TALK

Cluster structures and Cluster theories

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Joint with IGUSA and JOB D. ROCK

fdSeminar
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9-10am

Cluster structures and Cluster theories

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Joint with Igusa and JOB D. ROCK

Representations of quivers of type A

I, II, III, IV

Igusa, Rock, Todorov

JOB D. ROCK

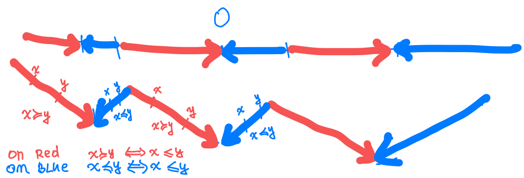
- Quivers of type A:

A_n $\leftarrow \leftarrow \leftarrow \cdots \leftarrow \bullet$ $\qquad \rightarrow \leftarrow \rightarrow \rightarrow \cdots \leftarrow$

A_∞ $\leftarrow \leftarrow \leftarrow \cdots$ $\qquad \leftarrow \rightarrow \rightarrow \cdots$

$A_{\infty\infty}$ $\cdots \leftarrow \leftarrow \cdots$ $\qquad \cdots \leftarrow \rightarrow \leftarrow \cdots$

$A_{\mathbb{R}}$



- **Representations of quivers (I)**

$A_n, A_\infty, A_{\infty\infty}$ – standard repr: $\{V_i\}_{i\in Q_0}$ $\{V_a\}_{a\in Q_1}$

$A_{\mathbb{R}}$

$$\begin{array}{ccc} V = \left(\{V_x\}_{x\in\mathbb{R}}, \right. & \left. \{V_x \xrightarrow{V(x,y)} V_y\}_{x\geq y}\right) \\ \downarrow \qquad \downarrow & \downarrow \qquad\qquad \downarrow \\ W \quad \{W_x\} & W_x \xrightarrow{W(x,y)} W_y \end{array}$$

Indecomposable representations

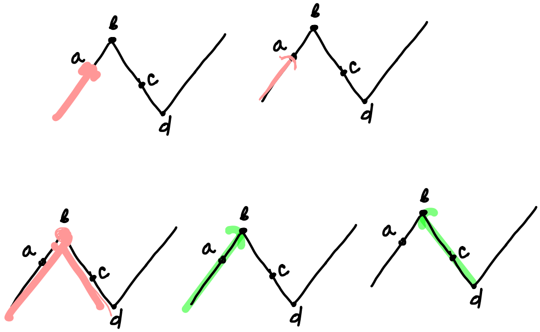
2

$V_{(a,b)}$, $V_{(a,b]}$, $V_{[a,b)}$, $V_{[a,b]}$ $\quad -\infty < a,b < \infty$

$V_{(-\infty,b)}$, $V_{(-\infty,b]}$, $V_{(a,\infty)}$, $V_{[a,\infty)}$, $V_{(-\infty,\infty)}$

$V_{|a,b|}(x) = \mathbb{C}$ all maps $V(x,y) = \mathrm{Id}$.

- Indecomp. projectives



$P_a = V_{(-\infty,a]}$

$P_{a)} = V_{(-\infty,a)}$

$P_b = V_{(-\infty,d]}$

$P_{b)} = V_{(-\infty,b)}$

$P_{(b} = V_{(-b,d)}$

$P_c = V_{[c,d]}$

$P_{(c} = V_{(c,d]}$

$P_d = V_{[d,d]}$

- Simples: $V_{[a,a]} = S_a$

$V_{[a,a]} = S_a$

- Some projective presentation.

$$0 \to P_{a)} \to P_a \to V_{[a,a]} \to 0$$

- $\mathrm{Rep}(A_{\mathbb{R}})$ = category of pointwise finite reps

Thm (·Crawley-Boevey ·Botnan ·Igusa, Rock, Todorov) Every pointwise finite repr. M is isomorphic $\bigoplus$ indecomp. represent. $V_{|a,b|}$

