A surface and a threefold with equivalent singularity categories

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Introduction

Let $0 \neq f \in \mathbb{C}[\![z_0, \ldots, z_d]\!] =: S$

A <u>matrix factorization</u> (MF) of f is a pair

$$(A, B) \in \mathrm{Mat}_n(S) \times \mathrm{Mat}_n(S)$$

satisfying

$$A \cdot B = f \cdot \mathrm{Id}_n = B \cdot A$$

Examples:

$$(\mathrm{Id}_n, f\cdot \mathrm{Id}_n) \text{ and } (f\cdot \mathrm{Id}_n, \mathrm{Id}_n)$$

are trivial matrix factorizations

Example: $f = z_0^2 + z_1^2 + z_2^2 + z_3^2$

Step 1: Consider $g = z_0^2 + z_1^2$

$g = (z_0 + iz_1)(z_0 - iz_1) =: XY$ is MF

Step 2:

$$\begin{pmatrix} -X & (z_2 - iz_3) \\ (z_2 + iz_3) & Y \end{pmatrix}, \begin{pmatrix} -Y & (z_2 - iz_3) \\ (z_2 + iz_3) & X \end{pmatrix}$$

is MF for f.

Remark:

(a) A similar approach yields a Matrix fact.

$$A \cdot A = (z_0^2 + z_1^2 + z_2^2 + z_3^2) \cdot Id_4$$

which is used in Dirac's dicovery of the Dirac equation.

(b) More recently, theoretical physicists are interested in understanding and classifying MFs in relation with string theory.

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is MF for f.

Observation (Knörrer 1987):

Let $0 \neq g \in \mathbb{C}[[z_0, \dots, z_d]]$

The construction in Step 2 defines a map

$$MF(g) \longrightarrow MF(g + z_{d+1}^2 + z_{d+2}^2)$$

This is a bijection (up to sums of trivial MFs).

More precisely, MFs (A,B) of f can be seen as 2-periodic "curved complexes"

$$\cdots \xrightarrow{A} S^n \xrightarrow{B} S^n \xrightarrow{A} S^n \xrightarrow{B} \cdots$$

In analogy with homotopy categories of complexes one can define the <u>homotopy category of MFs</u> $HMF(f)$.

<u>Thm</u> (<u>Knörrer '87</u>) There is a triangle equiv.

$$HMF(f) \xrightarrow{\sim} HMF(f + z_{d+1}^2 + z_{d+2}^2)$$

The Buchweitz – Orlov Singularity Category

X quasi-proj. variety $/\mathbb{C}$

$$\mathrm{Perf}(X) \hookrightarrow D^b(\mathrm{Coh}\,X) \twoheadrightarrow D_{sg}(X) := \frac{D^b(\mathrm{Coh}\,X)}{\mathrm{Perf}(X)}$$

'smooth part' consisting of bounded complexes of vector bundles

Singularity category measures complexity of singularities of X

Thm (Auslander-Buchsbaum & Serre)

X smooth $\iff D_{sg}(X) = 0$

__Thm__ (Orlov) If X has isolated singularities

$$D_{sg}(X) \cong \bigoplus_{s \in \mathrm{Sing}(X)} D_{sg}(\hat{\mathcal{O}}_s)$$

(up to taking direct summands)

where

$$D_{sg}(R) := \frac{D^b(\text{mod-}R)}{K^b(\text{proj-}R)}$$

__singularity category__ of a noetherian ring R.

Thm (Buchweitz, Eisenbud)

Let $0 \neq f \in \mathbb{C}[[z_0, \dots, z_d]] =: S$

$$HMF(f) \cong D_{sg}(S/(f))$$

Def: Two $\binom{\text{possibly}}{\text{noncomm.}}$ algebras R, S are called *singular equivalent* if there is a triangle equivalence

$$D_{sg}(R) \cong D_{sg}(S)$$

"*Trivial examples*": If $\operatorname{gldim} R < \infty$ and $\operatorname{gldim} S < \infty$

(a) $D_{sg}(R) \cong 0 \cong D_{sg}(S)$

(b) $D_{sg}(T \times R) \cong D_{sg}(T) \oplus D_{sg}(R) \cong D_{sg}(T \times S)$

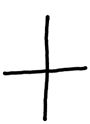
(c) $D^b(\text{mod-}A) \cong D^b(\text{mod-}B) \Rightarrow D_{sg}(A) \cong D_{sg}(B)$

