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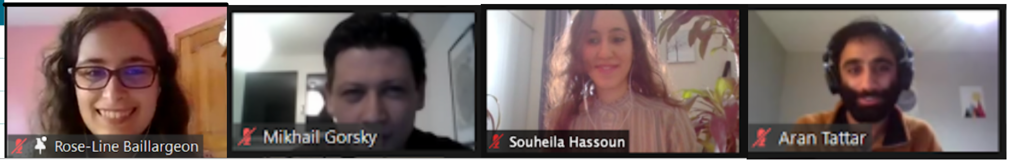
On length functions for an exact category

Thomas Brüstle
Université de Sherbrooke, CA

June 17th, 2021 @ 9:00 am in **your** local time zone

The notion of an exact category has been introduced by Quillen to axiomatize the homological properties of extension-closed subcategories of abelian categories. It allows to define and study homological properties of an exact category, and to define its derived category. However, it turns out that the fundamental concept of length, as known for modules, is less suitable to be studied in the context of an exact category. We aim in this talk to present some recent developments showing for which kind of exact categories an analogue of the Jordan-Hölder property holds, and what one can expect from the notion of length in general. We also present results on the lattice structure of the set of all exact structures that can be attached to a fixed additive category.

Some of the presented results are joint work with Rose-Line Baillargeon, Mikhail Gorsky, Souheila Hassoun and Aran Tattar.



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The TRAC Seminar - Théorie de Représentations et ses Applications et Connections

External homepage

Wed Jun 30 11:00	The FDSeminar organizers	Panel discussion: online meetings - past and future?
Wed Jun 30 11:15	The OCAS organizers	Panel discussion: online meetings - past and future?
Wed Jun 30 11:30	Angeleri-Hügel, Keller, Martsinkovsky	Panel discussion: online meetings - past and future?

Plan:

1. Exact categories
2. Projective objects
3. Simple objects
4. Length
5. Lattice of exact structures

(9:05)

1 Exact categories

An exact category is a pair $(\mathcal{A}, \mathcal{E})$ where

$\mathcal{A}$ is an additive category

$\mathcal{E}$ is a distinguished class of short exact sequences

exact structure

(+ some axioms)

Extreme cases:

(a) $\mathcal{E}_{min} = \{$split exact sequences$\}$ $\quad \mathcal{E}_{min} \subseteq \forall \mathcal{E}$

(b) $\mathcal{E}_{max} = \{$all short exact sequences$\}$ $\quad \forall \mathcal{E} \subseteq \mathcal{E}_{max}$

if $\mathcal{A}$ is abelian

① Exact categories

An **exact category** is a pair $(\mathcal{A}, \mathcal{E})$ where

$\mathcal{A}$ is an additive category

$\mathcal{E}$ is a distinguished class of short exact sequences $X \overset{i}{\rightarrowtail} Y \overset{d}{\twoheadrightarrow} Z$

exact structure

(i = admissible mono, d = admissible epi)

Axiomatic setting (Heller '60, Quillen '73, Keller '90):

Ask $\mathrm{Ext}_{\mathcal{E}}(Z,X) = \{0 \to X \overset{i}{\rightarrowtail} Y \overset{d}{\twoheadrightarrow} Z \to 0 \text{ ses in } \mathcal{E}\}/\cong$ to define an additive bifunctor $\mathcal{A} \times \mathcal{A} \to \mathcal{A}b$

$0 \in$ ... $\hat{=}$ split seq. $X \rightarrowtail X \oplus Z \twoheadrightarrow Z$

$$\begin{tikzcd}
X \arrow[r, tail, "i'"] \arrow[d, equal] & Y' \arrow[r, two heads] \arrow[d] \arrow[dr, phantom, "PB"] & Z' \arrow[d, "\forall f"] & \text{in } \mathcal{E} \\
X \arrow[r, tail, "i"] \arrow[d, "\forall g"'] \arrow[dr, phantom, "PO"] & Y \arrow[r, two heads, "d"] \arrow[d] & Z \arrow[d, equal] & \text{in } \mathcal{E} \arrow[u, Rightarrow] \arrow[d, Rightarrow] \\
X' \arrow[r, tail, "i''"] & Y'' \arrow[r, two heads] & Z & \text{in } \mathcal{E}
\end{tikzcd}$$

plus: composition of admissible monos is an admissible mono

composition of admissible epis is an admissible epi

($\mathrm{Ext}_{\mathcal{E}}$ is a closed bifunctor)

⇝ compare "relative homology" by Buchsbaum '58, Butler, Horrocks '61, ...

