Geometric models of Ginzburg algebras

Merlin Christ, FD Seminar, July 22nd 2021

Based on

arxiv: 2101.01939, arxiv: 2107.10091

Plan

1) Introduction: gentle algebras and Ginzburg algebras

2) Gluing for gentle algebras

3) Gluing for Ginzburg algebras

Why care about these algebras?

- Categorification of cluster algebras

- Relation to Fukaya categories

- Related to each other via the Jacobian algebra.

1) <u>Gentle algebras</u>

KQ/I is gentle if

* Q is a quiver w/ vertices of valency ≤ 4
* $I \subset KQ$ ideal generated by paths of length 2, s.t. for all $a \in Q_1$ there exist

  at most one $b \in Q_1$ w/ $0 \neq ab \in I$

  —— " —— w/ $0 \neq ab \notin I$

  —— " —— w/ $0 \neq ba \in I$

  —— " —— w/ $0 \neq ba \notin I$

KQ/I can be infinite dimensional

<u>Examples</u>

* $Q = 1 \xrightarrow{a} 2 \xrightarrow{b} 3$ $\quad (I = (0))$
* $Q = 1 \xrightarrow{a} 2 \xrightarrow{b} 3$ $\quad (I = (ba))$
* $Q =$
$$\begin{tikzcd} & 2 \arrow[dr] & \\ 1 \arrow[ur] & & 3 \arrow[ll] \end{tikzcd}$$
* $Q = 1 \rightrightarrows 2$ Kronecker quiver

Geometric (surface) model for $\mathcal{D}^{perf}(kQ/I)$

[HKK, LP, BS, OPS]

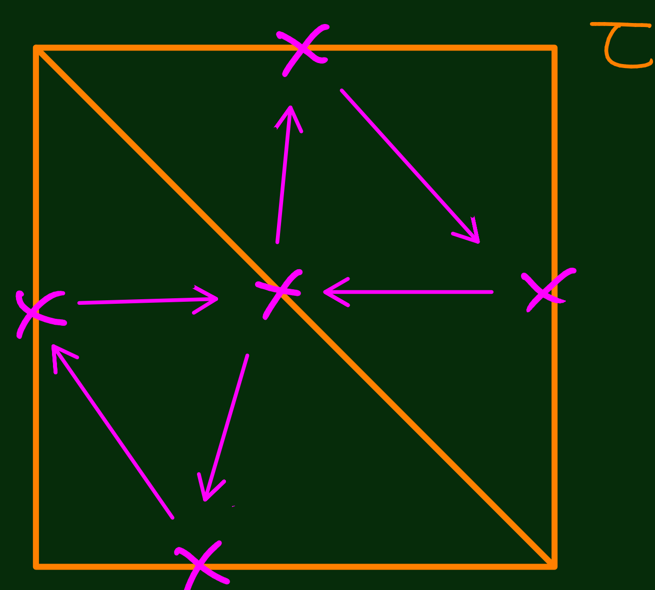
- describe all indecomposables in terms of (homotopy classes of) curves in an oriented marked surface with $M \subset \partial S$
- describe Hom's in terms of intersections
- ...

Relative Ginzburg algebras of triangulated surfaces

Fix an oriented marked surface w/ triangulation $\mathcal{T}$

closed or w/ boundary (vertices of $\mathcal{T}$ = marked points)

Define the quiver $Q_{\mathcal{T}}$ with



* vertices = edges of $\mathcal{T}$

use only internal edges for non-relative Ginzburg algebra

* arrows = clockwise 3-cycle $T(f)$ for each face f.

Form the graded quiver $\widetilde{Q}_\tau$ with

* Vertices of $\widetilde{Q}_\tau$ = vertices of Q_τ

and arrows

* $a: i \to j$ degree 0 for $a: i \to j \in (Q_\tau)_1$
* $a^*: j \to i$ degree 1 for $a: i \to j \in (Q_\tau)_1$
* $\ell_i: i \to i$ degree 2 for $i \in (Q_\tau)_0$ given by an internal edge

<u>Definition</u>

The relative Ginzburg algebra $\mathcal{G}_\tau = (k\widetilde{Q}_\tau, d)$ is the (non-complete) path algebra of $\widetilde{Q}_\tau$ with

* $d(a) = 0$

potential

* $d(a^*) = \partial_a \sum_{\text{faces}} T(f)$ (cyclic derivative)
* $d(\ell_i) = \sum_{a \in (Q_\tau)_1} p_i [a, a^*] p_i$

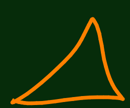
τ lazy path

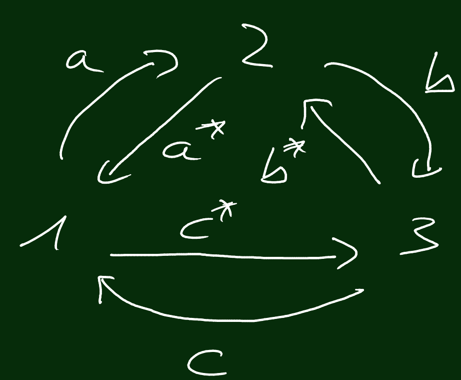
Remark

1) The potential $\sum_f T(f)$ is in most cases degenerate if S has no internal marked points.

