Invariants related to the finitistic dimension

Based on two preprints:

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Main Questions:

- What is the value of the finitistic dimension?
- Can it be "pinned down" by basic invariants?

By basic, mean: structural invariant of rings, essentially computable (maybe by software ...)

Setup: Λ is a two-sided Noetherian ring, e.g. an Artin algebra.

$$\operatorname{findim} \Lambda := \sup\{\operatorname{pd} M \mid M \in \operatorname{mod} \Lambda,\ \operatorname{pd} M < \infty\}$$

f.g. right modules

$$\operatorname{Findim} \Lambda := \sup\{\operatorname{pd} M \mid M \in \operatorname{Mod} \Lambda,\ \operatorname{pd} M < \infty\}$$

all right modules

These are the little and big finitistic dimensions.

Use $\operatorname{findim} \Lambda^{op}$, $\operatorname{Findim} \Lambda^{op}$, ... for left version.

Examples of basic invariants

- $\operatorname{grade} M := \inf\{ i \geq 0 \mid \operatorname{Ext}^i(M, \Lambda) \neq 0 \}$
- $\operatorname{depth} \Lambda := \sup_S \operatorname{grade} S$, over all simples S in $\operatorname{mod} \Lambda$.

For (R, m, K) a local commutative noeth. ring, then

$$\operatorname{depth} R = \inf\{ i \geq 0 \mid \operatorname{Ext}^i(K, R) \neq 0 \}$$

recovers the usual notion.

In general, we have an inequality

$$\operatorname{depth} \Lambda \leq \operatorname{findim} \Lambda^{op}$$ [Jans, ~60's]

We may have

$$\operatorname{depth} \Lambda \ll \operatorname{findim} \Lambda^{op}$$ [Huisgen-Zimmermann, Ringel]

Example of Nakayama algebras due to Ringel w/ $\operatorname{depth} \Lambda = 2$, $\operatorname{findim} \Lambda^{op} = n \geq 2$ arbitrary.

However, sometimes the depth <u>does</u> capture the finitistic dimension ...

Ex. (1) Commutative noetherian rings R

$$\operatorname{depth} R = \operatorname{findim} R \quad \left[\begin{array}{c}\text{Auslander-}\\ \text{Buchsbaum}\end{array}\right]$$

(R not necessarily local, but reduces to local case.)

Anything non-commutative?

Ex.: 2 Rings of finitistic dimension zero.

Thm (Bass, ~60's). Let Λ be two-sided noeth.

Then $\operatorname{depth} \Lambda = 0 \iff \operatorname{findim} \Lambda^{op} = 0$.

So $\operatorname{depth} \Lambda = \operatorname{findim} \Lambda$ works in dimension zero.

The depth is not a bad starting point.

Rephrase Bass's Theorem :

$$\operatorname{depth} \Lambda = 0 \iff \operatorname{Hom}(S,\Lambda) \neq 0 \quad \forall \underset{\text{simple}}{S}$$

$$\iff S \overset{i}{\hookrightarrow} \Lambda_\Lambda \quad \forall \underset{\text{simple}}{S}$$

$$\iff S \text{ is a syzygy} \quad \forall \underset{\text{simple}}{S}$$

Syzygies: submodules of projectives.

$$M \rightsquigarrow \Omega M := \operatorname{Ker}(P \twoheadrightarrow M), \quad P \text{ projective}$$

$$\Omega : \underline{\mathrm{mod}}\,\Lambda \to \underline{\mathrm{mod}}\,\Lambda$$

↖ module category modulo projectives

Bass's Thm: $\operatorname{fin\,dim} \Lambda^{op} = 0 \iff$ every simple $S \in \mathrm{mod}\,\Lambda$ is a syzygy.

Let's look at other examples ...

Def^n: Λ is Iwanaga-Gorenstein if
$\operatorname{injdim} \Lambda < \infty$ and $\operatorname{injdim} \Lambda^{op} < \infty$.

(In this case $\operatorname{injdim} \Lambda = \operatorname{injdim} \Lambda^{op}$
by Zaks.)

Ex.: 3 Iwanaga-Gorenstein rings

Thm (Iwanaga): In this case

$$\operatorname{findim} \Lambda = \operatorname{findim} \Lambda^{op} = \operatorname{injdim} \Lambda.$$

Can we relate Iwanaga's and Bass's Theorems?