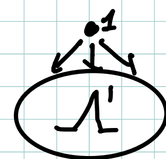
① Exact categories $(\mathcal{A}, \mathcal{E})$: Examples

(c) $\mathcal{A} \subseteq \operatorname{mod} R$ extension-closed subcategory

$\mathcal{E} = \mathcal{E}_{max}|_{\mathcal{A}}$ (e.g. $\mathcal{A} = \mathcal{T}$ torsion class, $\mathcal{A} = \mathcal{F}(\Delta)$ Δ-filtered modules, ...)

(d) $\mathcal{A}$ = BANach spaces (not abelian: f epi $\Leftrightarrow$ Im f dense) but quasi-abelian

$\mathcal{E} = \mathcal{E}_{max}$

(e) one-point extension: $\Lambda =$  $= \begin{bmatrix} k & 0 \\ M & \Lambda' \end{bmatrix}$

$\mathcal{A} = \operatorname{mod} \Lambda$

$\mathcal{E}_{\Lambda'} = \{ \eta = 0 \to X \to Y \to Z \to 0 \text{ ses} \mid \eta|_{\Lambda'} \text{ splits} \}$

Dräxler
Reiten
Smalø
Solberg
1999

$(\operatorname{mod} \Lambda, \mathcal{E}_{max})$ reduce

$(\operatorname{mod} \Lambda, \mathcal{E}_{\Lambda'})$ ⋮

$(\operatorname{mod} \Lambda, \mathcal{E}_{min})$

Resembles Roiter's bocs reduction technique

② Projective objects in $(\mathcal{A}, \mathcal{E})$

$P \in \mathcal{A}$ is $\mathcal{E}$-projective $\iff \operatorname{Ext}_{\mathcal{E}}(P, -) = 0$

$\iff \operatorname{Hom}(P, -)$ exact on $\mathcal{E}$

$\mathcal{P}(\mathcal{E}) = \{\mathcal{E}\text{-projective objects}\}$

Examples: 1) $\mathcal{E} = \mathcal{E}_{min} \Rightarrow \mathcal{P}(\mathcal{E}) = \mathcal{A}$ semisimple

2) $\mathcal{E} = \mathcal{E}_{\Lambda'}$ 1-pt extension $\Rightarrow \operatorname{mod} \Lambda' \subseteq \mathcal{P}(\mathcal{E})$

Now do $\mathcal{E}$-relative homological algebra:
Study $\mathcal{P}(\mathcal{E})$-resolutions,
introduce homotopy category $K(\mathcal{A}, \mathcal{E})$,
derived category $D(\mathcal{A}, \mathcal{E})$

② Projective objects in $(\mathcal{A},\mathcal{E})$

From [Krause, Homological theory of representations]:

Assume: $\mathcal{A}$ idempotent complete, $\mathcal{P}(\mathcal{E}) = \operatorname{add} T$,
every $X \in \mathcal{A}$ admits a finite $\mathcal{P}(\mathcal{E})$-resolution.

Then: T is a tilting object in $(\mathcal{A},\mathcal{E})$, and for $\Lambda = \operatorname{End}_{\mathcal{A}}(T)$:

$(\operatorname{Ext}^n(T,T) = 0$ for $n \neq 0$, $\operatorname{thick}(T) = \mathcal{A})$

thick(T) satisfies 2 out of 3 property: $0 \to X \to Y \to Z \to 0$ is in thick(T) if two of X,Y,Z are in thick(T)

$$\begin{tikzcd}
K^b(\mathcal{P}(\mathcal{E})) \arrow[r, "\operatorname{Hom}(T,-)", "\simeq"'] \arrow[d, phantom, "\cong" rotate=90] & K^b(\operatorname{proj} \Lambda) \arrow[d, phantom, "\cong" rotate=90] \\
D^b(\mathcal{A},\mathcal{E}) \arrow[r, "\operatorname{Hom}(T,-)", "\simeq"'] & D^{perf}(\Lambda)
\end{tikzcd}$$

triangle equivalence

does not "see" $\mathcal{E}$

example:

$\mathcal{A} = \operatorname{mod} R$ representation-finite
$\mathcal{E} = \mathcal{E}_{min}$ $\Big\} \Rightarrow$ T = all indecomposables, Λ = Auslander algebra of R

