$\mathbb{F}_1$-Representations and Hall Algebras

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Preliminary Report

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Overview

1. Background, results from Szczesny's 2011 paper [11].
   - "Doesn't a field need at least two elements?!?!"
   - Defining representations over $\mathbb{F}_1$.
   - Associated Hall algebras.
2. Results from Jun, Sistko's paper [4] (to appear in Algebras Represent. Theory).
   - Representation type over $\mathbb{F}_1$.
   - Combinatorial description of categories, Hall algebras.
3. New results:
   - New Hall algebra computations.
   - Refining techniques to compute Euler characteristics of quiver Grassmannians.

Part I

Background, Szczesny's results.

Background

All quotes are from Oliver Lorscheid's expository work [9].

1. What is $\mathbb{F}_1$-geometry?

   "$\mathbb{F}_1$-geometry is a recent area of mathematics that emerged from certain heuristics in combinatorics, number theory and homotopy theory that could not be explained in the frame work of Grothendieck's scheme theory."

2. The "field" $\mathbb{F}_1$ is generally left undefined. Instead,

   "what is needed for the aims of $\mathbb{F}_1$-geometry is a suitable category of schemes over $\mathbb{F}_1$."

3. Several candidates for such categories exist, depending on the context.

The category $\mathrm{Vect}(\mathbb{F}_1)$

In [11], Szczesny defines a category of $\mathbb{F}_1$-vector spaces:

$\mathbb{F}_1$-vector spaces

1. An object of $\mathrm{Vect}(\mathbb{F}_1)$ is a finite pointed set $(V, 0_V)$.
2. The dimension of V is $\dim_{\mathbb{F}_1}(V) = |V| - 1$.

$\mathbb{F}_1$-linear maps

A morphism $f : V \to W$ in $\mathrm{Vect}(\mathbb{F}_1)$ is a map of pointed sets such that $f|_{V \setminus f^{-1}(0_W)}$ is an injection.

Basic Properties

$\mathrm{Vect}(\mathbb{F}_1)$ has:

1. Quotients, subobjects, (co)kernels, 1^{st} Iso. Theorem
2. Zero objects/maps,
3. Two symmetric monoidal structures $V \oplus W$ and $V \otimes W$.

Quiver representations over $\mathbb{F}_1$

Let Q be a finite quiver, $C(Q)$ the free category on Q. Then

Representations of Q over $\mathbb{F}_1$

1. $\mathrm{Rep}(Q, \mathbb{F}_1)$ is the category of functors

$$M : C(Q) \to \mathrm{Vect}(\mathbb{F}_1)$$

   with natural transformations as morphisms.
2. The dimension of an object M in $\mathrm{Rep}(Q, \mathbb{F}_1)$ is $\sum_{u \in Q_0} \dim_{\mathbb{F}_1}(M(u))$.

Nilpotent representations over $\mathbb{F}_1$

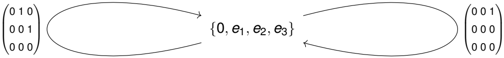
$\mathrm{Rep}(Q, \mathbb{F}_1)_{\mathrm{nil}}$ is the full subcategory of functors M for which there exists a natural number n such that for all paths $p = \alpha_1 \cdots \alpha_m$ of length $m \geq n$,

$$M(\alpha_m) \circ \cdots \circ M(\alpha_1) = 0.$$

$\mathbb{F}_1$-Representations and Binary Matrices

1. 2-Loop quiver:

$Q =$



2. Kronecker quiver:

$$Q = \ 1 \rightrightarrows 2 \ .$$

$$\{0, e_1, e_2\} \overset{\begin{pmatrix} 1 & 0 \end{pmatrix}}{\underset{\begin{pmatrix} 0 & 1 \end{pmatrix}}{\rightrightarrows}} \{0, e_1\} \ .$$

Basic properties [11]