2) $\mathfrak{g}_\tau = H_0(\mathfrak{g}_\tau)$ is a gentle algebra (generalizing [ABCP]) and finite dim. if S has no internal marked points.

Examples

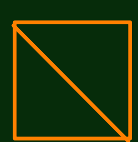
1) $\tau =$ 

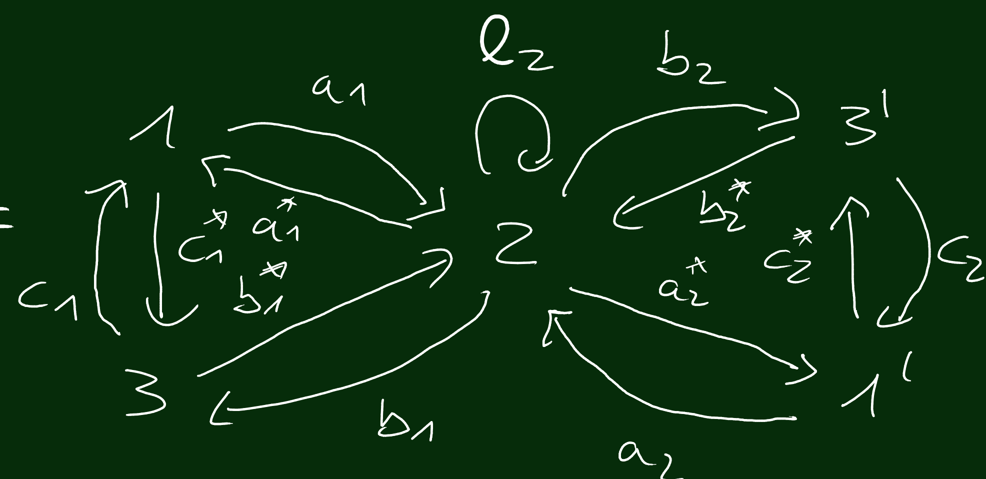
$\mathfrak{g}_\Delta =$ 

$d(a^*) = cb$

$d(b^*) = ac$

$d(c^*) = ba$

2) $\tau =$ 

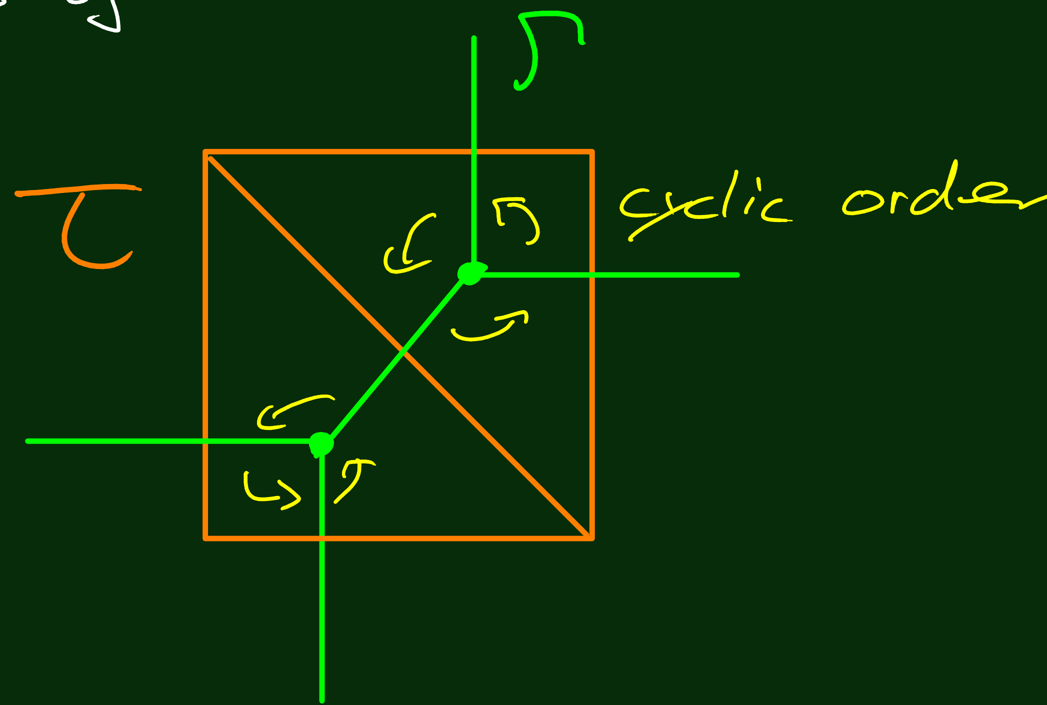
$\mathfrak{g}_\tau =$ 

$d(a_i^*) = c_i b_i$ for $i = 1, 2$ + cyclic permutations

$d(\ell_2) = a_1 a_1^* + a_2 a_2^* - b_1^* b_1 - b_2^* b_2$

Dual ribbon graph Γ of τ : triangulation

* vertices Γ_0 = faces of τ
* edges Γ_1 = edges of τ



Geometric model for (non-relative) Ginzburg algebra of triangulated surface (w/o interior marked points)

[Qiu, Zhou]

- Describe (some) modules in terms of curves in $S \setminus (M \cup \Gamma_0)$ including 3-spherical simples and projectives
- Describe Hom's in terms of intersections
- ...

