An algebraic variety related to τ-tilting theory

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Reminder on τ-tilting theory

I'm going to begin by recalling some of the basic results of τ-tilting theory, initiated by Adachi, Iyama, and Reiten.

Let A be a basic finite-dimensional algebra over a field with n isoclasses of simple modules.

A module M is called τ-rigid if $\mathrm{Hom}(M, \tau M) = 0$.

There is a simplicial complex on the (iso-classes of) τ-rigid indecomposable modules. $\{X_1, \ldots, X_r\}$ is a simplex iff $X_1 \oplus \cdots \oplus X_r$ is τ-rigid.

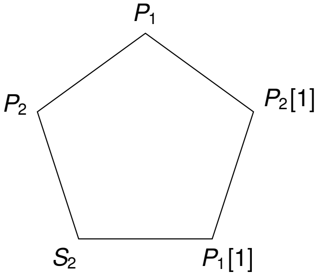
This can be enlarged by adding n additional vertices, usually denoted $P_i[1]$, where the P_i are the indecomposable projective modules.

We extend the notion of τ-rigidity to sums including the $P_i[1]$ by defining $\tau P_i[1] = I_i$ and $\mathrm{Hom}(P_i[1], M) = 0$ for any module M.

Example: A_2

Recall the vertices of the simplicial complex are indecomposables X such that $\mathrm{Hom}(X, \tau X) = 0$; faces are pairwise compatible collections where X and Y are compatible if $\mathrm{Hom}(X, \tau Y) = 0$ and $\mathrm{Hom}(Y, \tau X) = 0$.

$$\begin{tikzcd}
 & P_2 \arrow[dr] & & P_1[1] & \\
P_1 \arrow[ur] & & S_2 & & P_2[1]
\end{tikzcd}$$



g-vectors and the τ-tilting fan

Given any A-module M, we can take a minimal projective presentation

$$Q \to P \to M \to 0.$$

The g-vector of M is the n-tuple whose i-th coordinate is the number of occurrences of P_i in P minus the number of occurrences of P_i in Q.

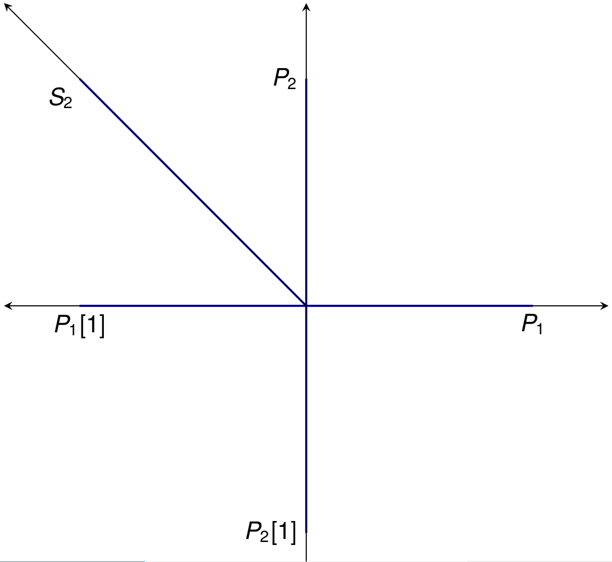
We define the g-vector of $P_i[1]$ to be $-e_i$.

We define the τ-tilting fan of A to be the fan whose rays are the g-vectors of the τ-rigid indecomposable objects, and whose cones correspond to rigid objects.

Theorem (Demonet–Iyama–Jasso)

The above definition yields a fan. If A has only finitely many τ-rigid indecomposables, it is proper (covers $\mathbb{R}^n$).

The τ-tilting fan for A_2



Brief motivation of τ-tilting theory

τ-tilting theory is an analogue to tilting theory, but has the property (familiar from cluster algebras) that every maximal cone in the τ-tilting fan can be "mutated" in exactly n ways; each ray can be replaced by another ray to give another maximal cone.

The maximal cones in the τ-tilting fan correspond bijectively to functorially finite torsion classes (and to all torsion classes, if there are only finitely many).