First properties of $\mathrm{Rep}(Q, \mathbb{F}_1)$

1. $\mathrm{Rep}(Q, \mathbb{F}_1)$ a zero object, sub/quotient objects, kernels and cokernels.
2. $\mathrm{Rep}(Q, \mathbb{F}_1)$ has a symmetric monoidal structure $V \oplus W$.
3. The 1^{st} Isomorphism, Jordan-Hölder, and Krull-Schmidt Theorems hold.
4. For any field k, there is a base-change functor

$$- \otimes_{\mathbb{F}_1} k : \mathrm{Rep}(Q, \mathbb{F}_1) \to \mathrm{Rep}(Q, k)$$

All the above statements also hold for $\mathrm{Rep}(Q, \mathbb{F}_1)_{\mathrm{nil}}$.

Extra properties for $\mathrm{Rep}(Q, \mathbb{F}_1)_{\mathrm{nil}}$

1. $\mathrm{Rep}(Q, \mathbb{F}_1) = \mathrm{Rep}(Q, \mathbb{F}_1)_{\mathrm{nil}}$ iff Q is acyclic.
2. Simples in $\mathrm{Rep}(Q, \mathbb{F}_1)_{\mathrm{nil}} \Leftrightarrow$ vertices of Q.

Hall algebras of $\mathbb{F}_1$-representations

$\mathrm{Rep}(Q, \mathbb{F}_1)$ has an associated Hall algebra H_Q.

1. $H_Q =$ finitely-supported functions $\mathrm{Iso}(Q) \to \mathbb{C}$.
2. $M, N \in \mathrm{Iso}(Q) \Rightarrow \delta_M \delta_N = \sum_{R \in \mathrm{Iso}(Q)} \frac{P^R_{M,N}}{a_M a_N} \delta_R$, where
   1. $P^R_{M,N} = \#\{s.e.s. 0 \to N \to R \to M \to 0\}$
   2. $a_M = \#\,\mathrm{Aut}(M)$.
3. H_Q is a Hopf algebra with coproduct $\Delta(f)(\delta_M, \delta_N) = f(\delta_{M \oplus N})$.
4. $H_Q \cong U(\mathfrak{n}_Q)$, where $\mathfrak{n}_Q =$ Lie algebra of primitive elements.

The same holds for nilpotent representations, and we denote the associated Hall algebra $H_{Q,\mathrm{nil}}$.

Basic properties cont. [11]

Theorem 1 [11]

Let Q be a quiver without self-loops and underlying graph $\overline{Q}$. Let $\mathfrak{g}(\overline{Q}) = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ denote the associated symmetric Kac-Moody algebra. Then there exists a Hopf algebra homomorphism $\rho : U(\mathfrak{n}_+) \to H_Q$.

Further results from [11]:

1. If $\overline{Q}$ is a tree, then ρ is surjective.
2. If Q is of type $\mathbb{A}_n$, then ρ is an isomorphism.
3. If Q is the Jordan quiver, then $H_{Q,\mathrm{nil}}$ is the ring of symmetric functions.
4. $H_{Q,\mathrm{nil}}$ is computed when Q is $\tilde{\mathbb{A}}_n$ with the equi-orientation.

Part II

Results from Jun Sistko [4].

Highlights of [4]

Representation type over $\mathbb{F}_1$

1. Classify the Q of finite and bounded representation type over $\mathbb{F}_1$.
2. Describe the remaining Q's through representation embeddings.

Coefficient Quivers and Hall algebras

To any $\mathbb{F}_1$-representation M one can associated a coefficient quiver (Γ_M, c_M).

1. Interpret $\mathrm{Rep}(Q, \mathbb{F}_1)$ and $\mathrm{Rep}(Q, \mathbb{F}_1)_{\mathrm{nil}}$ in terms of coefficient quivers.
2. Give a combinatorial description of H_Q and $H_{Q,\mathrm{nil}}$.