2) Gluing for gentle algebras

Describe $\mathcal{D}(kQ/I)$ as colimit of

constructible cosheaf of stable ∞-categories

on a ribbon graph Γ with vertices on ∂S.

locally constant on strata = edges / vertices of Γ

Define poset $\mathrm{Entry}(\Gamma)$ w/

* objects vertices and edges of Γ
* morphism $e \to v$ if edge e incident to vertex v.

$\Gamma = \overset{v}{\bullet} \underline{\;e\;} \overset{v'}{\bullet}$

$\mathrm{Entry}(\Gamma) = v' \leftarrow e \rightarrow v$

Constructible cosheaf on Γ:

functor $\mathcal{F}: \mathrm{Entry}(\Gamma) \to \underbrace{\mathrm{LinCat}_k}$

k-linear ∞-categories, colimits modeled by dgCat_k w/ Morita model structure

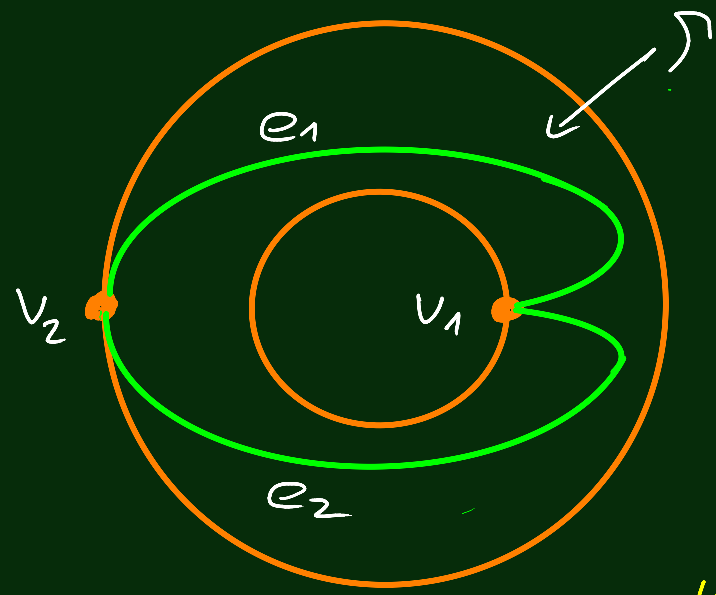
* $\mathcal{F}(v) = \mathcal{D}(A_n)$ for v vertex of valency n w/

$$A_n = 1 \to 2 \to 3 \to \cdots \to n$$

← all composites are zero

* $\mathcal{F}(e) = \mathcal{D}(A_1) = \mathcal{D}(k)$ for each edge e.