Thm (Buchweitz): Let Λ be Iwanaga-Gorenstein of dimension d. Then $\forall M \in \operatorname{mod} \Lambda$, $\Omega^d M$ is an "∞-syzygy"

i.e. $\exists N_1, N_2, \ldots,$ $\quad \Omega^d M \simeq \Omega^{d+1} N_1 \simeq \Omega^{d+2} N_2 \simeq \ldots$

In particular, looking at simples S we can find a syzygy module ΩN_S "further"

$$\begin{array}{ccc} S & & \Omega N_S \\ \Omega S & & \Omega^2 N_S \\ \vdots & & \vdots \\ \Omega^d S & \simeq & \Omega^{d+1} N_S \end{array}$$

Can find syzygy at "higher level"

Aim: Turn this into an invariant, try to compute finitistic dimension with it.

Language: If $M \simeq \Omega N$ is a syzygy, we say that M can be "delooped".

analogy with topology.

Bass: $\operatorname{findim} \Lambda^{op} = 0 \iff$ every simple S can be delooped.

Buchweitz: Λ Iwanaga-Gorenstein $\Longrightarrow$ every simple S can "eventually" be delooped.

<u>Def^n</u>: The delooping level of a module $M \in \operatorname{mod} \Lambda$ is $\operatorname{dell} M := \inf\{ i \geq 0 \mid \Omega^i M \overset{\oplus}{\leq} \Omega^{i+1} N \text{ for some } N \}$

direct summand, needed for technical reasons.

Def^n: The delooping level of Λ is

$$\operatorname{dell} \Lambda := \sup_S \operatorname{dell} S \quad , \quad S \text{ simple in } \operatorname{mod} \Lambda.$$

Facts: • $\operatorname{dell} \Lambda = 0 \iff S$ is a syzygy $\forall S$

• $\operatorname{dell} \Lambda \leq n \iff \Omega^n S \overset{\oplus}{\leq} \Omega^{n+1} N \quad \forall S$

• Λ Iwanaga-Gorenstein $\xRightarrow{\text{Buchweitz's Thm}} \operatorname{dell} \Lambda = \operatorname{inj\,dim} \Lambda$.

Prop.: We have inequalities

$$\operatorname{depth} \Lambda \overset{\text{Jans'}}{\leq} \operatorname{fin\,dim} \Lambda^{op} \leq \operatorname{dell} \Lambda.$$

If Λ is an Artin algebra, then

$$\operatorname{depth} \Lambda \leq \operatorname{fin\,dim} \Lambda^{op} \leq \operatorname{Fin\,dim} \Lambda^{op} \leq \operatorname{dell} \Lambda. \quad \square$$

So depth Λ and dell Λ serve to bound the values of fin dim Λ^{op} (resp. Fin dim Λ^{op}).

Remark: Similar invariants due to

- Goodearl–Huisgen-Zimmermann (repetition index)
- Cibils (syzygy quiver for monomial algebras)

Part II

Is $\operatorname{dell} \Lambda$ a "basic invariant" of Λ?

computable?

Answer: Yes!

For any simple S, need to find $N \in \operatorname{mod} \Lambda$ w/ $\Omega^n S \overset{\oplus}{\leq} \Omega^{n+1} N$ for some $n \geq 0$.

$\rightsquigarrow$ Only test the universal N

(for each level)

Recall: $\Omega : \underline{\mathrm{mod}}\,\Lambda \longrightarrow \underline{\mathrm{mod}}\,\Lambda$ syzygy functor,

$\mathrm{Tr} : \underline{\mathrm{mod}}\,\Lambda \longrightarrow \underline{\mathrm{mod}}\,\Lambda^{op}$ transpose, $\mathrm{Tr}^2 = \mathrm{id}$.

<u>Thm</u> (Auslander-Reiten, 92):

Ω has a left adjoint given by $\Sigma := \mathrm{Tr}\,\Omega\,\mathrm{Tr}$.

<u>Pf</u>: Use identity $\underline{\mathrm{Hom}}(M,N) \cong \mathrm{Tor}_1(N, \mathrm{Tr}M)$.

Then $\underline{\mathrm{Hom}}(\Sigma M, N) \cong \mathrm{Tor}_1(N, \Omega\mathrm{Tr}M)$

$\cong \mathrm{Tor}_1(\Omega N, \mathrm{Tr}M) \cong \underline{\mathrm{Hom}}(M, \Omega N)$. $\square$

For any $X \in \underline{\mathrm{mod}}\,\Lambda$, $\eta_k : X \longrightarrow \Omega^k \Sigma^k X$.

unit of adjunction (Σ^k, Ω^k)

Prop.: The following are equivalent for $M \in \underline{\mathrm{mod}}\,\Lambda$:

(a) $\operatorname{dell} M \leq n$.