$\mathcal{E} = \mathcal{E}_{max} \Rightarrow \operatorname{add} T = \operatorname{proj} R$, $\Lambda \cong R$

② Projective objects in $(\mathcal{A}, \mathcal{E})$

Define Grothendieck group $K_0(\mathcal{A}, \mathcal{E}) = \mathbb{Z}^{(\mathrm{Ind}\,\mathcal{A})} \Big/ \begin{matrix} [Y]-[X]-[Z] \\ \forall\ X \rightarrowtail Y \twoheadrightarrow Z \text{ in } \mathcal{E} \end{matrix}$

Assume: $\mathcal{A}$ idempotent complete, $\mathcal{P}(\mathcal{E}) = \operatorname{add} T$,
every $X \in \mathcal{A}$ admits a finite $\mathcal{P}(\mathcal{E})$-resolution.

Then: $K_0(\mathcal{A}, \mathcal{E}) \cong K_0(\Lambda)$ with $\Lambda = \operatorname{End}_{\mathcal{A}}(T)$

In particular, $K_0(\mathcal{A}, \mathcal{E})$ is a free abelian group, with basis $T_1, \dots, T_n$
$(T = T_1 \oplus \dots \oplus T_n)$

exact structure: $\mathcal{E}_{min} \subseteq \mathcal{E} \subseteq \mathcal{E}_{max}$ for $\mathcal{A} = \operatorname{mod} R$

$\mathcal{E}$-projectives: $\operatorname{mod} R \supseteq \mathcal{P}(\mathcal{E}) \supseteq \operatorname{proj} R$

Grothendieck group: $K_0^{split}(R) \geq K_0(\mathcal{A}, \mathcal{E}) \geq K_0(R)$

Roiter's
Docs reduction

larger rank $\rightsquigarrow$ finer invariant
($\hookrightarrow$ B. Blanchette, persistence theory)

(9:25)

③ Simple objects in $(\mathcal{A}, \mathcal{E})$

$K_0(\mathcal{A},\mathcal{E}) \cong K_0(\Lambda)$ ← admits a basis by simple Λ-modules

(every $M \in \operatorname{mod}\Lambda$ has composition series
Jordan-Hölder gives uniqueness)

⇝ $[M] = \underline{\dim}\, M$ multiplicities of simples
length $\ell(M) = \dim M$

Definition: $S \in \mathcal{A}$ is $\mathcal{E}$-*simple* if $A \overset{i}{\rightarrowtail} S$ (admissible monic) implies $A = 0$ or i isomorphism.

$\mathcal{E}$-composition series = sequence of admissible monics i_j

$$0 \overset{i_0}{\rightarrowtail} X_1 \overset{i_1}{\rightarrowtail} X_2 \rightarrowtail \cdots \rightarrowtail X_n = X$$

with all quotients Y_j $\mathcal{E}$-simple: $X_j \overset{i_j}{\rightarrowtail} X_{j+1} \overset{d_j}{\twoheadrightarrow} Y_j$

③ Simple objects in $(\mathcal{A}, \mathcal{E})$

Definition: $S \in \mathcal{A}$ is $\mathcal{E}$-simple if $A \overset{i}{\rightarrowtail} S$ (admissible monic) implies $A = 0$ or i isomorphism.

Example: For $\mathcal{E} = \mathcal{E}_{min}$, $\mathcal{E}$-simple = indecomposable

$\rightsquigarrow$ Schur Lemma fails for $\mathcal{E}$-simples (End S need not be a division ring)

Definition: $f: X \to Y$ is admissible if $f = i \circ d$ (ad. monic $\circ$ ad. epic)

$\mathcal{E}$-Schur Lemma [Hassoun, Roy '19] Let X, Y be $\mathcal{E}$-simple objects. Then every non-zero admissible morphism $f: X \to Y$ is invertible.

$\rightsquigarrow$ $\mathrm{End}^{ad}(X)$ would be a division ring (if it were a ring)

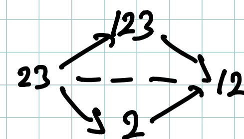
[BHT'20] $\mathrm{End}^{ad}(X) \subseteq \mathrm{End}(X)$ subring $\forall X \iff \mathcal{A}$ abelian, $\mathcal{E} = \mathcal{E}_{max}$

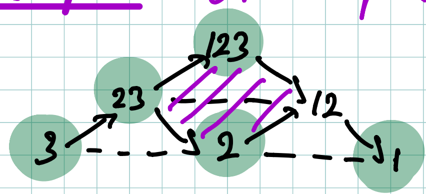
(cannot consider $\mathrm{End}^{ad}(T)$ for T tilting ...)