A preorder on quivers

Definition

For any Q, there is a function $\mathrm{NI}_Q : \mathbb{N} \to \mathbb{Z}_{\geq 0}$,

$$\mathrm{NI}_Q(n) = \#\left(\{\ n\text{-dim. indecomp. in } \mathrm{Rep}(Q, \mathbb{F}_1)_{\mathrm{nil}}\ \}/\cong\right).$$

Definition

We write $Q \leq_{\mathrm{nil}} Q'$ if and only if there exists $C \in \mathbb{R}_{>0}$ and $D \in \mathbb{N}$ such that

$$\mathrm{NI}_Q(n) \leq C \cdot \mathrm{NI}_{Q'}(Dn) \text{ for } n >> 0.$$

We write $Q \approx_{\mathrm{nil}} Q'$ if and only if $Q \leq_{\mathrm{nil}} Q'$ and $Q' \leq_{\mathrm{nil}} Q$.

Example

For $n \geq 0$, let $\mathbb{L}_n$ denote the quiver with one vertex and n loops.

1. $\mathbb{L}_0 \not\approx_{\mathrm{nil}} \mathbb{L}_1$ and $\mathbb{L}_1 \not\approx_{\mathrm{nil}} \mathbb{L}_2$.
2. $\mathbb{L}_m \approx_{\mathrm{nil}} \mathbb{L}_n$ for all $m, n \geq 2$.

Finite Type over $\mathbb{F}_1$

Finite type over $\mathbb{F}_1$

Q has **finite type** over $\mathbb{F}_1$ if $\mathrm{NI}_Q(n) = 0$ for $n >> 0$.

Theorem [4]

Let Q be a connected quiver. Then the following are equivalent.

1. Q has finite type.
2. $Q \approx_{\text{nil}} \mathbb{L}_0$.
3. $\overline{Q}$ is a tree.

Note: (3) $\Rightarrow$ (1) was proven in [11].

Bounded Type over $\mathbb{F}_1$

Bounded type over $\mathbb{F}_1$

Q has **bounded type** over $\mathbb{F}_1$ if $\mathrm{NI}_Q = O(1)$ (big-O notation).

Theorem [4]

Let Q be a connected quiver. Then the following are equivalent:

1. Q is of bounded type.
2. $Q \leq_{\mathrm{nil}} \mathbb{L}_1$.
3. $\overline{Q}$ is a tree or a cycle.

Furthermore, $Q \approx_{\mathrm{nil}} \mathbb{L}_1$ if and only if $\overline{Q}$ is a cycle.

Unbounded quivers

Definition

A connected quiver Q is called a **proper pseudotree** if Q is not a tree or a cycle and $H_1(\overline{Q}, \mathbb{Z}_2) = \mathbb{Z}_2$.

Theorem [4]

Let Q be a connected quiver that is not of bounded type over $\mathbb{F}_1$. Then there exists a fully faithful, exact functor

$$\mathrm{Rep}(Q', \mathbb{F}_1)_{\mathrm{nil}} \to \mathrm{Rep}(Q, \mathbb{F}_1)_{\mathrm{nil}},$$

where Q' is either a proper pseudotree or $\mathbb{L}_2$. If Q is not a proper pseudotree, then $Q \approx_{\mathrm{nil}} \mathbb{L}_2$.

Coefficient quivers

Definition

Let $M = (\{M_u\}_{u \in Q_0}, \{f_\alpha\}_{\alpha \in Q_1})$ be an object in $\mathrm{Rep}(Q, \mathbb{F}_1)$. The **coefficient quiver** of M is a pair (Γ_M, c_M), where Γ_M is a quiver and $c_M : \Gamma_M \to Q$ is a quiver map. The vertex set of Γ_M is

$$(\Gamma_M)_0 = \bigsqcup_{u \in Q_0} M_u \setminus \{0\},$$

with $c_M(M_u \setminus \{0\}) = u$ for all $u \in Q_0$. For $\alpha \in Q_1$, there is an arrow $u \xrightarrow{\tilde{\alpha}} v$ in Γ_M with $c_M(\tilde{\alpha}) = \alpha$ if and only if $v = f_\alpha(u)$.

Note

It is often convenient to think of Γ_M as a quiver colored by the vertices and arrows of Q.