However, for commutative A & B $\Rightarrow$ $A \cong B$

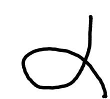
(d) If A and B are commutative and "analytically isomorphic" we can have $D_{sg}(A) \cong D_{sg}(B)$ $\boxed{\text{BUT IN GENERAL}}$ $D_{sg}(A) \not\cong D_{sg}(B)$

e.g. $A = \dfrac{\mathbb{C}[x,y]}{(xy)}$, $B = \dfrac{\mathbb{C}[[x,y]]}{(xy)}$

e.g. $A = \mathbb{C}[x,y]/(xy)$



$B = \dfrac{\mathbb{C}[x,y]}{(y^2 - x^2 - x^3)}$



Non-trivial singular equivalences between commutative rings

(A) $D_{sg}\left(\dfrac{S}{(f)}\right) \cong D_{sg}\left(\dfrac{S[[y_1, \ldots, y_{2n}]]}{(f + y_1^2 + \ldots + y_{2n}^2)}\right)$

[Knörrer '87]

(B) [D. Yang, Y. Kawamata, K.–Karmazyn]
all ~ 2015

$$D_{sg}\left(\mathbb{C}[|z_1, z_2|]^{\frac{1}{m}(1,1)}\right) \cong D_{sg}\left(\frac{\mathbb{C}[|z_1, \dots, z_{m-1}|]}{(z_1, \dots, z_{m-1})^2}\right)$$

where $\mathbb{C}[|z_1, \dots, z_d|]^{\frac{1}{m}(a_1, \dots, a_d)}$ is the invariant ring of the following group action:

$$z_i \longmapsto \varepsilon_m^{a_i} z_i \quad \left(\begin{array}{c}\text{for a primitive} \\ m\text{-th root of unity} \\ \varepsilon_m \in \mathbb{C}\end{array}\right)$$

<u>Remarks:</u>

(i) Until recently this was a complete list of <u>non-trivial</u> singular equivalences between <u>comm.</u> rings!

(ii) All these equivalences preserve the parity of the Krull dimension.

(iii) Knörrer's equivalences are the only known examples where both rings have positive Krull dim.

Thm (K. '21)

There is a singular equivalence

$$D_{Sg}\left(\mathbb{C}[[z_1, z_2, z_3]]^{\frac{1}{2}(1,1,1)}\right) \simeq D_{Sg}\left(\mathbb{C}[[z_1, z_2]]^{\frac{1}{4}(1,1)}\right).$$

Remark:

The Krull dimensions of these invariant rings are 3 and 2, respectively.
In particular, this singular equivalence does not preserve the parity of Krull dimensions.

Proof

Def: A noetherian $\mathbb{C}$-algebra R is *syzygy simple of order m* if there exist $S \in \text{mod-}R$ s.th.

(s1) For every $M \in \text{mod-}R$ there is $n \in \mathbb{N}$ s.th. $\Omega^n(M) \in \underline{\text{add}}\text{-}S$.

(s2) $\underline{\text{add}}\text{-}S \cong \text{mod-}\mathbb{C}$

(s3) $\Omega(S) \cong S^{\oplus m}$ in $\underline{\text{mod}}\text{-}R$.

Prop: Let R & S be syzygy simple of order m. Then

$$D_{sg}(R) \cong D_{sg}(S)$$

Examples: (a) $$K_m := \frac{\mathbb{C}[z_1,\dots,z_m]}{(z_1,\dots,z_m)^2}$$

is syzygy simple of order m,

with $$S = \frac{\mathbb{C}[z_1,\dots,z_m]}{(z_1,\dots,z_m)}$$ simple

(b) [cf. Iyama-Wemyss, "GL(2) McKay-Corr."]

$$R_m := \mathbb{C}[[z_1,z_2]]^{\frac{1}{(m+1)}(1,1)}$$

is syzygy simple of order m.

(c) $$R := \mathbb{C}[[z_1,z_2,z_3]]^{\frac{1}{2}(1,1,1)}$$

is syzygy simple of order 3.

Here, $S = \Omega(\omega_R)$.

[uses Auslander-Reiten theory]

In combination with the proposition above this shows all non-trivial singular equivalences between comm. rings except for Knörrer's

Proof

of

Proposition

Blackbox (Heller):

$\mathcal{A}$ category with $E : \mathcal{A} \longrightarrow \mathcal{A}$

there is a
universal way of
"enlarging" $\mathcal{A}$ s.th.
E "becomes"
autoequivalence

$$S(\mathcal{A}, E)$$

"stabilization" of $(\mathcal{A}, E)$

Thm [Keller-Vossieck '87]

R noetherian. There is a Δ-equiv.

$$\mathcal{S}(\underline{\mathrm{mod}}\text{-}R, \Omega) \cong D_{sg}(R)$$

⌜it is the "stabilization" of

$$F : \underline{\mathrm{mod}}\text{-}R \longrightarrow D_{sg}(R).$$ ⌟

Lemma: If R syzygy simple

$$(\underline{\mathrm{add}}\text{-}S, \Omega) \subset (\underline{\mathrm{mod}}\text{-}R, \Omega)$$

is a left triangulated subcategory and the "stabilization" is a Δ-equiv.

$$\mathcal{S}(\underline{\mathrm{add}}\text{-}S, \Omega) \cong \mathcal{S}(\underline{\mathrm{mod}}\text{-}R, \Omega)$$