③ Simple objects in $(\mathcal{A},\mathcal{E})$

Definition: $S\in\mathcal{A}$ is $\mathcal{E}$-simple if $A \overset{i}{\rightarrowtail} S$ (admissible monic) implies $A=0$ or i isomorphism.

Example: $\mathcal{A} = \mathrm{rep}(1\to 2\to 3)$, $\mathcal{E}$ given by





$23 \xrightarrow{i} 123\oplus 2$ admissible

$23 \hookrightarrow 123$ not admissible

$\mathcal{E}$-simples: $1, 2, 3, 23, 12, 123$

$\mathcal{E}$-projectives: $3, 23, 123, 2, 1$

$\mathcal{E}$-simples do not form a basis for $K_0(\mathcal{A},\mathcal{E})$

$\not\Updownarrow$

Λ-simples $\in K_0(\mathrm{End}\, T)$, with $K_0(\mathcal{A},\mathcal{E}) \cong K_0(\mathrm{End}\, T)$

Theorem [Enomoto '19] Assume $n = \mathrm{rank}\, K_0(\mathcal{A},\mathcal{E}) < \infty$. Then $(\mathcal{A},\mathcal{E})$ satisfies Jordan-Hölder property $\Leftrightarrow$ $n = \#\ \mathcal{E}$-simples

③ Simple objects in $(\mathcal{A}, \mathcal{E})$

Definition: $(\mathcal{A}, \mathcal{E})$ satisfies Jordan-Hölder (JHP) $\Longleftrightarrow$ any two $\mathcal{E}$-composition series have same length, same quotients (up to order, iso)

All proofs (I know) of JHP (for groups, modules, ...) use some form of Noether's isomorphism theorem

$$\begin{tikzcd}[column sep=small] & L+N \arrow[dl, dash] \arrow[dr, dash] & \\ L \arrow[dr, dash] & & N \arrow[dl, dash] \\ & L \cap N & \end{tikzcd}$$

Intersection, sum is <u>not</u> given by axioms of an exact category (pull-back of admissible mono $\begin{tikzcd} L\cap N \arrow[r, tail] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner\,PB"] & N \arrow[d, tail] \\ L \arrow[r, tail] & M \end{tikzcd}$ need not exist)

Theorem [BHT'20]

$L \cap N$, $L+N$ exist, and are admissible subobject $\forall\, L, N \rightarrowtail M$

$\Longleftrightarrow$ $(\mathcal{A}$ is abelian, $\mathcal{E} = \mathcal{E}_{max})$

③ Simple objects in $(\mathcal{A}, \mathcal{E})$

This proof technique cannot be extended beyond abelian case.

(JHP) may still be valid when $\mathcal{E} \neq \mathcal{E}_{max}$:

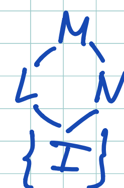
Example: Assume $\mathcal{A}$ Krull-Schmidt category, $\mathcal{E} = \mathcal{E}_{min}$.

$\mathcal{E}$-composition series $= 0 \subset X_1 \hookrightarrow X_1 \oplus X_2 \hookrightarrow \cdots \hookrightarrow X_1 \oplus \cdots \oplus X_n = X$

(JHP) $\Leftrightarrow$ $X_1 \oplus \cdots \oplus X_n = X$ is unique up to iso, permutation $\Leftrightarrow$ Krull-Schmidt property

What about $\mathcal{E}_{min} \subseteq \mathcal{E} \subseteq \mathcal{E}_{max}$?

In [BHT'20], we introduce



- generalized intersection – can be a set of objects
- generalized radical $\mathrm{rad}_{\mathcal{E}}(M)$
- (AW) Artin–Wedderburn condition: $\mathrm{rad}_{\mathcal{E}}(M) = \{0\} \Leftrightarrow M$ $\mathcal{E}$-semisimple

Thm: $\left.\begin{array}{l} (\mathcal{A}, \mathcal{E}) \text{ Krull-Schmidt} \\ \text{and satisfies (AW)} \end{array}\right\} \Rightarrow$ (JHP) satisfied.