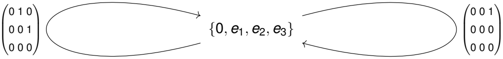
Similar ideas in the literature

1. **Coefficient Quivers (Ringel [10]):** Γ_M is the coefficient quiver of $M \otimes_{\mathbb{F}_1} k$ with respect to the obvious basis.
2. **Over-Quivers (Kinser [5]):** Γ_M is a quiver over Q with the map $c_M : \Gamma_M \to Q$.

An example

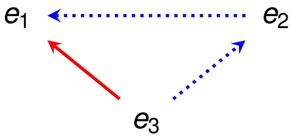
Consider again the following representation of $\mathbb{L}_2$:

$M =$



Then the colored quiver follows below, where the right loop is in solid red and the left loop is in dotted blue:

$\Gamma_M =$



Basic properties of Γ_M

Definition

A quiver map $c : \Gamma \to Q$ is a **winding** if for all $\alpha, \beta \in \Gamma_1$, $c(\alpha) = c(\beta) \Rightarrow s(\alpha) \neq s(\beta)$ and $t(\alpha) \neq t(\beta)$.

Lemma [4]

Let M be an object of $\mathrm{Rep}(Q, \mathbb{F}_1)$. Then the following hold:

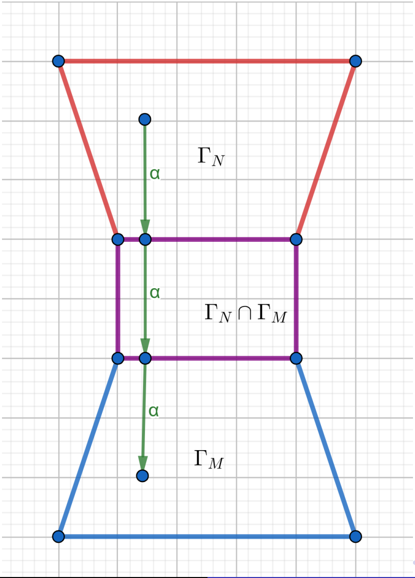
1. $c_M : \Gamma_M \to Q$ is a winding.
2. M is nilpotent $\Leftrightarrow$ Γ_M is acyclic.
3. M is indecomposable $\Leftrightarrow$ Γ_M is connected.
4. M is simple $\Leftrightarrow$ Γ_M is strongly connected.

Proposition [4]

Let $c : \Gamma \to Q$ be a winding. Then there is an $\mathbb{F}_1$-representation M of Q (unique up to isomorphism) such that $c_M = c$.

Morphisms in $\mathrm{Rep}(Q, \mathbb{F}_1)$ and coefficient quivers

A morphism $M \to N$ as gluing Γ_M to Γ_N (compare to [6]:



Morphisms in $\text{Rep}(Q, \mathbb{F}_1)$ and coefficient quivers

Definitions

Let $c : \Gamma \to Q$ be a winding.

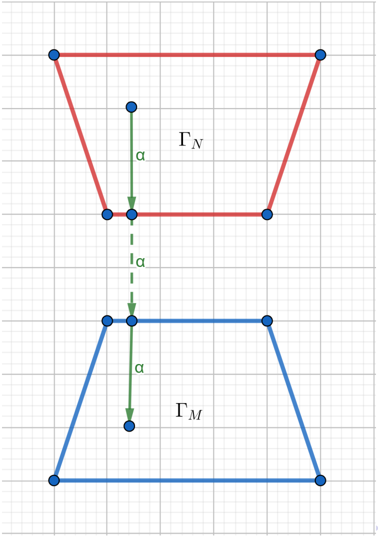
1. A full subquiver S of Γ is said to be **successor-closed** (resp. **predecessor-closed**) $\Leftrightarrow$ for any arrow $\alpha \in \Gamma_1$, $t(\alpha) \in S_1 \Rightarrow s(\alpha) \in S_1$ (resp. $s(\alpha) \in S_1 \Rightarrow t(\alpha) \in S_1$).
2. Let $c' : \Gamma' \to Q$ be another winding. A quiver isomorphism $\phi : \Gamma \to \Gamma'$ is a **coefficient isomorphism** if $c' \circ \phi = c$.

