THE HEISENBERG CATEGORY OF A CATEGORY

QUANTUM MECHANICS $\rightsquigarrow$

INFINITE-DIMENSIONAL HEISENBERG ALGEBRA H_k (k - BASE FIELD, CHAR $k=0$)

GENERATORS: $a(n)$ $\quad n\in\mathbb{Z}$ $\qquad a(0)=1$

RELATIONS: $[a(m),a(n)]=\delta_{n,-m}\,m$

NB: WE CAN THINK OF H_k AS THE RING OF DIFFERENTIAL OPERATORS ON $k[x_1,x_2,\dots]$ VIA

$$a(n)=\frac{\partial}{\partial x_n},\quad a(-n)=n x_n \quad \forall n>0 \qquad (\dagger)$$

FOCK SPACE REPRESENTATION F_k:

$k[x_1,x_2,\dots]$ WITH H_k ACTING AS IN $(\dagger)$. IRREDUCIBLE, FAITHFUL

ALTERNATIVE PRESENTATION OF H_k:

GENERATORS: $p^{(n)}, q^{(n)}$ $\quad n>0$

RELATIONS: $q^{(m)}p^{(n)}=\sum_{i=0}^{\min(m,n)} p^{(n-i)}q^{(m-i)}$

e.g. $q^{(3)}p^{(2)}=p^{(2)}q^{(3)}+p^{(1)}q^{(2)}+q^{(1)}$

$$\sum_{n\geq 0} p^{(n)}z^n=\exp\Big(\sum_{m\geq 1}\frac{a(-m)}{m}z^m\Big)$$

$$\sum_{n\geq 0} q^{(n)}z^n=\exp\Big(\sum_{m\geq 1}\frac{a(m)}{m}z^m\Big)$$

$p^{(1)}=a(-1) \qquad q^{(1)}=a(1)$

IN THESE TERMS, $F_k=H_k/\langle q^{(n)}\mid n\neq 0\rangle$ (LEFT IDEAL)

HEISENBERG ALGEBRA H_M OF A LATTICE (M,χ):

M: FREE ABELIAN GROUP OF FIN. RANK.

χ: BILINEAR FORM $M\times M\longrightarrow\mathbb{Z}$.

H_M: GENERATORS $p_a^{(n)}, q_a^{(n)}$ $\quad \forall a\in M,\ n\geq 0$

RELATIONS:

- $p_a^{(0)}=q_a^{(0)}=1$
- $p_{a+b}^{(n)}=\sum_{i=0}^{n} p_a^{(i)}p_b^{(n-i)}$ AND $q_{a+b}^{(n)}=\sum_{i=0}^{n} q_a^{(i)}q_b^{(n-i)}$
- $p_a^{(n)}p_b^{(m)}=p_b^{(m)}p_a^{(n)}$ AND $q_a^{(n)}q_b^{(m)}=q_b^{(m)}q_a^{(n)}$

$$\bullet\ q_a^{(n)} p_b^{(m)} = \sum_{i=0}^{\min(m,n)} s^i(\chi(a,b))\, p_b^{(m-i)} q_a^{(n-i)}$$

HERE $s^i(r) = \frac{1}{i!}(r+i-1)(r+i-2)\dots r$ DIM $S^i\mathbb{C}^r$ $r>0$

SO IF $\chi(a,b)=0$, $q_a^{(n)}, p_a^{(n)}$ COMMUTE WITH $q_b^{(m)}, p_b^{(m)}$

THUS $H_R = H_{\mathbb{Z}}$ WITH $\chi(1,1)=1$.