Example: S = twice marked annulus



$$\begin{tikzcd} & e_1 \arrow[dl] \arrow[dr] & \\ v_1 & & v_2 \\ & e_2 \arrow[ul] \arrow[ur] & \end{tikzcd}$$

$\parallel$

$\mathfrak{F} : \mathrm{Entry}(\Gamma) \longrightarrow \mathrm{LinCat}_k$

is the diagram

$$\begin{tikzcd} & \mathcal{D}(k) \arrow[dl, "k \mapsto (k \xrightarrow{\sim} k)"'] \arrow[dr, "k \mapsto (k \xrightarrow{\sim} k)"] & \\ \mathcal{D}(A_2) & & \mathcal{D}(A_2) \\ & \mathcal{D}(k) \arrow[ul, "k \mapsto (0 \to k)"] \arrow[ur, "k \mapsto (0 \to k)"'] & \end{tikzcd}$$

Colimit:

$\mathcal{D}(K_2)$ w/ $K_2 = \bullet \rightrightarrows \bullet$ Kronecker quiver

$\cong$

$\mathcal{D}(\mathrm{Coh}\,\mathbb{P}^1)$

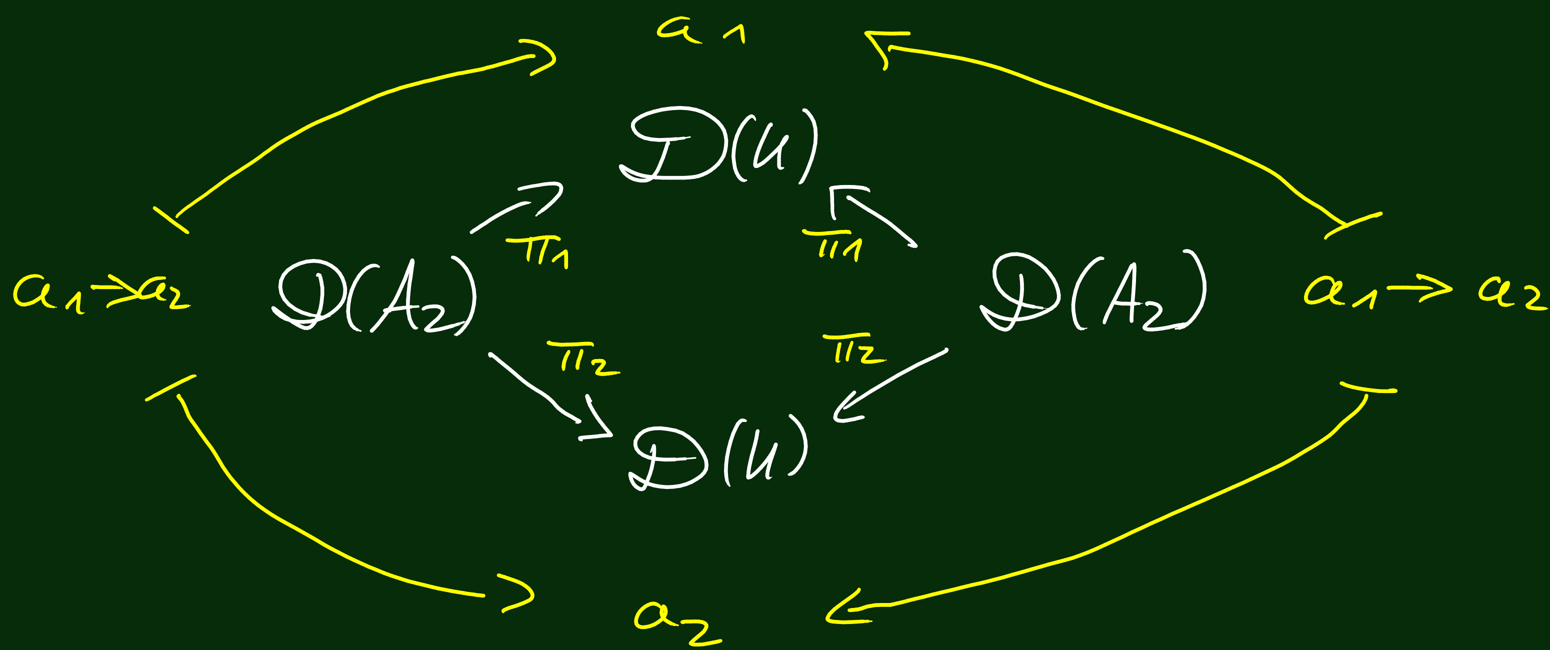
<u>Goal</u>: Use the gluing construction to construct objects and morphisms in $\mathcal{D}(kQ/I)$ from local data.