Continuous derived category:

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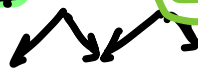
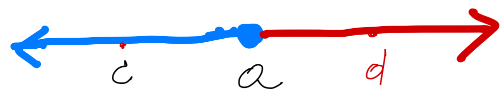
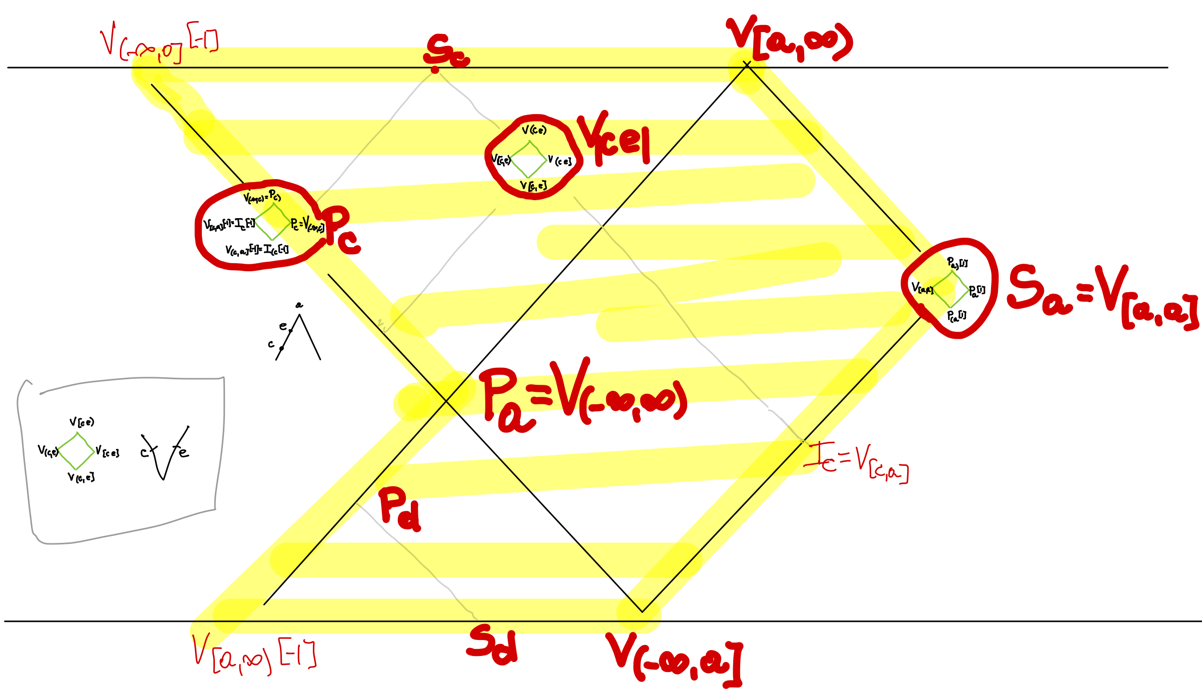


- $A_{\mathbb{R}}$ – cont. quiver of type A
- rep $A_{\mathbb{R}}$ – pointwise finite represent. of $A_{\mathbb{R}}$
- rep $A_{\mathbb{R}}$ is hereditary category
- $D^b(A_{\mathbb{R}})$ – derived category of bdd complexes
- D^b "=" $\bigcup_{i\in\mathbb{Z}} (\text{rep } A_{\mathbb{R}})[i]$

Example



$A_{\mathbb{R}}$



Continuous mg-cluster category (4)

- $D^b(\operatorname{rep} A_{\mathbb{R}})$
- $\tilde{D}$ = double $D^b(\operatorname{rep} A_{\mathbb{R}})$
  - objects are (M,N), $M,N \in D^b(\operatorname{rep} A_{\mathbb{R}})$.
  - morph $\operatorname{Hom}_{\tilde{D}}((M,N),(M',N'))$ (all Homs).
- $F : \tilde{D} \longrightarrow \tilde{D}$

  $F(M,N) := (N[1], M[1])$
- $\boxed{C(A_{\mathbb{R}}) := \tilde{D}/F}$ mg-cluster cat.