NAKAJIMA-GROJNOWSKI HEISENBERG ACTION (MID 90s):

X - SMOOTH PROJECTIVE SURFACE

$X^{[n]}$ - HILBERT SCHEME OF n POINTS ON X SMOOTH, RESOLVES X^n/S_n

THEOREM (N, G, 90s):

$H_{H^*(X)}$ ACTS ON $\bigoplus_{n\geq 0} H^*(X^{[n]})$ IDENTIFYING IT WITH $F_{H^*(X)}$

THE CONSTRUCTION IS VERY GEOMETRIC, SUGGESTING IT SHOULD LIFT TO:

COH. RING GROTHENDIECK RING DERIVED CATEGORY

$$H^*(X,\mathbb{Q}) \xleftarrow[\text{ALG. PART}]{\text{REFINES}} K_0(X) \xleftarrow{\text{PROJECTS TO}} D(X)$$

CUP PAIRING

ISO. CLASSES OF SHVS

$0 \to F \hookrightarrow G \twoheadrightarrow H \to 0 \Rightarrow [G]=[F]+[H]$

$\chi([F],[G]) = \sum(-1)^i \dim \mathrm{Ext}^i(F,G)$

COMPLEXES OF SHVS LOCALISED BY QUASI-ISOS

KRUG'S SYMMETRIC QUOTIENT STACK ACTION (2015)

THEOREM (HAIMAN, BRIDGELAND-KING-REID, ~2000):

X - SMOOTH PROJECTIVE SURFACE.

$$D(X^{[n]}) \simeq D^{S_n}(X^n) \simeq D(S^nX) \qquad S^nX := [X^n/S_n]$$

S_n-EQUIVARIANT DERIVED CAT.

SYMMETRIC QUOTIENT STACK

THEOREM (KRUG, 2015):

X - SMOOTH PROPER VARIETY

$H_{K_0(X)}$ ACTS ON $\bigoplus_{n\geq 0} K_0(S^nX)$

KRUG'S ACTION IS CONSTRUCTED ON THE LEVEL OF DERIVED CATEGORIES, HOWEVER THERE IT IS ONLY A WEAK ACTION. THE HEISENBERG RELATIONS ONLY HOLD UP TO AN ISOMORPHISM OF FUNCTORS.

Categorification (an overview)

Algebra A $\rightsquigarrow$ Monoidal (additive, abelian, triag.) category $\mathcal{A}$ with $K_0(\mathcal{A}) \simeq A$

"Generating objects of $\mathcal{A}$ categorify generating elements of A, monoidal structure categorifies the product, and morphisms of $\mathcal{A}$ encode the relations between generators in A"

A-module M $\rightsquigarrow$ Category $\mathcal{M}$ and a monoidal functor $\mathcal{A} \to \mathrm{Fun}(\mathcal{M}, \mathcal{M})$ such that $K_0(\mathcal{M}) \simeq M$ in A-mod.

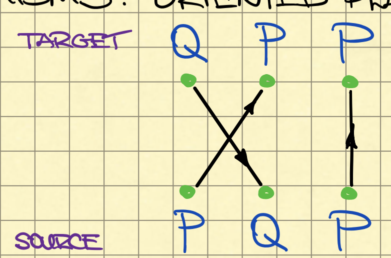
Khovanov's categorification of H_k (2010):

Define additive, k-linear monoidal category $\mathcal{H}_k$ by:

Ob $\mathcal{H}_k$: finite words on symbols P and Q + direct sums

Monoidal structure: product $\otimes$ is concatenation
identity $\mathbb{1}$ is the empty word.