Remarkable fact from ∞-category theory

The colimit of a constructible cosheaf $\mathcal{F} : \mathrm{Entry}(\Gamma) \to \mathrm{LinCat}_u$ is equivalent to the limit of the right adjoint diagram

Constructible sheaf on Γ $\longrightarrow$ $\mathrm{Radj}(\mathcal{F}) : \mathrm{Entry}(\Gamma)^{op} \to \mathrm{LinCat}_u$

Right adjoint of $\mathcal{F}$:



Limits of ∞-categories are well behaved:

* objects are sections of the diagram
* morphisms are natural transformations between sections

Sections of $\mathrm{Rad}_j(\mathcal{F})$

$$\begin{tikzcd}
(\mathcal{O} \to K) \arrow[r] \arrow[dr] & \mathcal{O} & (\mathcal{O} \to K) \arrow[l] \arrow[dl] \\
& K &
\end{tikzcd}$$

$= \mathcal{O} \in \mathcal{D}(\mathrm{Coh}\,\mathbb{P}^1)$

line bundle

$$\begin{tikzcd}
(K \xrightarrow{\cdot 1} K) \arrow[r] \arrow[dr] & K & (K \xrightarrow{\cdot \lambda} K) \arrow[l] \arrow[dl] \\
& K &
\end{tikzcd}$$

$= K_\lambda \in \mathcal{D}(\mathrm{Coh}\,\mathbb{P}^1)$

skyscraper sheaf

glue

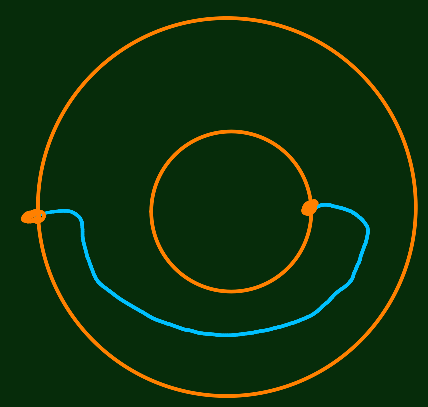
Local sections

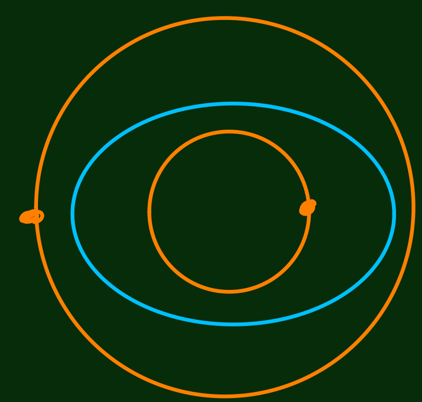
$$\begin{tikzcd}
(K \to \mathcal{O}) \arrow[r] \arrow[dr] & K \\
& \mathcal{O}
\end{tikzcd}$$

$$\begin{tikzcd}
(\mathcal{O} \to K) \arrow[r] \arrow[dr] & \mathcal{O} \\
& K
\end{tikzcd}$$

$$\begin{tikzcd}
(K \xrightarrow{\sim} K) \arrow[r] \arrow[dr] & K \\
& K
\end{tikzcd}$$

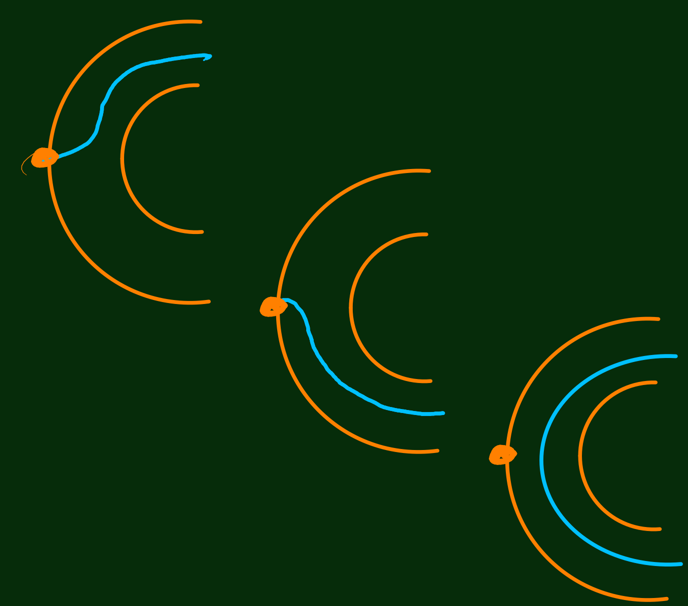
Geometric interpretation





glue

Curve segments



3) Gluing for Ginzburg algebras

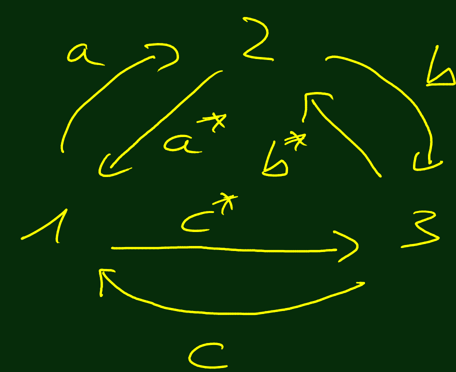
τ ideal triangulation of S

Γ dual ribbon graph

Define cosheaf $\mathcal{F} : \mathrm{Entry}(\Gamma) \to \mathrm{LinCat}_k$ w/

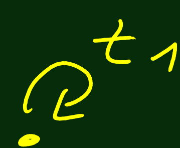
* $\mathcal{F}(v) = \mathcal{D}(\mathcal{G}_\Delta)$

