatrix} \arrow[dr] & \\
1 \arrow[ur] & & 2 \arrow[ur] & & 3
\end{tikzcd}$$

$$\begin{tikzcd}
 & & P(3) \arrow[dr] & & \\
 & P(2) \arrow[ur] \arrow[dr] & & \tau^{-}P(2) \arrow[dr] & \\
P(1) \arrow[ur] & & \tau^{-}P(1) \arrow[ur] & & \tau^{-2}P(1)
\end{tikzcd}$$

Preprojective algebras

- $\overline{Q}$: double of Q.
- $\rho := \sum_{a \in Q_1} aa^* - a^*a \in K\overline{Q}$.
- $\rho_i := e_i\rho = \rho e_i$. $= \sum_{s(a)=i} aa^* - \sum_{t(a)=i} a^*a$

$\forall\, i \xrightarrow{a} j$ in Q: $i \underset{a^*}{\overset{a}{\rightleftarrows}} j$

$\deg a = 0$
$\deg a^* = 1$

Definition (Gelfand-Ponomarev)

The preprojective algebra of Q is $\Pi := K\overline{Q}/\langle\rho\rangle = K\overline{Q}/\langle \rho_i \mid i \in Q_0\rangle$

Theorem (Crawley-Boevy)

$\Pi \simeq T_{KQ}\,\mathrm{Ext}^1_{KQ}(D(KQ), KQ)$. $\Pi_0 = KQ$ $\Pi_m = \tau^{-m} KQ$

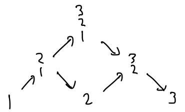
Ex Q: $1 \xrightarrow{a} 2 \xrightarrow{b} 3$ $\overline{Q}$: $1 \underset{a^*}{\overset{a}{\rightleftarrows}} 2 \underset{b^*}{\overset{b}{\rightleftarrows}} 3$

$aa^* = 0$
$a^*a = bb^*$
$b^*b = 0$

$\Pi e_1 \cong D(e_3\Pi)$: $\begin{matrix}1\\2\\3\end{matrix}$, deg $0, 1, 2$

$\Pi e_2 \cong D(e_2\Pi)$: $\begin{matrix} & 2 & \\ 1 & & 3\\ & 2 & \end{matrix}$

$\Pi e_3 \cong D(e_1\Pi)$: $\begin{matrix}3\\2\\1\end{matrix}$



Calabi-Yau properties

Theorem (Auslander-Reiten, Crawley-Boevy)

1. *If Q is Dynkin, then*

$$\dim \Pi < \infty, \quad \Pi\colon \textit{selfinjective}, \quad \Pi - \underline{\mathrm{mod}}\colon 2\text{-}\textit{Calabi-Yau}.$$

2. *If Q is non-Dynkin, then*

$$\dim \Pi = \infty, \quad \mathrm{gl.dim}\,\Pi = 2, \quad \mathsf{D}_{\mathsf{fd}}(\Pi)\colon 2\text{-}\textit{Calabi-Yau}.$$

In ① $\mathrm{add}_{KQ}\Pi = KQ\text{-mod}$ Ex $Q:\ 1 \xrightarrow{a} 2 \xrightarrow{b} 3$

In ② $\mathrm{add}_{KQ}\Pi \subsetneq KQ\text{-mod}$ Ex $Q:\ 1 \overset{a}{\underset{b}{\rightrightarrows}} 2$

preprojective modules

$$\begin{tikzcd}[column sep=small] & P(2) \arrow[dr, shift left] \arrow[dr, shift right] & & \tau^{-}P(2) & \cdots \\ P(1) \arrow[ur, shift left] \arrow[ur, shift right] & & \tau^{-}P(1) \arrow[ur, shift left] \arrow[ur, shift right] & & \cdots \end{tikzcd}$$

Quiver Heisenberg algebras

$[x,[x,y]]=0 \quad [y,[x,y]]=0$

- For $\alpha \in \overline{Q}_1$ set $\eta_\alpha = [\alpha, \rho] = \alpha\rho - \rho\alpha$.

Definition (H-Minamoto)

The quiver Heisenberg algebra of Q is $\Lambda := K\overline{Q}/\langle \eta_\alpha \mid \alpha \in \overline{Q} \rangle$.

Remark

- *More general version modified by parameters* $v \in (K^\times)^{Q_0}$.
- *Etingof-Rains: central extensions of preprojective algebras.*
- *Cachazo-Katz-Vafa:* N*-quiver algebras.*

Note $\rho \in Z(\Lambda)$ $\quad \Lambda/\langle \rho \rangle = kQ/\langle \rho \rangle = \Pi$

$$\rightsquigarrow \quad \Lambda \xrightarrow{\rho} \Lambda \longrightarrow \Pi \longrightarrow 0$$

exact sequence of Λ-Λ-bimodules.