(9:40)

④ Length

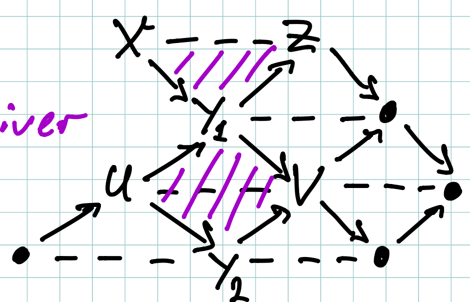
$(\mathcal{A},\mathcal{E})$ satisfies Jordan-Hölder property $\Big\} \Rightarrow$ Define $\mathcal{E}$-length $l_{\mathcal{E}}(X)$ to be the length of an$_{\text{(unique)}}$ $\mathcal{E}$-composition series of X

Have $l_{\mathcal{E}}(Y) = l_{\mathcal{E}}(X) + l_{\mathcal{E}}(Z)$ for $0 \to X \to Y \to Z \to 0$ in $\mathcal{E}$

General case: Composition series can have different lengths.

Example: $\mathcal{A} = KQ$ with $Q = \bullet \leftarrow \bullet \longrightarrow \bullet \longrightarrow \bullet$

AR-quiver



$\mathcal{E}$ given by $X \to Y_1 \to Z$,

$U \to Y_1 \oplus Y_2 \to V$

U, V $\mathcal{E}$-simple $\Big\} \Rightarrow$ $0 \to U \to Y_1 \oplus Y_2$ $l=2$

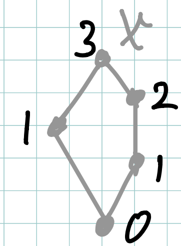
$0 \to X \to Y_1 \to Y_1 \oplus Y_2$ $l=3$

2 composition series

④ Length for $(\mathcal{A}, \mathcal{E})$

Definition: Set $\ell_{\mathcal{E}}(X) = \sup(\text{lengths of } \mathcal{E}\text{-composition series of } X)$

Similar: height in poset $\mathcal{P}_{\mathcal{E}}(X)$



X has finite $\mathcal{E}$-length $\iff$ all proper $\mathcal{E}$-filtrations of X have common bound on length

Proposition: [BHT '20] Let $(\mathcal{A}, \mathcal{E})$ be exact category, $X \in \mathcal{A}$. Then

X is $\mathcal{E}$-Artin & $\mathcal{E}$-Noetherian $\Rightarrow$ X admits a finite composition series

Corollary: Let $(\mathcal{A}, \mathcal{E})$ be a JH-exact category. Then

X is $\mathcal{E}$-Artin & $\mathcal{E}$-Noetherian $\iff$ X has finite $\mathcal{E}$-length

④ Length for $(\mathcal{A}, \mathcal{E})$

Definition: Set $l_{\mathcal{E}}(X) = \sup(\text{lengths of } \mathcal{E}\text{-composition series of } X)$

$\rightsquigarrow$ $l_{\mathcal{E}}$ is a super-additive function:

$$l_{\mathcal{E}}(Y) \geq l_{\mathcal{E}}(X) + l_{\mathcal{E}}(Z) \quad \forall\ 0 \to X \to Y \to Z \to 0 \text{ in } \mathcal{E}$$

For two exact structures $\mathcal{E} \subseteq \mathcal{E}'$ on $\mathcal{A}$, we get

$$l_{\mathcal{E}}(X) \leq l_{\mathcal{E}'}(X) \quad \forall X \in \mathcal{A}$$

$\mathcal{A} = \operatorname{mod} R$, $l_{\mathcal{E}_{max}}$ = abelian length

$\rotatebox{90}{$\leq$}$

$l_{\mathcal{E}}$ — reduce $l_{\mathcal{E}}(X)$ until every indecomposable has length 1

$\rotatebox{90}{$\leq$}$

$l_{\mathcal{E}_{min}}$ = Krull-Schmidt length (#direct summands)

(9:50)

⑤ Lattice of exact structures

Assume $\mathcal{A}$ essentially small!

How many exact structures $\mathcal{E}$ exist on a fixed $\mathcal{A}$?
What is the structure of $\mathrm{Ex}(\mathcal{A}) = \{\text{all exact structures on } \mathcal{A}\}$
It clearly is a poset under containment $\mathcal{E} \subseteq \mathcal{E}'$.