You can think of the fan as being realized in the space of possible (King) stability conditions on the module category; the above bijection becomes natural from that viewpoint

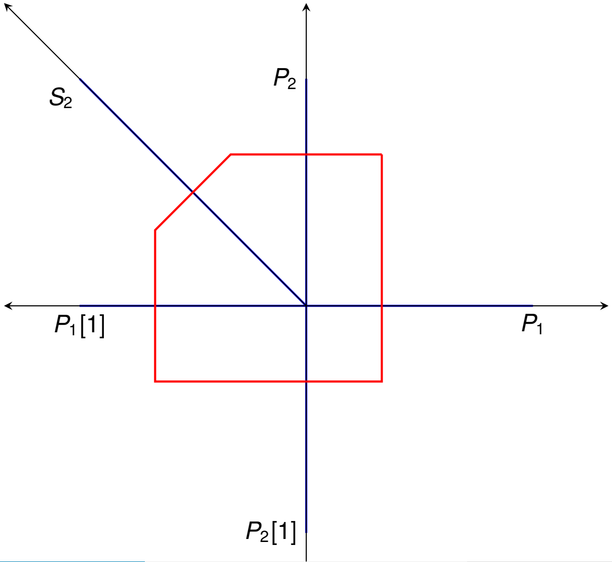
Dual polytope to the τ-tilting fan

Given a complete fan, you can ask if there is a dual polytope to the fan. This is a polytope the outer normals of whose facets are given by rays of the fan, and whose facets correspond to cones of the fan.

A full-dimensional polytope always defines a dual fan in this way, but a fan need not admit a dual polytope.

Fei has shown that in the case that A admits only finitely many τ-rigid indecomposable modules, the τ-tilting fan admits a dual polytope. (It can be realized as a Harder-Narasimhan polytope; this won't be especially relevant for us, so I won't discuss it further.)

A_2 dual polytope



Compatibility degree

We begin by making a simple observation: two τ-rigid indecomposables appear in a τ-tilting cone fan together if and only if their compatibility degree is zero, where the compatibility degree is defined by:

$$c(X, Y) = \dim \mathrm{Hom}(X, \tau Y) + \dim \mathrm{Hom}(Y, \tau X)$$

Recall that we define $\tau(P_i[1]) = I_i$, and that $\mathrm{Hom}(P_i[1], M) = 0$ for M any module.

For example, in type A_n, indecomposable τ-rigid objects are either compatible (i.e., have compatibility degree zero) or else have compatibility degree 1.

The τ-rigidity of an indecomposable can can also described in terms of compatibility degree: an indecomposable X is τ-rigid iff $c(X, X) = 0$.

The u-equations

From now on, we assume that A is a finite-dimensional algebra over $\mathbb{C}$ and admits only finitely many isomorphism classes of modules.

Write $\mathcal{I}$ for the collection of isomorphism classes of indecomposable modules and $P_i[1]$.

For each $M \in \mathcal{I}$, we introduce a variable u_M. Further, for each $M \in \mathcal{I}$, we introduce an equation:

$$u_M + \prod_{N \in \mathcal{I}} u_N^{c(M,N)} = 1$$

Since we have written down as many equations as variables, generically we would expect the solutions to consist of points. However, as we shall see, this is not the case.

Define $\mathcal{V}$ to be the affine variety cut out by these equations over $\mathbb{C}$.

The one case which we can easily write down is A_1, where the two u-equations each give $u_{P_1} + u_{P_1[1]} = 1$.

The totally non-negative part of $\mathcal{V}$

Recall that the equations cutting out $\mathcal{V}$ are:

$$u_M + \prod_{N \in \mathcal{I}} u_N^{c(M,N)} = 1$$

We are interested in the locus inside $\mathcal{V}$ where the variables u_M are all real and non-negative. We call this $\mathcal{V}_{\geq 0}$, the totally non-negative part of $\mathcal{V}$. Let us try to make some qualitative observations about it.

If $u_M \geq 0$ for all M, then both terms in each equation are non-negative, so both must be at most 1. Thus, we see that in this region, $0 \leq u_M \leq 1$ for all M.

The boundaries of $\mathcal{V}_{\geq 0}$ (i.e., where some inequality becomes an equality) are when some $u_M = 0$ If M is not τ-rigid, then u_M occurs in both terms in the equation for M, and thus for non τ-rigid M, u_M can never be zero.

Stratification according to the τ-tilting fan

We can stratify $\mathcal{V}_{\geq 0}$ according to which coordinates are zero.

In order for $\mathcal{V} \cap \{u_{M_1} = \cdots = u_{M_r} = 0\}$, we must have $M_1, M_2, \ldots, M_r$ mutually compatible. In other words, they most correspond to a face of the τ-tilting fan. For σ a cone of the τ-tilting fan, write $\mathcal{V}^{\sigma}_{\geq 0}$ for the subset of $\mathcal{V}_{\geq 0}$ with the u-variables corresponding to the generators of σ being zero, and the others positive.