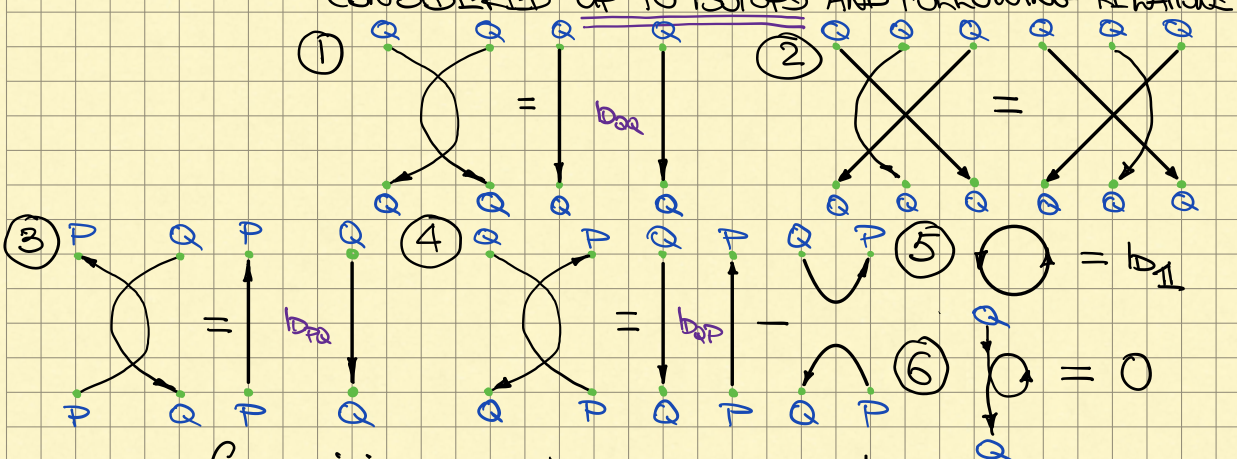
Morphisms: oriented planar diagrams, e.g.



Upwards oriented at P endpoints

Downwards oriented at Q endpoints

Considered up to isotopy and following relations:



Composition: vertical concatenation

EXAMPLE: RELATIONS ③ - ⑥ ENSURE THAT

$$\begin{tikzcd} PQ & \oplus & \mathbb{1} \\ & QP \arrow[ul, "X"] \arrow[ur, "\cap"'] & \end{tikzcd} \quad \text{AND} \quad \begin{tikzcd} & QP & \\ PQ \arrow[ur, "X"] & \oplus & \mathbb{1} \arrow[ul, "\cup"'] \end{tikzcd}$$

ARE MUTUALLY INVERSE ISOMORPHISMS. THUS IN $K_0(\mathcal{H}_R)$ WE HAVE

$$\overset{q^{(1)}}{[Q]}\overset{p^{(1)}}{[P]} = \overset{p^{(1)}}{[P]}\overset{q^{(1)}}{[Q]} + 1 \qquad \text{THE HEISENBERG RELATION}$$

OTOH, RELATIONS ①-② GIVE ACTION OF S_n ON P^n AND Q^n.
WE SET $p^{(n)}$ TO BE THE DIRECT SUMMAND OF $[P^n]$ DEFINED BY THE SYMMETRISING IDEMPOTENT OF S_n, AND SIMILARLY FOR $q^{(n)}$.

THEOREM (K, 2010): THESE DEFINE $H_R \hookrightarrow K_0(\mathcal{H}_R)$.
CONJECTURE: THIS IS AN ISO

CAUTIS-LICATA (2010): CONSTRUCT SIMILAR CATEGORIFICATION FOR THE HEISENBERG ALGEBRA OF AN ADE ROOT LATTICE.

THE HEISENBERG CATEGORY OF A CATEGORY: (GYENGE-KOPPENSTEINER-L, '21).

$\mathcal{A}$ - SMOOTH AND PROPER DG CATEGORY "NON-COMMUTATIVE SMOOTH & PROPER VARIETY"

E.G. THE DG ENHANCEMENT OF $D^b_{coh}(X)$ FOR A SMOOTH & PROPER ALG. VARIETY X

- WE VIEW SUCH $\mathcal{A}$ AS THE ENHANCEMENT OF TRIANGULATED CATEGORY $D_c(\mathcal{A})$, THE DERIVED CATEGORY OF PERFECT $\mathcal{A}$-MODULES.