Hassoun, Langford presented this question (and a conjecture) at the 2018 ICRA.

<u>Enomoto '18</u>: Assume $\mathcal{A}$ is additively finite.
Then any $\mathcal{E} \in \mathrm{Ex}(\mathcal{A})$ is uniquely determined by the Auslander-Reiten sequences which $\mathcal{E}$ contains.

In particular, $\mathrm{Ex}(\mathcal{A})$ is a boolean lattice.

$$|\mathrm{Ex}(\mathcal{A})| = 2^m, \quad m = \#\text{AR-sequences in } \mathcal{A}$$

$\mathcal{E}_{max} \longrightarrow \cdots m \text{ steps} \cdots \longrightarrow \mathcal{E}_{min}$ split exact

⑤ Lattice of exact structures

View exact structure $\mathcal{E}$ as bimodule $\mathrm{Ext}_{\mathcal{E}} : \mathcal{A} \times \mathcal{A}^{op} \to Ab$

When $\mathcal{A}$ is abelian, $\mathrm{Ext}_{\mathcal{E}} \subseteq \mathrm{Ext}^1_{\mathcal{A}}(-,-)$ sub(bi-)module.

Rump '11: Any additive $\mathcal{A}$ admits a unique maximal exact structure $\mathcal{E}_{max}$. Call the corresponding bimodule $\mathrm{Ext}^1_{\mathcal{A}}$.

Consequence: $\mathrm{Ex}(\mathcal{A})$ is a complete lattice,
with meet $\mathcal{E} \wedge \mathcal{E}' = \mathcal{E} \cap \mathcal{E}'$
and join $\mathcal{E} \vee \mathcal{E}' = \bigcap \{\mathcal{E}'' : \mathcal{E} \subseteq \mathcal{E}'', \mathcal{E}' \subseteq \mathcal{E}''\}$

Remember: composition of admissible monos is an admissible mono
composition of admissible epis is an admissible epi
($\mathrm{Ext}_{\mathcal{E}}$ is a closed subbimodule of $\mathrm{Ext}^1_{\mathcal{A}}$)

5 Lattice of exact structures

Well-known: AR-sequences lie in the socle of $\mathrm{Ext}^1_{\mathcal{A}}(-,-)$

$\mathrm{Ext}_{\mathcal{E}}$ is closed subbimodule of $\mathrm{Ext}^1_{\mathcal{A}}$ $\Rightarrow$ $\mathrm{Ext}_{\mathcal{E}}$ is uniquely determined by its socle (maximal sub-bimodule containing these AR-sequences) in case $\mathcal{A}$ is additively finite.

[BBGH'20]: What about the non-closed sub-bimodules?

$$\mathrm{Ex}(\mathcal{A}) \xleftrightarrow[\text{lattice iso}]{1-1} \text{lattice of closed sub-bimodules of } \mathrm{Ext}^1_{\mathcal{A}}$$

$$\cap \qquad\qquad\qquad\qquad \cap$$

$$\mathrm{WEx}(\mathcal{A}) \xleftrightarrow[\text{lattice iso}]{1-1} \mathrm{Sub}(\mathrm{Ext}^1_{\mathcal{A}})$$

weakly exact structures: all axioms except closedness

all sub-bimodules of $\mathrm{Ext}^1_{\mathcal{A}}$, modular lattice

5 Lattice of exact structures

$$\begin{array}{ccc}
\mathrm{Ex}(\mathcal{A}) & \xleftrightarrow{1-1} & \text{lattice of closed sub-bimodule of } \mathrm{Ext}^1_{\mathcal{A}} \\
\cap\!\!| & & \cap\!\!| \\
\mathrm{WEx}(\mathcal{A}) & \xleftrightarrow{1-1} & \mathrm{Sub}(\mathrm{Ext}^1_{\mathcal{A}})
\end{array}$$

weakly exact structures: all axioms except closedness

all sub-bimodules of $\mathrm{Ext}^1_{\mathcal{A}}$, modular lattice

Warning: $\mathrm{Ex}(\mathcal{A}) \subseteq \mathrm{WEx}(\mathcal{A})$

subset, but not a sublattice:

lattice		lattice
inclusion	=	inclusion
meet $\wedge$	=	meet $\wedge$
join $\vee$	$\neq$	join $+$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