Theorem [4]

Let M and N be $\mathbb{F}_1$-representations of a quiver Q. Then the morphisms $M \to N$ are in bijective correspondence with coefficient isomorphisms $\phi : \mathcal{C} \to \mathcal{D}$, where $\mathcal{C} \subset \Gamma_M$ is successor-closed and $\mathcal{D} \subset \Gamma_N$ is predecessor-closed.

Short exact sequences and coefficient quivers

A short exact sequence $0 \to M \to R \to N \to 0$ as stacking Γ_M and Γ_N:



Short exact sequences and coefficient quivers

Definition

Let $c : \Gamma \to Q$ be a winding. Consider Γ as a quiver colored by the vertices and arrows of Q. For any $\alpha \in Q_1$, a vertex $v \in \Gamma$ is called an α-**source** (resp. α-**sink**) if there is no α-colored arrow ending at (resp. starting at) v.

Theorem [4]

Let $0 \to M \to R \to N \to 0$ be a short exact sequence of $\mathbb{F}_1$-representations of Q. Then Γ_R is obtained from the disjoint union $\Gamma_M \sqcup \Gamma_N$ by adding certain α-colored arrows from α-sinks of Γ_N to α-sources of Γ_M. Under this decomposition, Γ_M is a predecessor-closed subquiver of Γ_R and Γ_N is a successor-closed subquiver of Γ_R.

A combinatorial description for the Hall algebras

Corollary [4]

Let Q be a finite quiver. Then H_Q is generated (as an algebra) by connected coefficient quivers. Similarly, $H_{Q,\text{nil}}$ is generated by connected acyclic coefficient quivers.

Note

In the formula

$$\delta_M \delta_N = \sum_R \frac{P^R_{M,N}}{a_M a_N} \delta_R,$$

$P^R_{M,N} \neq 0 \Leftrightarrow \Gamma_R$ admits a decomposition as in the previous theorem.

Part III

Current research.

New Hall algebra computations

Problem: If Q and Q' are two different orientations of a single graph, how close are their $\mathbb{F}_1$-representation theories?

Theorem (Jun Sistko 2021)

Let Q and Q' be two orientations of a single tree T. Then $H_Q \cong H_{Q'}$.

Note: Szczensy *only* proves surjectivity of $\rho : U(\mathfrak{n}_+) \to H_Q$.

Theorem (Jun Sistko 2021)

Let Q be an acyclic quiver of type $\tilde{\mathbb{A}}_n$, with $\mathfrak{n}_Q$ the Lie algebra of primitive elements of $H_{Q,\text{nil}}$. Furthermore, let $\mathfrak{n}$ denote the Lie algebra of primitive elements in $H_{Q',\text{nil}}$, where Q' is the equioriented quiver of type $\tilde{\mathbb{A}}_n$. Then $\mathfrak{n}_Q/Z(\mathfrak{n}_Q) \cong \mathfrak{n}$.

Note: By [4], this describes (nilpotent) Hall algebras of bounded type over $\mathbb{F}_1$.

Euler characteristics of quiver Grassmannians

Results from Literature

1. Revelant to cluster algebras [1].
2. Euler characteristics for string modules computed by Cerulli-Irelli in [3].
3. Techniques expanded by Haupt in [2], applied to tree and band modules.
4. Similar ideas appear in Lorscheid's work [7, 8].

Note: Many of these representations are defined over $\mathbb{F}_1$!

Our goals

1. Further develop the techniques in [2, 3].
2. Characterize $\mathbb{F}_1$-reps where these techniques are applicable.
3. Find formulas for the associated Euler characteristics.

Gradings of representations [2]

Throughout, let M be an $\mathbb{F}_1$-representation of Q with colored quiver $(\Gamma, c) := (\Gamma_M, c_M)$. Let $\underline{d} = \underline{\dim}_{\mathbb{F}_1}(M)$ and $\underline{e} \leq \underline{d}$.