THE REASON TO WORK WITH DG CATEGORIES:

$S^n(D^b_{coh}(X)) \not\simeq D^b_{coh}(S^nX)$ BUT $\mathcal{A}$ ENHANCES $D^b_{coh}(X)$
$\Downarrow$
$S^n\mathcal{A}$ ENHANCES $D^b_{coh}(S^nX)$

- $D_c(\mathcal{A}) \simeq D_c(\overset{\text{H-INJ PERFECT } \mathcal{A}\text{-MODULES}}{HPerf\mathcal{A}})$

REPLACING $\mathcal{A}$ BY $HPerf\mathcal{A}$ WE CAN ASSUME THAT $\mathcal{A}$ ADMITS A HOMOTOPY SERRE FUNCTOR:

QUASI-EQUIVALENCE $S: \mathcal{A} \to \mathcal{A}$ + NATURAL QUASI-ISOS $Hom_{\mathcal{A}}(a,b) \simeq Hom_{\mathcal{A}}(b,Sa)^*$.

THEOREM (G-K-L, 2021):

THERE EXIST:

- A DG 2-CATEGORY $\mathcal{H}_{\mathcal{A}}$ "THE HEISENBERG CATEGORY OF $\mathcal{A}$"
  Ho(DG-CAT) ENRICHED
- A DG 2-CATEGORY $\mathcal{F}_{\mathcal{A}}$ "THE CATEGORICAL FOCK SPACE"
  - OBJECTS: $S^n\mathcal{A}$ $n \in \mathbb{Z}$
  - 1-MORPHISMS: ENHANCED FUNCTORS $S^n\mathcal{A} \to S^m\mathcal{A}$
  - 2-MORPHISMS: ENHANCED NATURAL TRANSFORMATIONS
- A 2-FUNCTOR $\Phi_{\mathcal{A}}: \mathcal{H}_{\mathcal{A}} \longrightarrow \mathcal{F}_{\mathcal{A}}$ "FOCK SPACE REPRESENTATION"
  HOMOTOPY LAX

SUCH THAT

- $H_{K_0^{num}(\mathcal{A})} \hookrightarrow K_0(\mathcal{H}_{\mathcal{A}})$ "$\mathcal{H}_{\mathcal{A}}$ CATEGORIFIES $H_{K_0^{num}(\mathcal{A})}$"
  CONJECTURE: THIS IS AN ISOMORPHISM (†)
- $F_{K_0^{num}(\mathcal{A})} \hookrightarrow K_0(\mathcal{F}_{\mathcal{A}})$ "$\mathcal{F}_{\mathcal{A}}$ CATEGORIFIES $F_{K_0^{num}(\mathcal{A})}$"
  IF (†) HOLDS, THEN THIS IS ALSO AN ISOMORPHISM.

THE CONSTRUCTION OF $\mathcal{H}_{\mathcal{A}}$:

OBJECTS: $n \in \mathbb{Z}$

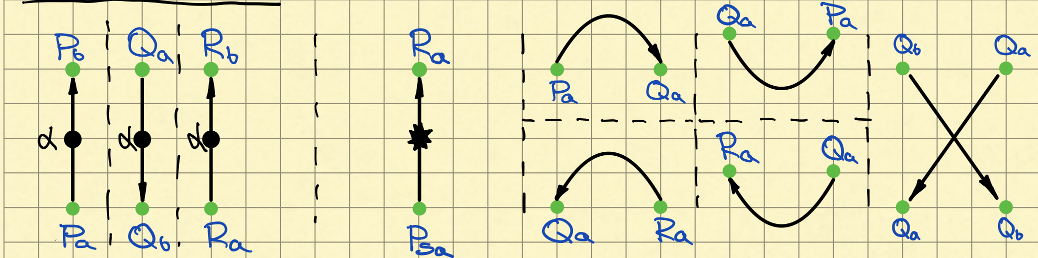
1-MORPHISMS: FREELY GENERATED BY

$P_a : n \to n+1$

$Q_a : n+1 \to n$ $\quad \forall a \in \mathcal{A},\ n \in \mathbb{Z}$

$R_a : n \to n+1$

2-MORPHISMS: PLANAR DIAGRAMS GENERATED BY

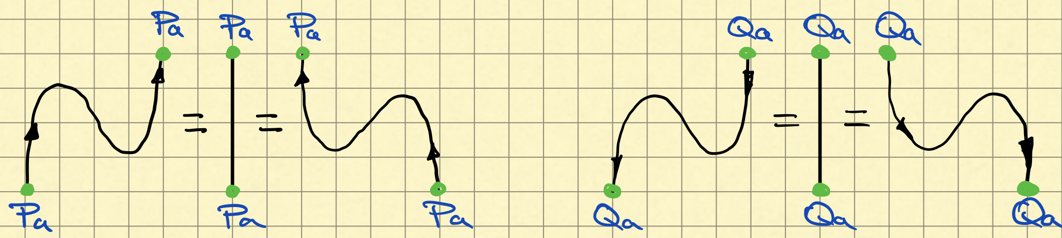


$\forall \alpha \in \mathrm{HOM}_{\mathcal{A}}(a,b)$ $\quad\quad \forall a,b \in \mathcal{A}$

DEG = DEG α $\quad\quad$ DEG = 0

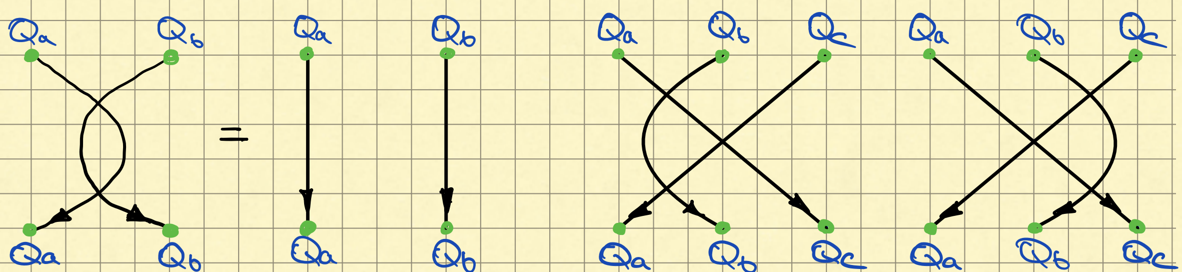
SUBJECT TO THE FOLLOWING RELATIONS:

① STRAIGHTENING / ADJUNCTION RELATIONS:

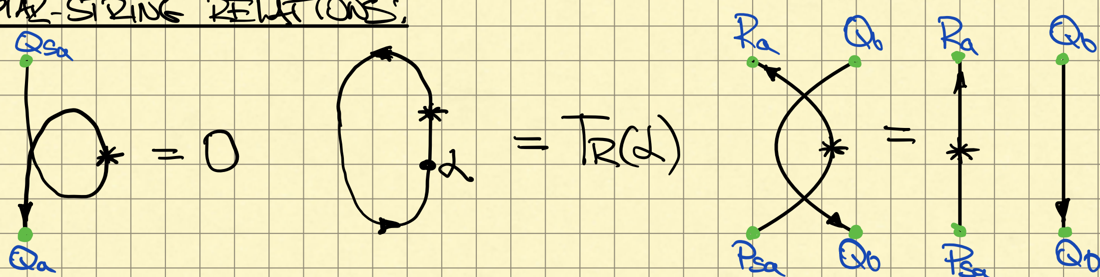


AND SAME FOR R_a

② SYMMETRIC GROUP RELATIONS:



③ STAR-STRING RELATIONS:



WE THEN TAKE PERFECT HULL AND DRINFELD QUOTIENT BY

TWO-SIDED IDEAL GENERATED BY

$$\mathrm{CONE}\left(P_{sa} \xrightarrow{\;*\;} R_a\right)$$

$$\mathrm{CONE}\left(P_b Q_a \oplus \mathrm{Hom}_{\mathfrak{h}}(a,b) \otimes_k \mathbb{1} \xrightarrow{[\times,\ \psi_2]} Q_a P_b\right).$$