(b) $\Omega^n M$ is a summand of $\Omega^{n+1} N$ for some N.

(c) $\Omega^n M$ is a summand of $\Omega^{n+1}\Sigma^{n+1}\Omega^n M$.

(d) $\Omega^n M \xrightarrow{\eta_{n+1}} \Omega^{n+1}\Sigma^{n+1}\Omega^n M$ is a split monomorphism. $\square$

$N := \Sigma^{n+1}\Omega^n M$ is a universal choice for level n.

$\implies$ dell Λ is essentially computable if finite.

For Artin algebras, Λ/J contains all simples.

Neatly packaged:

$$\operatorname{dell}\Lambda = \inf\Big\{ n \geq 0 \;\Big|\; \underbrace{\Omega^n(\Lambda/J) \xrightarrow{\eta_{n+1}} \Omega^{n+1}\Sigma^{n+1}\Omega^n(\Lambda/J)}_{\text{is a split monomorphism}} \Big\}.$$

<u>Part III</u>: How can we study

$$\Omega^n S \xrightarrow{\eta_{n+1}} \Omega^{n+1}\Sigma^{n+1}\Omega^n S \quad ?$$

Answer: Read the classics!

Auslander–Bridger "Stable module theory"

(Σ, Ω) adjoint endofunctors $\Longrightarrow$ $\exists$ adjunctions (Σ^k, Ω^k) $\forall k \geq 1$

For any $X \in \underline{\mathrm{mod}}\,\Lambda$, these fit into a "tower"

$$X \to \Omega\Sigma X \to \Omega^2\Sigma^2 X \to \Omega^3\Sigma^3 X \to \cdots$$

For $X = \Omega^n M$, want to study first $(n+1)$ stages

$$\begin{tikzcd} \Omega^n M \arrow[r] \arrow[rrrr, bend right, "\eta_{n+1}"'] & \Omega\Sigma\Omega^n M \arrow[r] & \cdots \arrow[r] & \Omega^n\Sigma^n\Omega^n M \arrow[r] & \Omega^{n+1}\Sigma^{n+1}\Omega^n M \end{tikzcd}$$

Want to know if left invertible

Auslander–Bridger studied this tower

(under the notation $D_n^2 := \Omega^n \Sigma^n$)

Let $X \xrightarrow{\sigma} X^{**}$ be the eval map $\left((-)^* := \mathrm{Hom}(-,\Lambda) \right)$

$tX := \mathrm{Ker}(\sigma)$

<u>Fact</u>: $\exists$ commutative diagram

$$
\begin{tikzcd}
X \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowleft"] & X/tX \arrow[r] \arrow[d, "\cong"] \arrow[dr, phantom, "\circlearrowleft"] & X^{**} \arrow[d, "\cong"] \\
X \arrow[r] & \Omega\Sigma X \arrow[r] & \Omega^2\Sigma^2 X \arrow[r] & \cdots
\end{tikzcd}
$$

So this tower is related to (higher) reflexivity of X.

We want $X = \Omega^n M$ $\rightsquigarrow$ interested in (higher) reflexivity of syzygy modules!

Auslander-Bridger studied this, introduced conditions

$$\operatorname{grade} \operatorname{Ext}^i(M, \Lambda) \geq i \quad \forall\, 1 \leq i \leq n$$

naturally a Λ^{op}-module

Classical problem:

Are 2^{nd} syzygy modules reflexive?

(Reflexive modules are always 2^{nd} syzygies)

In general, No.

Auslander–Bridger:

$\Omega^2 M$ is reflexive $\Longleftrightarrow$ $\operatorname{grade} \operatorname{Ext}^2(M, \Lambda) \geq 1$.

slightly weaker than the above condition at $n=2$

When $\Omega^2 M$ is reflexive, can ascend first two stages

$$
\begin{tikzcd}
\Omega^2 M \arrow[rr, "\sigma", "\cong"'] \arrow[d, equal] & & (\Omega^2 M)^{**} \arrow[d, phantom, sloped, "\cong"] & \\
\Omega^2 M \arrow[r, "\cong"] & \Omega\Sigma\Omega^2 M \arrow[r, "\cong"] \arrow[u, phantom, "\circlearrowright"] & \Omega^2\Sigma^2\Omega^2 M \arrow[r] & \Omega^3\Sigma^3\Omega^2 M
\end{tikzcd}
$$

grade $\mathrm{Ext}^2(M,\Lambda) \geq 1$ not strong enough to get up here.