1. A **grading** of M (equiv. c) is a map $\partial : \Gamma_0 \to \mathbb{Z}$.
2. If $\partial_1, \ldots, \partial_n$ are gradings of $M \Rightarrow$ a grading ∂ is $(\partial_1, \ldots, \partial_n)$**-nice**, if whenever two arrows $\alpha, \beta \in \Gamma_1$ satisfy:

$$c(\alpha) = c(\beta),$$

$$\partial_i(s(\alpha)) = \partial_i(s(\beta)), \ i = 1, \ldots, n$$

$$\partial_i(t(\alpha)) = \partial_i(t(\beta)), \ i = 1, \ldots, n$$

   we have $\partial(t(\alpha)) - \partial(s(\alpha)) = \partial(t(\beta)) - \partial(s(\beta))$.
3. A **nice grading** (or $\emptyset$-nice grading) is where $c(\alpha) = c(\beta) \Rightarrow \partial(t(\alpha)) - \partial(s(\alpha)) = \partial(t(\beta)) - \partial(s(\beta))$.

Example

Let $Q = \mathbb{L}_2$, with arrow set $Q_1 = \{\alpha_1, \alpha_2\}$. Let M be the representation with quiver

$$\Gamma_M = \bullet \xrightarrow{\alpha_1} \bullet \xrightarrow{\alpha_2} \bullet \xleftarrow{\alpha_1} \bullet \xleftarrow{\alpha_2} \bullet .$$

1. A nice grading ∂_0 on M:

$$0 \xrightarrow{+1} 1 \xrightarrow{+2} 3 \xleftarrow{+1} 2 \xleftarrow{+2} 0 .$$

2. A ∂_0-nice grading ∂_1 on M:

$$0 \xrightarrow{+1} 1 \xrightarrow{+1} 2 \xleftarrow{-1} 3 \xleftarrow{-1} 4 .$$

Nice gradings and quiver Grassmannians (cont.)

Assumption

Suppose that for all $i \leq n$, ∂_i is a $(\partial_1, \ldots, \partial_{i-1})$-nice grading (with ∂_1 a nice grading).

Gradings of representations cont. [2]

Let $\chi_{\underline{e}}(M \otimes_{\mathbb{F}_1} \mathbb{C})$ denote the Euler characteristic of the quiver Grassmannian $\mathrm{Gr}^Q_{\underline{e}}(M \otimes_{\mathbb{F}_1} \mathbb{C})$.

1. $(\partial_1, \ldots, \partial_n) \Rightarrow X^{\partial_1, \ldots, \partial_n} \subset \mathrm{Gr}^Q_{\underline{e}}(M \otimes_{\mathbb{F}_1} \mathbb{C})$ locally-closed, same Euler characteristic.
2. If for each $x, y \in \Gamma_0$, there exists $i \leq n$ with $\partial_i(x) \neq \partial_i(y)$, then

$$\chi_{\underline{e}}(M \otimes_{\mathbb{F}_1} \mathbb{C}) = |\{N \leq M \mid \dim_{\mathbb{F}_1}(N) = \underline{e}\}|.$$

3. Euler characteristic counts the "$\mathbb{F}_1$-points" of the quiver Grassmannian!

Representations with finite nice length

Question

When does there exists a sequence $\partial_1, \ldots, \partial_n$?

Nice length (Jun Sistko 2021)

1. A **nice sequence** for M = a collection $\underline{\partial} = \{\partial_i\}_{i \geq 0}$ s.t. ∂_i is a $(\partial_0, \ldots, \partial_{i-1})$-nice grading for all i (∂_0 is nice).
2. The **nice length** of M = the smallest n s.t. there is a nice sequence $\underline{\partial}$ with the property that for all $x, y \in \Gamma_0$, $\partial_i(x) \neq \partial_i(y)$ for some $i \leq n$. We write $\mathrm{nice}(M) = n$ (and $\mathrm{nice}(M) = \infty$ if no such n exists).
3. $\mathrm{nice}(M) < \infty \Rightarrow$ the previous formula holds for all $\underline{e} \leq \underline{d}$.