  v vertex of Γ

$\mathcal{G}_\Delta =$ 

* $\mathcal{F}(e) = \mathcal{D}(k[t_1])$

  e edge of Γ

$k[t_1] =$ 

$|t_1| = 1$

polynomial algebra

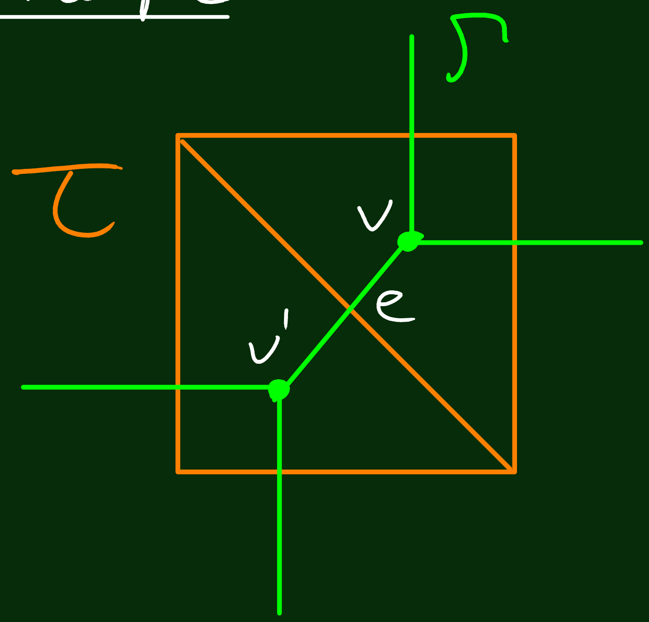
* $\mathcal{F}(e \to v) = \varphi_! : \mathcal{D}(k[t_1]) \longrightarrow \mathcal{D}(\mathcal{G}_\Delta)$

$$\varphi : k[t_1] \longrightarrow \mathcal{G}_\Delta$$
$$* \longmapsto 2$$
$$t_1 \longmapsto \pm(a a^* - b^* b)$$