It is also clear that if σ is a maximal cone of the τ-tilting fan generated by rays corresponding to $M_1, \ldots, M_n$, then $u_{M_1} = u_{M_2} = \cdots = u_{M_n} = 0$, $u_N = 1$ for all other N, defines a point in $\mathcal{V}^{\sigma}_{\geq 0}$.

Theorem (Arkani-Hamed, Frost, Plamondon, Salvatori, T)

For each σ a cone of the τ-tilting fan, $\mathcal{V}^{\sigma}_{\geq 0}$ is non-empty.

Conjecture

$\mathcal{V}$ is irreducible as a variety.

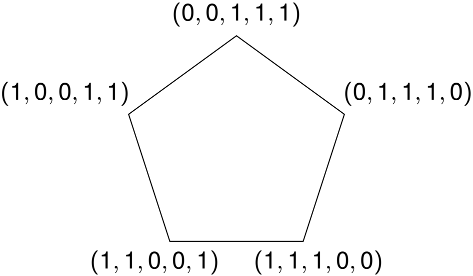
Conjecture

For each σ a cone of the τ-tilting fan, $\mathcal{V}^{\sigma}_{\geq 0}$ is homeomorphic to an open ball, and $\mathcal{V}_{\geq 0}$ is homeomorphic to a polytope dual to the τ-tilting fan, respecting their stratifications.

Example: A_2

The A_2 equations are:

$$
\begin{aligned}
u_{P_1} + u_{S_2} u_{P_1[1]} &= 1 & \qquad u_{P_2} + u_{P_1[1]} u_{P_2[1]} &= 1 \\
u_{S_2} + u_{P_2[1]} u_{P_1} &= 1 & \qquad u_{P_1[1]} + u_{P_1} u_{P_2} &= 1 \\
u_{P_2[1]} + u_{P_2} u_{S_2} &= 1 & &
\end{aligned}
$$



F-polynomials

We fix a description of our algebra A as $\mathbb{C}Q/I$, where Q is a quiver with vertices numbered 1 to n.

Let M be an A-module, which we can also think of as a representation of Q, with M_i being the vector space over vertex i.

The F-polynomial of M is defined to be

$$F_M(y_1, \ldots, y_n) = \sum_e \chi(\mathrm{Gr}(M, e)) y_1^{e_1} \ldots y_n^{e_n}$$

The sum is over dimension vectors $e \in \mathbb{Z}^n_{\geq 0}$. $\mathrm{Gr}(M, e)$ is defined to consist of a subspace of with dimension e_1 of M_1, a subspace with dimension e_2 of M_2, and so on, with the additional property that it defines a subrepresentation of M.

χ is the Euler characteristic.

Example: F-polynomials in type A_n

In type A_n the F-polynomials of indecomposable modules are easy to calculate, because the quiver Grassmannian whose Euler characteristic must be calculated is either empty or a point, depending on whether or not there is a subrepresentation of the corresponding dimension.

Thus, for the quiver $1 \leftarrow 2 \cdots \leftarrow n$, the F-polynomial of the indecomposable representation supported over vertices $i, i+1, \ldots, j$ is $1 + y_i + y_i y_{i+1} + \cdots + y_i y_{i+1} \ldots y_j$.

The can be a little more complicated if the quiver is not linearly oriented. For $1 \rightarrow 2 \leftarrow 3$, the F-polynomial corresponding to the sincere representation is $1 + y_2 + y_2 y_1 + y_2 y_3 + y_2 y_1 y_3$.

A mutation-style formula involving F-polynomials

We need a formula due to Domínguez and Geiss which is reminiscent of cluster mutation, but which applies for any finite-dimensional algebra.

Theorem (Domínguez–Geiss)

Let

$$0 \to \tau X \to E \to X \to 0$$

be an AR-sequence. Then

$$F_{\tau X} F_X = F_E + y^{\dim X}$$

There are also similar equations relating the F-polynomial of an indecomposable projective with that of its radical, and that of an indecomposable injective with its coradical.