Will define mg-clusters
mutations ...

Need pairwise compatibility conditions:

Consider repres of orbit and isom. classes of indecomposable objects V, W (WMA V, W in $\operatorname{rep} A_{\mathbb{R}}$)

Def • V is E-compatible with W if

$\langle g(V), g(W) \rangle \geq 0$, $\langle g(W), g(V) \rangle \geq 0$

where $g(V) = [P_0^V] - [P_1^V]$ $\in K^0(\operatorname{rep} Q)$

$g(W) = [P_0^W] - [P_1^W]$

$P_1^V \to P_0^V \to V \to 0$ $\qquad$ $P_1^W \to P_0^W \to W \to 0$

Def. • nq-cluster is a maximal E-compatible set of indecomposable objects, T

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• let T nq-cluster. X in T if
$\exists Y \not\cong X$ such that
$(T \setminus X) \cup Y$ is nq-cluster
then Y is called mutation of X
$(T \setminus X) \cup Y$ is called mutation of T

Example $C(A_{\mathbb{R}})$ cont. nq-cluster cont.

$T = P = \{$all indecomp. projective repr$\}$

$P = \{P_a, P_{a)}, P_{(c}\}$ nq-cluster

$g(P_a) = [P_a]$, $g(P_{a)}) = [P_{a)}]$

$\langle g(\ ), g(\ )\rangle \geq 0$

⟹ some objects can be mutated
⟹ some objects cannot be mutated

eg ① $P_{a)}$ can be mutated to $S_a = V_{[a,a]}$
$(P \setminus P_{a)}) \cup S_a$ is E-compatible ✓
proof - use $g(S_a) = [P_a] - [P_{a)}]$

eg ② P_a cannot be mutated

JOB D. ROCK — "cluster theory" 6

- $\mathcal{C}$ - skeletally small, Krull-Schmidt additive category.
- $\mathcal{P}$ - pairwise compatibility condition
- $\{T\}$ maximal $\mathcal{P}$-compatible sets (i.e. each pair of obj. is $\mathcal{P}$ compat.)
- <u>Assume</u>: for each X in T there is at most one $Y \not\cong X$ such that $\{T \setminus X\} \cup Y$ is $\mathcal{P}$-compatible.

Such T is called nq-cluster.

Remark 1 – Not every X in T can be mutated (as P_a in $\mathcal{C}(A_n)$)

Remark 2 – The same X might not be mutable from one T but might be mutable from a different T'.

More on such mutations and sequences of mutations
Karin Baur, Sira Gratz
Job D. Rock (IV)

"Cluster theory" – $\mathcal{T}_{\mathcal{P}}(\mathcal{C})$

- collection of nq-clusters maximal $\mathcal{P}$-compatible sets
- mutations when exist

$\mathcal{T}_{\mathcal{P}}(\mathcal{C})$

Objects – nq-clusters
Morphisms – generated by mutations

subcategory of Sets

Remark Any cluster structure (in the sense of BIRS) defines a "cluster theory".

Embeddings of cluster theories

Def. Let $\mathcal{T}_P(\mathcal{C})$ be a cluster theory defined by the pairwise compat. condition P and

Let $\mathcal{T}_Q(\mathcal{D})$ be a cluster theory defined by the pairwise compat. condition Q.