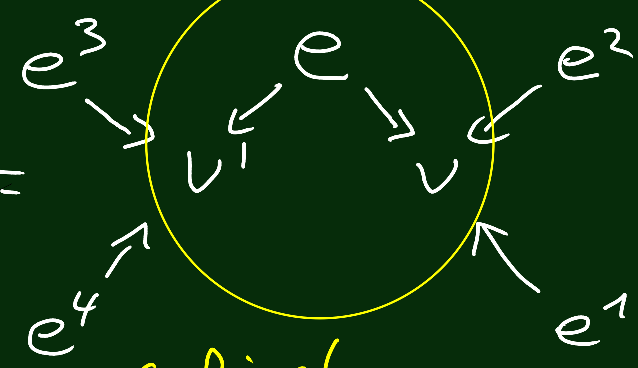
up to cyclic permutations of 1, 2, 3

compare with $d(\ell_i) = \sum p_i [a, a^*] p_i$

Example

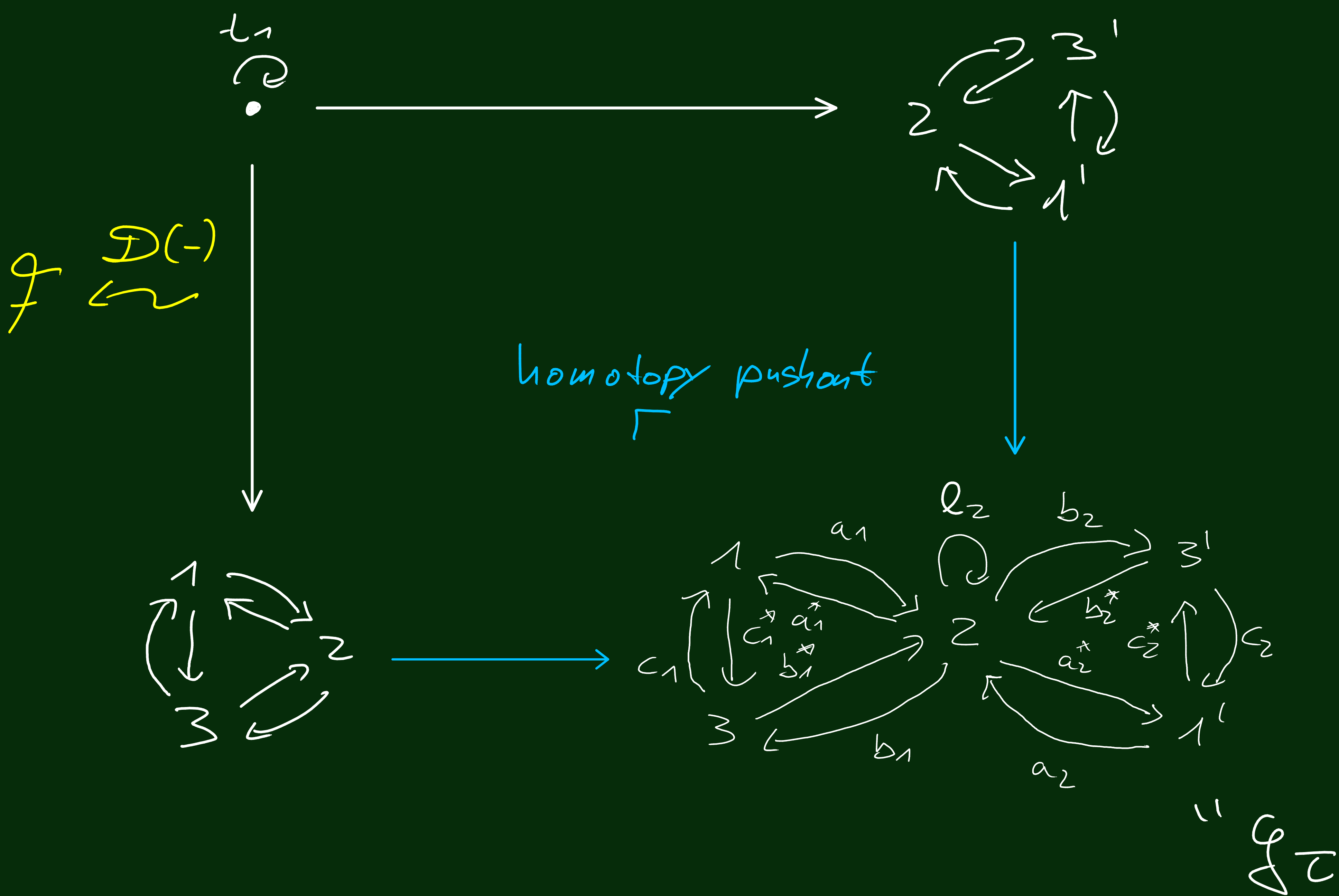


Entry(Γ) =



cofinal subcategory

⇝ same colimit



Theorem (C.)

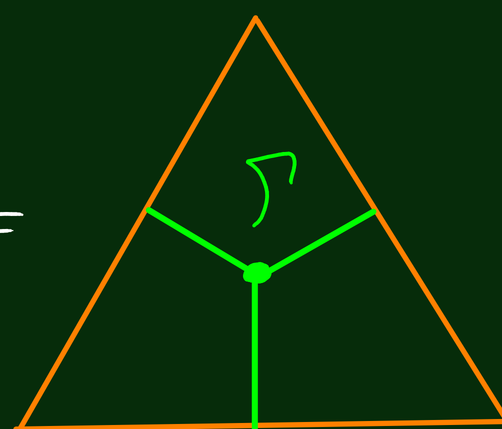
Let $\mathcal{T}$ be an ideal triangulation and Γ the dual ribbon graph. There exists a constructible cosheaf

$\mathcal{F}: \mathrm{Entry}(\Gamma) \to \mathrm{LinCat}_k$ as above with

$$\operatorname{colim} \mathcal{F} \simeq \mathcal{D}(\mathcal{G}_{\mathcal{T}})$$

$\rightsquigarrow$ Relative Ginzburg algebras glue to relative Ginzburg algebras

Geometric model for $\mathcal{G}_{\mathcal{T}}$

Step 1: determine sections for $\mathcal{T} =$ 

right adjoint $\mathrm{Radj}(\mathcal{F})$

$$\begin{tikzcd}
\mathcal{D}(k[t_1]) & & \mathcal{D}(k[t_1]) \\
& \mathcal{D}(\mathcal{G}_\Delta) \arrow[ul, "\mathrm{RHom}(P_3,-)"] \arrow[ur, "\mathrm{RHom}(P_2,-)"'] \arrow[d, "\mathrm{RHom}(P_1,-)"] & \\
& \mathcal{D}(k[t_1]) &
\end{tikzcd}$$