Example

Let $Q = \mathbb{L}_2$, with arrow set $Q_1 = \{\alpha_1, \alpha_2\}$. Let M be the representation with quiver

$$\Gamma_M = \bullet \xrightarrow{\alpha_1} \bullet \xrightarrow{\alpha_2} \bullet \xleftarrow{\alpha_1} \bullet \xleftarrow{\alpha_2} \bullet .$$

1. A generic nice grading on M:

$$x \xrightarrow{+\Delta_1} x+\Delta_1 \xrightarrow{+\Delta_2} x+\Delta_1+\Delta_2 \xleftarrow{+\Delta_1} x+\Delta_2 \xleftarrow{+\Delta_2} x .$$

   The start and finish always have the same image!
2. The existence of the sequence ∂_0, ∂_1 with ∂_1 injective implies that $\mathrm{nice}(M) = 1$.

Current work

The category $\mathrm{Rep}(Q, \mathbb{F}_1)^{\mathrm{nice}}_{\mathrm{nil}}$ (Sistko Jun 2021)

1. $\mathrm{Rep}(Q, \mathbb{F}_1)^{\mathrm{nice}}_{\mathrm{nil}}$ = full subcategory generated by M with $\mathrm{nice}(M) < \infty$.
2. Contains all tree representations and (primitive) "band" representations (recovering results from [2, 3]).
3. Closed under sub/quotient objects, certain types of gluing.
4. New families in $\mathrm{Rep}(Q, \mathbb{F}_1)^{\mathrm{nice}}_{\mathrm{nil}}$ identified.
5. $\mathrm{Rep}(Q, \mathbb{F}_1)^{\mathrm{nice}}_{\mathrm{nil}}$ characterized when Q is a pseudotree.

Hall algebra computations (Jun Sistko 2021)

1. $\mathrm{Rep}(Q, \mathbb{F}_1)^{\mathrm{nice}}_{\mathrm{nil}}$ has a Hall algebra $H^{\mathrm{nice}}_{Q,\mathrm{nil}}$.
2. $H^{\mathrm{nice}}_{Q,\mathrm{nil}} = H_{Q,\mathrm{nil}} / \langle \delta_M \mid \mathrm{nice}(M) = \infty \rangle$.
3. Explicit descriptions for $H^{\mathrm{nice}}_{Q,\mathrm{nil}}$ when Q is of bounded type.

Example

Let Q be an acyclic quiver of type $\tilde{\mathbb{A}}_n$.

Indecomposables of finite nice length

1. The indecomposables in $\mathrm{Rep}(Q, \mathbb{F}_1)_{\mathrm{nil}} = \mathrm{Rep}(Q, \mathbb{F}_1)$ are either strings or “bands” [4].
2. Strings have finite nice length (tree representations).
3. For each $m \geq 1$, there is a “band” representation B_n (decomposable over $\mathbb{C}$ if $m > 1$).
4. $\mathrm{nice}(B_m) < \infty \Leftrightarrow m = 1$.
5. $H_Q^{\mathrm{nice}} = H_Q / \langle \delta_{B_m} \mid m > 1 \rangle$.
6. $H_Q^{\mathrm{nice}} =$ the Hall algebra of “absolutely indecomposable” representations (i.e. M such that $M \otimes k$ is indecomposable for any $k = \overline{k}$).

Future Work

Related to this talk

1. If Q is of unbounded type and Q' is a different orientation, determine the relationship between $H_{Q,\mathrm{nil}}$ and $H_{Q',\mathrm{nil}}$.
2. Find combinatorial characterizations for $\mathrm{nice}(M) < \infty$.
3. Compute gradings efficiently.

Other directions

1. Classification for non-nilpotent simples?
2. Classification for absolutely indecomposable representations?
3. Connections to crystal bases?

Thanks!

Thanks for listening!

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