Solving the u-equations

Theorem (Arkani-Hamed, Frost, Plamondon, Salvatori, T)

The u-equations admit the following family of solutions. For X a non-productive module, let $0 \to \tau X \to E \to X \to 0$ be the AR-sequence ending at X. Then set

$$u_X = \frac{F_E}{F_{\tau X} F_X}$$

For X indecomposable projective, set

$$u_X = \frac{F_{\mathrm{rad}\, X}}{F_X}$$

For $X = P_i[1]$, set

$$u_X = \frac{y_i F_{I_i/S_i}}{F_{I_i}}.$$

The theorem follows from the results of Domínguez–Geiss already mentioned plus telescoping products.

Motivation from physics

If $A = \mathbb{C}A_n$ then the u-equations were worked out in 1969 by Kikkawa–Sakita–Virasoro. The totally positive part of the variety cut out by the u-equations has vertices which correspond to (as we have seen) maximal cones in the τ-tilting fan, which correspond to triangulations of an $(n+3)$-gon. Dually, these can be viewed as trivalent, planar, loopless Feynman diagrams. The solutions to the u-equations are therefore a way to capture the totality of the relevant (tree-level) Feynman diagrams in a single geometrical object.

In type A_n, this recovers the open string moduli space.

One of our motivations is to understand something similar for Feynman diagrams admitting loops (drawn on surfaces which may have interesting topology), obtaining integrals analogous to the integrals that appear in string theory.

Motivation from cluster algebras

If $A = \mathbb{C}Q$, Q Dynkin, then the solutions to the u-equations have been referred to as "cluster configuration spaces," see [Arkani-Hamed, He, Lam, T] and the follow-up [Arkani-Hamed, He, Lam].

It is possible to express solutions to the u-equations in terms of (a subset of) the Fock-Goncharov $\mathcal{X}$-variables. To a u-variable u_M, we have $u_M/(1-u_M)$ giving the value of a $\mathcal{X}$-variable.

In cases where the is a cluster category (as in these cases), the compatibility degree can also be expressed within the cluster category; in the Dynkin case it is the usual compatibility degree from cluster algebras.

Functoriality

Let $B = A/I$. Note that a τ-rigid B-module is still τ-rigid as an A-module. It appears that there is a natural surjective map from $\mathcal{V}_A$ to $\mathcal{V}_B$, which for a B-module M, sends u_M to a monomial in the u-variables for A.

Functoriality: $A_2 \to A_1 \times A_1$

Let $A = \mathbb{C}A_2$, and let B be the quotient isomorphic to $A_1 \times A_1$.

The A_2 equations are:

$$
\begin{aligned}
u^A_{P_1} + u^A_{S_2} u^A_{P_1[1]} &= 1 & \qquad u^A_{P_2} + u^A_{P_1[1]} u^A_{P_2[1]} &= 1 \\
u^A_{S_2} + u^A_{P_2[1]} u^A_{P_1} &= 1 & \qquad u^A_{P_1[1]} + u^A_{P_1} u^A_{P_2} &= 1 \\
u^A_{P_2[1]} + u^A_{P_2} u^A_{S_2} &= 1 & &
\end{aligned}
$$

The $A_1 \times A_1$ equations are:

$$
\begin{aligned}
u^B_{P_1} + u^B_{P_1[1]} &= 1 \\
u^B_{P_2} + u^B_{P_2[1]} &= 1
\end{aligned}
$$

One can easily verify that

$$
\begin{aligned}
u^B_{P_1} &= u^A_{P_1} & \qquad u^B_{P_2} &= u^A_{P_2} u^A_{S_2} \\
u^B_{P_1[1]} &= u^A_{S_2} u^A_{P_1[1]} & \qquad u^B_{P_2[1]} &= u^A_{P_2[1]}
\end{aligned}
$$

satisfy the u-equations for B given that the u^A satisfy the u-equations for A.

What other equations are satisfied by the points of $\mathcal{V}$?

In the case of the cluster configuration spaces, there is an additional (redundant) u-equation corresponding to every mutation, while the u-equations correspond to the "smallest" mutations. Are there similar extra u-equations in general?

The work of Cerulli Irelli, Esposito, Franzen, and Reineke suggests that the answer is yes; they give a variant of the Geiss–Domínguez result which starts, not from an AR sequence, but from a "generating extension," and it seems that this should also give rise to a corresponding u-equation.

Generalization to infinite representation type

For physics applications, it is desirable to generalize to infinite representation type algebras. This raises problems of convergence of the products in the u-equations.

It also seems to be important to understand (if possible) how to forget the non τ-rigid indecomposables.

Thank you!