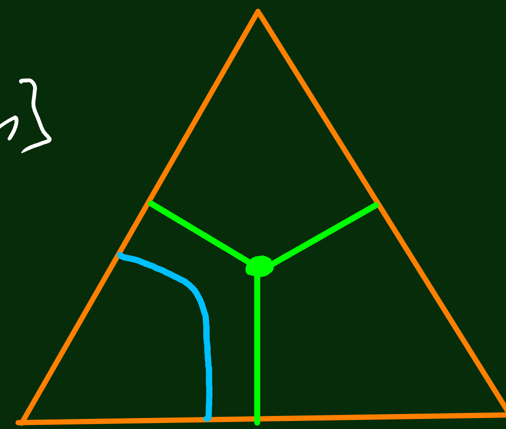
$\mathcal{G}_\Delta =$ 

P_i projective at i is $k[t_1]$-$\mathcal{G}_\Delta$-bimodule

Section	$\color{cyan}{\text{Curve}}$ + local value $L \in \mathcal{D}(k[t_1])$

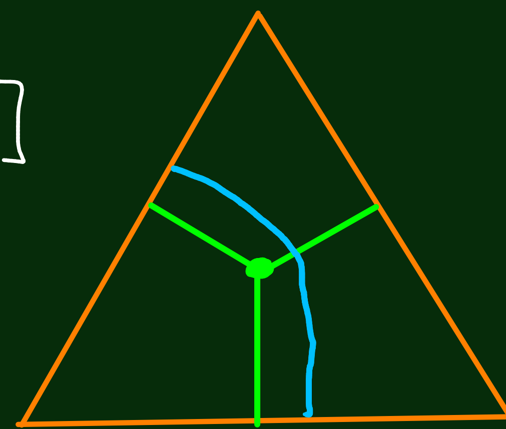
$$\begin{tikzcd}[column sep=small]
k[t_1] & & \mathcal{O} \\
& P_1 \arrow[ul] \arrow[ur] \arrow[d] & \\
& k[t_1] &
\end{tikzcd}$$

$L = k[t_1]$



$$\begin{tikzcd}[column sep=small]
k[t_1] & & \mathcal{O} \\
& \mathrm{cone}(P_2 \to P_3) \arrow[ul] \arrow[ur] \arrow[d] & \\
& k[t_1][1] &
\end{tikzcd}$$

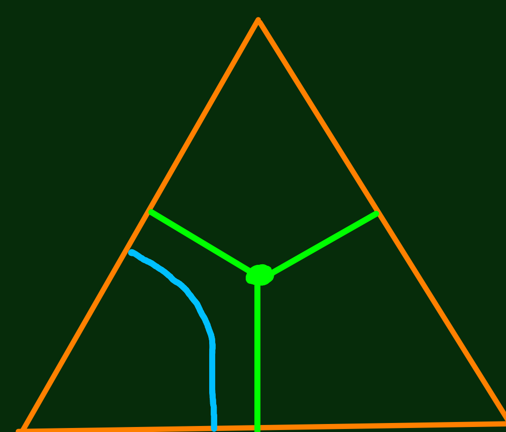
$L = k[t_1]$



Tensor products of the above w/ $k[t_1]$-module L, e.g.

$$\begin{tikzcd}[column sep=small]
k & & \mathcal{O} \\
& k \otimes_{k[t_1]} P_1 \arrow[ul] \arrow[ur] \arrow[d] & \\
& k &
\end{tikzcd}$$

$L = k$



Step 2: glue local sections

$\rightsquigarrow$ Produce section $M_\gamma^L \in \mathcal{D}(\mathcal{G}_\tau)$ for each

* $L \in \mathcal{D}(k[t_1])$
* γ curve in $S \setminus (M \cup \Gamma_0)$ with endpoints in $\partial S \setminus M$

Can also describe $\mathrm{Hom}(M_\gamma^L, M_{\gamma'}^{L'})$ in terms of intersections

Fun application of geometric model:

<u>Proposition (C_0)</u>

Suppose S has no interior marked points.
There exists an isomorphism of graded algebras

tensor algebra

$$H_*(\mathcal{G}_\tau) \simeq \mathcal{J}_\tau \otimes_k k[t_1].$$

Jacobian algebra $H_0(\mathcal{G}_\tau)$.