Want to go three stages up ...

(Analogous story for $\Omega^n M$, all $n \geq 1$)

Use this to study $\operatorname{dell} \Lambda$!

Let S simple, $j_S := \operatorname{grade} S$. Consider the condition

$$\operatorname{grade} \operatorname{Ext}^{j_S}(S, \Lambda) \geq j_S \qquad (*)$$

<u>Main Theorem (−)</u>: Let $S \in \operatorname{mod} \Lambda$ be simple with finite grade j_S.

If $(*)$ holds, then there is an isomorphism

$$\Omega^{j_S} S \xrightarrow[\cong]{\eta_{j_S+1}} \Omega^{j_S+1} \Sigma^{j_S+1} \Omega^{j_S} S.$$

In particular, $\operatorname{dell} S \leq \operatorname{grade} S$. $\square$

The Auslander–Bridger condition lets us "deloop" simples (cat level = grade).

Here $\operatorname{depth} \Lambda = \sup_S \operatorname{grade} S$

and $\operatorname{dell} \Lambda = \sup_S \operatorname{dell} S$

start to look similar...

Recall: $\operatorname{depth} \Lambda \le \operatorname{findim} \Lambda^{op} \le \operatorname{dell} \Lambda$.

$+ \operatorname{Findim} \Lambda^{op}$ if Artin

Corollary: Assume $(*)$ holds $\forall S$ with finite grade.

Then $\operatorname{depth} \Lambda = \operatorname{findim} \Lambda^{op} = \operatorname{dell} \Lambda$.

$+ \operatorname{Findim} \Lambda^{op}$ if Artin

Pf: By Theorem $\operatorname{dell} S \le \operatorname{grade} S$, $\therefore$ $\operatorname{dell} \Lambda \le \operatorname{depth} \Lambda$. $\square$

Example: Let R be commutative Noetherian, $S = R/m$ simple, $j_{R/m} = \text{grade } R/m$

Then $\operatorname{Ext}^j(\underbrace{\operatorname{Ext}^{j_{R/m}}(R/m, R)}_{R/m \text{ vector space}}, R) = 0$ for $j < j_{R/m}$.

Consequence: $\operatorname{depth} R = \operatorname{findim} R = \operatorname{dell} R$

for all commutative rings.

(local case): (R, m, K) with depth $d \Longrightarrow \Omega^d K \xrightarrow[\cong]{\eta_{d+1}} \Omega^{d+1}\Sigma^{d+1}\Omega^d K$.

Example : Rings with depth zero.

$$\operatorname{depth} \Lambda = 0 \iff \operatorname{grade} S = 0 \quad \forall S$$

Condition (*) is trivially true ($\operatorname{grade} S^* \geq 0$)

$$\Longrightarrow \operatorname{depth} \Lambda = \operatorname{findim} \Lambda^{op} = \operatorname{dell} \Lambda \; (= 0)$$

Bass's Theorem "$\operatorname{depth} \Lambda = 0 \iff \operatorname{findim} \Lambda^{op} = 0$"

is a limiting special case!

Part IV: Beyond Auslander–Bridger.

Restrict to Artin algebras ...

Have $\operatorname{depth} \Lambda \leq \operatorname{findim} \Lambda^{op} \leq \operatorname{Findim} \Lambda^{op} \leq \operatorname{dell} \Lambda$.

- Auslander-Bridger condition on simples $\Longrightarrow$ all equalities.
- First inequality often strict otherwise.
- Second inequality sometimes strict (Huisgen-Zimmermann, Smalø).

First finitistic dimension conjecture

What about $\mathrm{Findim}\,\Lambda^{op} \leq \mathrm{dell}\,\Lambda$?

Equality here seems to hold surprisingly often...

Have $\mathrm{Findim}\,\Lambda^{op} = \mathrm{dell}\,\Lambda$ in following cases:

- Artin algebras with radical square zero.
- Nakayama algebras (Ringel, Sen).
- Smalø's counterexamples to first finitistic dimension conjecture.
- Cohen-Macaulay Artin algebras (in the sense of Auslander-Reiten)

includes Iwanaga-Gorenstein algebras

No counterexample known.

Open Question:

Does $\operatorname{Findim} \Lambda^{op} = \operatorname{dell} \Lambda$ hold for all Artin algebras?

Thank

You!