A functor $F: \mathcal{T}_P(\mathcal{C}) \to \mathcal{T}_Q(\mathcal{D})$

is called embedding of cluster theories if F is
- injective on objects
- injective on mutations
- respects compatibility

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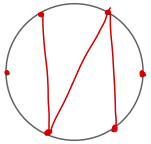
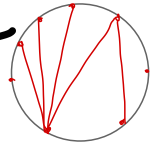
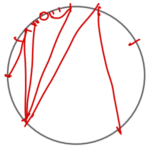
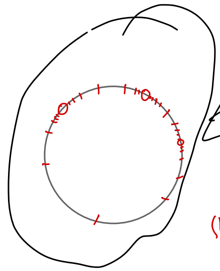
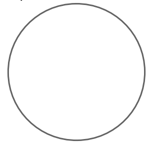
THEY

$m \leq n$



cyclic posets



Cont. cluster category

hyperbolic

ideal tiling of

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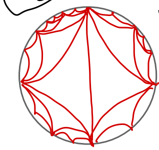
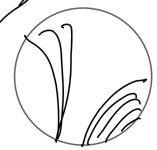


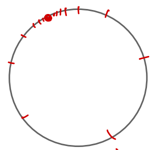


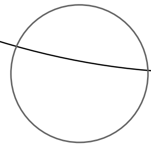
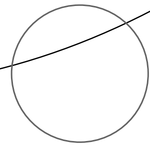
$$
\begin{tikzcd}
 & & \mathscr{T}_{\mathbf{N}_\infty}(\mathcal{C}(A_\infty)) \arrow[r, dashed] \arrow[d] \arrow[rrr, bend left] & \mathscr{T}_{\mathbf{N}_{(i,\infty)}}(\mathcal{C}(A_{\mathbf{N}_{(i,\infty)}})) \arrow[r, dashed] \arrow[d, dashed] & \mathscr{T}_{\mathbf{N}_{(j,\infty)}}(\mathcal{C}(A_{\mathbf{N}_{(j,\infty)}})) \arrow[d, dashed] \arrow[dr, dashed] & & \\
\mathscr{T}_{\mathbf{N}_m}(\mathcal{C}(A_m)) \arrow[r] & \mathscr{T}_{\mathbf{N}_n}(\mathcal{C}(A_n)) \arrow[ur] \arrow[dr] & & & & \mathscr{T}_{\mathbf{N}_\mathbb{R}}(C_\pi) \arrow[r] & \mathscr{T}_{\mathbf{E}}(\mathcal{C}(A_\mathbb{R})) \\
 & & \mathscr{T}_{\mathbf{N}_{\overline{\infty}}}(\mathcal{C}(A_{\overline{\infty}})) \arrow[r, dashed] \arrow[rrru, bend right] & \mathscr{T}_{\mathbf{N}_{\overline{(i,\infty)}}}(\mathcal{C}(A_{\mathbf{N}_{\overline{(i,\infty)}}})) \arrow[r, dashed] & \mathscr{T}_{\mathbf{N}_{\overline{(j,\infty)}}}(\mathcal{C}(A_{\mathbf{N}_{\overline{(j,\infty)}}})) \arrow[ur, dashed] & &
\end{tikzcd}
$$

(BMRRT
CALDERO-CHAPOTON-SCHIFFLER)

cluster theory
(not?)
cluster structure

COMPLETION



$$
\begin{tikzcd}
\mathscr{T}_{\mathbf{N}_m}(\mathcal{C}(A_m)) \arrow[r] & \mathscr{T}_{\mathbf{N}_n}(\mathcal{C}(A_n)) \arrow[r] \arrow[rr, bend left] & \mathscr{T}_{\mathbf{T}}(\mathcal{C}_\mathcal{Z}) \arrow[r, dashed] & \mathscr{T}_{\mathbf{E}}(\mathcal{C}(A_\mathbb{R}))
\end{tikzcd}
$$

Figure 1. The conjectured embeddings of cluster theories. The theories in the diagram are the ones presented. The integers m, n, i, j are such that $1 < m < n$ and $1 < i < j$. We make no further assumptions about m, n, i, j. The dashed arrows are the conjectured arrows. The solid arrows in the top diagram are from [R20+]. The solid arrows in the bottom diagram are from [KMMR21+].