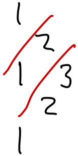
Example

Let $Q: 1 \xrightarrow{a} 2 \xrightarrow{b} 3$. Then $\overline{Q}$:

$$\begin{tikzcd} 1 \arrow[r, bend left, "a"] & 2 \arrow[l, bend left, "a^*"] \arrow[r, bend left, "b"] & 3 \arrow[l, bend left, "b^*"] \end{tikzcd}$$

$$\begin{aligned} \eta_a &= abb^* - 2aa^*a \\ \eta_b &= a^*ab - 2bb^*b \\ \eta_{a^*} &= 2a^*aa^* - bb^*a^* \\ \eta_{b^*} &= 2b^*bb^* - b^*a^*a \end{aligned}$$

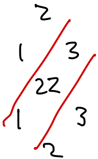
Λe_1



$\cong$

$D(e_1\Lambda)$

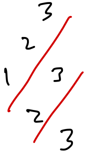
Λe_2



$\cong$

$D(e_2\Lambda)$

Λe_3



$\cong$

$D(e_3\Lambda)$

Main results

[Etingof-Rains] $\dim \Lambda = \sum (\dim M)^2$
Q : Dynkin.

Theorem (H-Minamoto)

As KQ-modules $\Lambda e_i \simeq \bigoplus_M M^{\oplus \dim e_i M}$, where the direct sum is taken over all indecomposable preprojective KQ-modules M.

Theorem (H-Minamoto, Etingof-Latour-Rains, Eu-Schedler)

1. *If Q is Dynkin, then*

$$\dim \Lambda < \infty, \quad \Lambda \colon \textit{symmetric}, \quad \Lambda - \underline{\mathrm{mod}} \colon 3\text{-}\textit{Calabi-Yau}$$

$$0 \to D(\Pi) \to \Lambda \xrightarrow{\rho} \Lambda \to \Pi \to 0 \quad \textit{is exact.}$$

2. *If Q is non-Dynkin, then*

$$\dim \Lambda = \infty, \quad \mathrm{gl.dim}\, \Lambda = 3, \quad \mathsf{D}_{\mathsf{fd}}(\Pi) \colon 3\text{-}\textit{Calabi-Yau}$$

$$0 \to \Lambda \xrightarrow{\rho} \Lambda \to \Pi \to 0 \quad \textit{is exact.}$$

Higher preprojective algebras

Let A: finite dimensional K-algebra with $\operatorname{gl.dim} A = n$.
Set $\nu = D\,\mathrm{RHom}(-, A)\colon\ \mathsf{D}^{\mathsf{b}}(A) \to \mathsf{D}^{\mathsf{b}}(A)$ and $\nu_n = \nu \circ [-n]$.

Definition (Iyama-Oppermann)

The $(n+1)$-preprojective algebra of A is $\Pi_{n+1}(A) := T_A \operatorname{Ext}^n_A(D(A), A)$

If $A = KQ$, then $\Pi_2(KQ) = \Pi$.

Definition (Iyama-Oppermann, H-Iyama-Oppermann)

1. A is n-hereditary if $H^i(\nu_n^j(A)) = 0$ for $j \in \mathbb{Z}$ and $i \notin n\mathbb{Z}$.
2. A is n-representation finite if there is an n-cluster tilting A-module.
3. A is n-representation infinite if $H^i(\nu_n^j(A)) = 0$ for all $j \leq 0$ and $i \neq 0$.

Theorem (H-Iyama-Oppermann)

If A is ring indecomposable, then A is n-hereditary if and only if A is n-representation finite or n-representation infinite.

Higher preprojective algebras

Theorem (Iyama-Oppermann, Amiot-Iyama-Reiten)

1. *If A is n-representation finite, then*

$$\dim \Pi_{n+1}(A) < \infty, \quad \Pi_{n+1}(A) : \textit{selfinjective},$$

$$\Pi_{n+1}(A) - \underline{\mathsf{mod}} : (n+1)\textit{-Calabi-Yau.}$$

2. *If A is n-representation infinite, then*

$$\dim \Pi_{n+1}(A) = \infty, \quad \mathrm{gl.dim}\, \Pi_{n+1}(A) = n+1,$$

$$\mathsf{D}_{\mathsf{fd}}(\Pi) : (n+1)\textit{-Calabi-Yau.}$$

In 1 $\Pi_{n+1}(A) \in A\text{-mod}$ is n-cluster tilting

In 2 $\nu_n^{-i}(A) = (\Pi_{n+1}(A))_i \in A\text{-mod}$

higher preprojective modules.

Q Is Λ a 3-preprojective algebra?

Quivers with potentials

Theorem (Keller)

Assume $A = KQ/I$, where I is admissible. Then there is a quiver with potential $(\tilde{Q}, W)$ such that $\Pi_3(A) \simeq \mathcal{P}(\tilde{Q}, W)$.