Since $\underline{\mathrm{add}}\text{-}S \cong \mathrm{mod}\text{-}\mathbb{C}$ is semisimple the functor

$$\Omega : \underline{\mathrm{add}}\text{-}S \longrightarrow \underline{\mathrm{add}}\text{-}S$$

is completely determined by

$$\Omega(S) \cong S^{\oplus m}.$$

Summing up, the Proposition follows from

$$\begin{tikzcd}
\mathcal{S}(\underline{\mathrm{add}}\text{-}S_R, \Omega_R) \arrow[r, phantom, "\cong"] \arrow[d, phantom, sloped, "\cong"] & \mathcal{S}(\underline{\mathrm{add}}\text{-}S_S, \Omega_S) \arrow[d, phantom, sloped, "\cong"] \\
\mathcal{S}(\underline{\mathrm{mod}}\text{-}R, \Omega_R) \arrow[d, phantom, sloped, "\cong"] & \mathcal{S}(\underline{\mathrm{mod}}\text{-}S, \Omega_S) \arrow[d, phantom, sloped, "\cong"] \\
D_{sg}(R) & D_{sg}(S)
\end{tikzcd}$$

More details on examples of syzygy simple algebras

__Thm__: The invariant rings

$Q = \mathbb{C}[[z_1, \dots, z_d]]^G$, $\quad G \subset GL(d, \mathbb{C})$

are Cohen-Macaulay $\quad$ finite subgrp.

__Lemma__: If R is a local Cohen-Macaulay ring of Krull dimension d and

$M \in \text{mod-}R$ then for all $n \geq d$

$\Omega^n(M)$ is a

maximal Cohen-Macaulay R-module

__Cor__: Understanding high syzygies

$\Omega^{\gg 0}(M)$, $M \in \text{mod-}R$

reduces to understanding syzygies of maximal Cohen-Macaulay modules $\text{MCM}(R)$. (which are again MCM!)

<u>Lemma</u>: Q invariant ring as above. Then there is an injective object
$$\omega_Q \in MCM(Q) \quad \text{"canonical module"}$$
In particular, for all $N \in MCM(Q)$:

$$0 \to \mathrm{Hom}_Q(N, \omega_Q) \to \mathrm{Hom}_Q(Q^n, \omega_Q) \to \mathrm{Hom}_Q(\Omega(N), \omega_Q) \to 0$$

is exact.

$\rightsquigarrow$ All morphisms factor

$$\begin{tikzcd} \Omega(N) \arrow[rr] \arrow[dr] & \arrow[d, phantom, "\circlearrowleft"] & \omega_Q \\ & Q^n \arrow[ur] & \end{tikzcd}$$

$$\rightsquigarrow \quad \underline{\mathrm{Hom}}_Q(\Omega(N), \omega_Q) = 0$$

Irreducible morphisms in $\underline{\mathrm{MCM}}(R)$:

[Auslander-Reiten]

$$R = \mathbb{C}[\![z_1, z_2, z_3]\!]^{\frac{1}{2}(1,1,1)}$$

$$M \;\substack{\longleftarrow \\ \longleftarrow \\ \longleftarrow}\; \omega_R$$

Lemma $\Rightarrow$ $\Omega(N) \in \underline{\mathrm{add}}_R(M)$ for all $N \in \mathrm{MCM}(R)$

and $\underline{\mathrm{End}}_R(M) \cong \mathbb{C}$

$\Rightarrow$ $\boxed{R \text{ is syzygy simple}}$

Applying $\Omega(-)$ to

$$0 \longrightarrow R \longrightarrow \omega_R^{\oplus 3} \longrightarrow M \longrightarrow 0$$

shows $\boxed{\Omega(M) \cong \Omega(\omega_R^{\oplus 3}) \cong M^{\oplus 3}}$

$$R_m := \mathbb{C}[[z_1, z_2]]^{\frac{1}{(m+1)}(1,1)}$$

$$N_1 \rightrightarrows N_2 \rightrightarrows \cdots \rightrightarrows N_{m-2} \rightrightarrows \omega_{R_m} \rightrightarrows S$$

$\boxed{\text{stable Auslander-Reiten quiver}}$

$\rightsquigarrow \exists$ morphisms $N_i \xrightarrow{\neq 0} \omega_{R_m}$ in $\underline{MCM}(R_m)$

& $\omega_{R_m} \xrightarrow{id \neq 0} \omega_{R_m}$

lemma $\rightsquigarrow$ $\Omega(M) \in \underline{\text{add}}\text{-}S$ for $M \in MCM(R_m)$

Moreover, $\underline{\text{End}}_{R_m}(S) \cong \mathbb{C}$

$\longrightarrow$ $\boxed{R_m \text{ syzygy simple}}$ (of order m)
[uses K-theory]