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A Note on upcoming revisions to [R20+].

Job D. Rock

R will expand their work [R20+] to even more type A cluster theories. In [R20+], the following cluster theories are included.

- $\mathscr{T}_{\mathbf{N}_n}(\mathcal{C}(A_n))$ from [BMR+06, CCS06]. The compatibility condition $\mathbf{N}_n$ is non-crossing diagonals on an $(n+3)$-gon.
- $\mathscr{T}_{\mathbf{N}_\infty}(\mathcal{C}(A_\infty))$ from [HJ12]. The compatibility condition $\mathbf{N}_\infty$ is non-crossing diagonals on the ∞-gon.
- $\mathscr{T}_{\mathbf{N}_{\overline{\infty}}}(\mathcal{C}(A_{\overline{\infty}}))$ from [BG18]. The compatibility condition $\mathbf{N}_{\overline{\infty}}$ is non-crossing diagonals on the completed ∞-gon.
- $\mathscr{T}_{\mathbf{N}_{\mathbb{R}}}(C_\pi)$ from [IT15a]. The compatibility condition $\mathbf{N}_{\mathbb{R}}$ is non-crossing geodesics on $\mathbb{R}$-gon (the hyperbolic plane).
- $\mathscr{T}_{\mathbf{E}}(\mathcal{C}(A_{\mathbb{R}}))$ from [IRT20+]. The compatibility condition $\mathbf{E}$ is based on the Euler product of projective resolutions.

By (n,∞)-gon we mean the circle with infinitely marked points and n accumulation points. The accumulation points are not included in the (n,∞)-gon and all accumulation points are two-sided. By completed (n,∞)-gon, or $\overline{(n,\infty)}$-gon, we mean the (n,∞)-gon with the n accumulation points. One may think of the ∞-gon as the $(1,\infty)$-gon and the completed ∞-gon as the $\overline{(1,\infty)}$-gon.

In [KMMR21+], the authors describe a way to construct a cluster theory starting with a category $\mathcal{D}$ that is triangulated equivalent to $\mathcal{D}_\pi$ from [IT15a]. The authors use a subcategory $\mathcal{Z} \subset \mathcal{D}$ to construct a category $\mathcal{C}_{\mathcal{Z}} \subset \mathcal{D}$. The compatibility condition is denoted $\mathbf{T}$ and is based on tilting rectangles in $\mathcal{C}_{\mathcal{Z}}$.

R will include the following cluster theories in the revised version of [R20+].

- $\mathscr{T}_{\mathbf{N}_{(n,\infty)}}(\mathcal{C}(A_{\mathbf{N}_{(n,\infty)}}))$ from [IT15b]. The compatibility condition $\mathbf{N}_{(n,\infty)}$ is non-crossing diagonals on the (n,∞)-gon. One may think of $\mathbf{N}_\infty$ as the special case where $n=1$.
- $\mathscr{T}_{\mathbf{N}_{\overline{(n,\infty)}}}(\mathcal{C}(A_{\mathbf{N}_{\overline{(n,\infty)}}}))$ from [PY21]. The compatibility condition $\mathbf{N}_{\overline{(n,\infty)}}$ is non-crossing diagonals on the $\overline{(n,\infty)}$-gon. One may think of $\mathbf{N}_{\overline{\infty}}$ as the special case where $n=1$.
- $\mathscr{T}_{\mathbf{T}}(\mathcal{C}_{\mathcal{Z}})$ from [KMMR21+]. The compatibility condition $\mathbf{T}$ is based on tilting rectangles $\mathcal{C}_{\mathcal{Z}}$.

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Conjecture (R.). *There exist commutative diagrams of embeddings of cluster theories as in Figure 1.*

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Thank you