Ex $\quad Q : 1 \xrightarrow{a} 2 \xrightarrow{b} 3 \qquad I = \langle ab \rangle \qquad A = KQ/_I$

$\operatorname{gldim} A = 2$

$$\begin{tikzcd} & 2 \arrow[dr, "b"] & \\ 1 \arrow[ur, "a"] \arrow[rr, dotted] & & 3 \end{tikzcd}$$

$$\tilde{Q} : \begin{tikzcd} & 2 \arrow[dr, "b"] & \\ 1 \arrow[ur, "a"] & & 3 \arrow[ll, "c"] \end{tikzcd} \qquad W = abc$$

$$\Pi_3(A) = \mathcal{P}(\tilde{Q}, W) = K\tilde{Q}/\langle \partial_\alpha W \mid \alpha \in \tilde{Q}_1 \rangle$$

$$\partial_a W = bc \qquad \partial_b W = ca \qquad \partial_c W = ab$$

Rmk $\quad \Lambda = K\overline{Q}/\langle \eta_\alpha \mid \alpha \in \overline{Q}_1 \rangle = \mathcal{P}(\overline{Q}, -\tfrac{1}{2}\rho^2)$

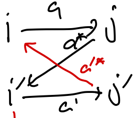
problem

$\deg(\rho^2) = 2.$

Double cover

$$\Lambda^{[2]} := \begin{bmatrix} \Lambda_0 & \Lambda_1 \\ 0 & \Lambda_0 \end{bmatrix} \oplus \begin{bmatrix} \Lambda_2 & \Lambda_3 \\ \Lambda_1 & \Lambda_2 \end{bmatrix} \oplus \begin{bmatrix} \Lambda_4 & \Lambda_5 \\ \Lambda_3 & \Lambda_4 \end{bmatrix} \oplus \cdots$$

$$B := \Lambda^{[2]}_0 = \begin{bmatrix} \Lambda_0 & \Lambda_1 \\ 0 & \Lambda_0 \end{bmatrix} = \begin{bmatrix} KQ & \Lambda_1 \\ 0 & KQ \end{bmatrix}$$



Define the quiver $\overline{Q}^{[2]}$ by

- two vertices i, i' for all $i \in Q_0$,
- deg four arrows $a\colon i \xrightarrow{0} j$, $a'\colon i' \xrightarrow{0} j'$, $a^*\colon j \xrightarrow{0} i'$, $a'^*\colon j' \xrightarrow{1} i$, for all arrows $a\colon i \to j$ in Q.

$$\rho := \sum_{a \in Q_1} aa^* - a^*a' \in K\overline{Q}^{[2]} \qquad \rho' := \sum_{a \in Q_1} a'a'^* - a'^*a \in K\overline{Q}^{[2]}$$

Proposition

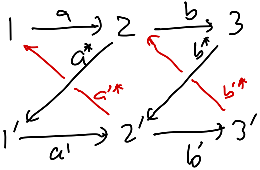
We have $\Lambda^{[2]} = \mathcal{P}(\overline{Q}^{[2]}, -\rho\rho')$ *and* $B = \mathcal{P}(\overline{Q}^{[2]}, -\rho\rho')_0$

Example

Let $Q: 1 \xrightarrow{a} 2 \xrightarrow{b} 3$. Then $\overline{Q}$:
$$\begin{tikzcd} 1 \arrow[r, bend left, "a"] & 2 \arrow[l, bend left, "a^*"] \arrow[r, bend left, "b"] & 3 \arrow[l, bend left, "b^*"] \end{tikzcd}$$
and

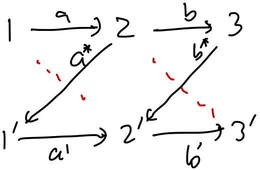
$$-\rho\rho' = -2aa^*a'a'^* + abb^*a'^* + a'b'b'^*a^* - 2bb^*b'b'^*$$

$\overline{Q}^{[2]}$



$W = -\rho \cdot \rho'$

B



$abb^* = 2aa^*a'$

$a^*a'b' = 2bb^*b'$

2-hereditary algebras

Theorem (H-Minamoto)

1. $\mathrm{gl.dim}\, B = 2$ *and* $\Pi_3(B) \simeq \Lambda^{[2]}$. *Moreover,* B *is* 2*-hereditary.*
2. *If* Q *is Dynkin, then* B *is* 2*-representation finite.*
3. *If* Q *is non-Dynkin, then* B *is* 2*-representation infinite.*

In 2 $\Lambda^{[2]} \in B\text{-mod}$ is 2-cluster tilting

$\bar{Q}^{[2]} \rightsquigarrow$ quiver of $\mathrm{add}\,\Lambda^{[2]} \overset{2\text{-CT}}{\subseteq} B\text{-mod}$

Ex $1 \to 2 \to 3$

$$\begin{tikzcd}
P(1) \arrow[r] & P(2) \arrow[r] \arrow[dl] & P(3) \arrow[dl] \\
P(1') \arrow[r] & P(2') \arrow[r] \arrow[dl] & P(3') \arrow[dl] \\
\tau_2^- P(1) \arrow[r] & \tau_2^- P(2) \arrow[r] \arrow[dl] & \tau_2^- P(3) \arrow[dl] \\
\tau_2^- P(1') \arrow[r] & \tau_2^- P(2') \arrow[r] & \tau_2^- P(3')
\end{tikzcd}$$