Periodic trivial extension algebras and fractionally Calabi–Yau algebras

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A : finite dimensional k-algebra / field $k = \overline{k}$ (for simplicity)

$\Omega = \Omega_A : \underline{\mathrm{mod}}\, A \longrightarrow \underline{\mathrm{mod}}\, A$ syzygy $\quad 0 \to \Omega X \to P \to X \to 0$

projective cover of X

- Behaviour of $\{\Omega^n X\}_{n \geq 1}$

$A^e := A \otimes_k A^{op}$

<u>Def</u> (BP) A : periodic $\overset{\text{def}}{\Longleftrightarrow}$ $\exists n \geq 1$, $\underbrace{\Omega^n_{A^e}(A)}_{\exists \phi \in \mathrm{Aut}_k(A)} \simeq A$ as A^e-mod
(tBP) twisted $\qquad {}_1A_\phi$

<u>Fact</u> (FP) $\exists n \geq 1$, $\Omega^n_A \simeq \mathrm{id}$ as functors on $\underline{\mathrm{mod}}\, A$
(tFP) $\exists \phi \qquad \phi^*$

(OP) ——— , $\forall X \in \mathrm{mod}\, A$: indecomp. non-proj. $\Omega^n_A X \simeq X$
(tOP) $\exists \phi \qquad X_\phi$

(SP) ——— , $\forall S$: simple A-mod. $\Omega^n_A S \simeq S$
(tSP) $\exists \phi \qquad S_\phi$

$$\mathrm{BP} \Rightarrow \mathrm{FP} \Rightarrow \mathrm{OP} \Rightarrow (\mathrm{SP} \Leftrightarrow \mathrm{tBP} \Leftrightarrow \mathrm{tFP} \Leftrightarrow \mathrm{tOP} \Leftrightarrow \mathrm{tSP}) \Rightarrow \text{selfinj.}$$

[Green–Snashall–Solberg]

- $T(A) := A \oplus DA$: trivial extension alg

<u>Ex</u> [Brenner–Butler–King 02] Q : Dynkin quiver $T(kQ)$: trivial

period $T(kQ) = \begin{cases} h-1 & \text{if } \mathrm{char}\, k = 2 \text{ and } Q \in \{A_1, D_{2n}, E_7 \text{ or } E_8\} \\ 2h-2 & \text{else} \end{cases}$

h = Coxeter #

<u>Ex</u> [Buchweitz 98] Q : Dynkin quiver, $\Pi(kQ)$: preprojective algebra
$\Pi(kQ)$ is 6-periodic

<u>Ex</u> [Dugas 10] A : representation-finite selfinjective $\Rightarrow$ periodic

<u>Ex</u> [Erdmann–Skowronski] Many examples of representation-tame selfinj. alg.

<u>Aim</u> ① A : fin. dim k-alg. Give criterions for (tw.) periodicity of $T(A)$.
As an application, construct a families of (tw.) periodic algebras.

② Study (Periodicity conjecture) [Erdmann–Skowronski] twisted periodic $\Rightarrow$ periodic
Equivalently, ϕ in (tBP) has finite order in $\mathrm{Out}_k(A)$.

Key notion : Calabi-Yau property

A: fin. dim. Gorenstein k-alg. (i.e. inj. dim A_A, inj. dim ${}_AA$ are finite)

$\Rightarrow$ per A has a Serre functor $\nu = - \overset{L}{\underset{A}{\otimes}} DA$

Def $\ell, m \in \mathbb{Z}$, $\ell \geq 1$

A: $\underbrace{(m,\ell)\text{-CY}}_{\text{twisted}}$ $\underbrace{\overset{\text{def}}{\iff}}_{\exists \phi \in \mathrm{Aut}_k(A)}$ $\nu^{\ell} \simeq \underbrace{[m]}_{\circ \phi^*}$ as functors on per A

(or $\underbrace{\frac{m}{\ell}\text{-CY}}_{\text{tw.}}$) $\iff$ $H^i\left((DA)^{\overset{L}{\otimes}_A \ell}\right) \simeq \begin{cases} A \text{ as } A^e\text{-mod} & i = -m \\ 0 & \text{else} \end{cases}$

CY dim $A := (m,\ell)$ if ℓ is minimal possible

Call A $\underbrace{}_{\text{tw.}}$ fractionally CY if it is $\underbrace{}_{\text{tw.}}$ $\frac{m}{\ell}$-CY for $\exists\, \ell, m$

Ex
- symmetric alg $\iff$ $\frac{0}{1}$-CY If basic, selfinj alg $\iff$ tw. $\frac{0}{1}$-CY
  $\phi =$ Nakayama auto.
- Q: Dynkin CY dim $KQ = \begin{cases} (\frac{h}{2}-1, \frac{h}{2}) & \text{if } Q = A_1, D_{2n}, E_7, E_8 \\ (h-2, h) & \text{else} \end{cases}$
  [Miyachi-Yekutieli, CDIM]
- A: canonical alg of type (2222), (333), $(2,4,4)$, $(2,3,6)$
  $\Rightarrow$ CY dim $A = (p,p)$ $p = \mathrm{lcm}(p_1, \cdots, p_n)$
- frac. CY alg. are closed under derived equivalences,
  taking tensor products [Herschend-I]

Q [HI] If gl. dim $A < \infty$, twisted frac. CY $\iff$ frac. CY
Equivalently, ϕ above has finite order in $\mathrm{Out}_k(A)$
(Fails if gl. dim $A = \infty$, e.g. selfinj. algebras)

Thm 1 A: fin. dim. k-alg, $k = \overline{k}$ (or more gen. $A/\mathrm{rad}A$ is a separable k-alg)

① (BP) $T(A)$ is periodic $\iff$ (CY) A is frac. CY and gl. dim $A < \infty$

② (tBP) $T(A)$ is tw. periodic $\iff$ (tCY) A is tw. frac. CY and gl. dim $A < \infty$

$\iff$ (C1) $\forall X \in \mathrm{mod}\, T(A)$ has complexity at most one

$(\cdots \to P_1 \to P_0 \to X \to 0$: min. proj. resol, $(\dim_k P_i)_{i \geq 0}$ is bounded)

$\iff$ (CT) $\exists\, r, d \geq 1$ s.t. $T_r(A) = \begin{bmatrix} A & DA & & 0 \\ & A & \ddots & \\ & 0 & \ddots & DA \\ DA & & & A \end{bmatrix}$ has a d-cluster tilting module

Sketch Key observation

$$\mathrm{gl.dim}\,A < \infty \overset{\text{[Happel]}}{\Longrightarrow} \begin{tikzcd} D^b(A) \arrow[r, "\sim"] \arrow[d, "\nu"] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{mod}^{\mathbb{Z}}\, T(A) \arrow[r] \arrow[d, "{[-1](1)}"] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{mod}\, T(A) \arrow[d, "{[-1]}"] \\ D^b(A) \arrow[r, "\sim"] & \mathrm{mod}^{\mathbb{Z}}\, T(A) \arrow[r] & \mathrm{mod}\, T(A) \end{tikzcd}$$

$$\left(\begin{array}{l} A : \overset{\text{tw.}}{\tfrac{m}{\ell}\text{-CY}} \\ \mathrm{gl.dim}\,A < \infty \end{array} \right) \Rightarrow \left(\begin{array}{l} [\ell+m] \overset{\circ\phi^*}{\simeq} (\ell) \text{ as} \\ \text{functors on } \mathrm{mod}^{\mathbb{Z}}\, T(A) \end{array} \right) \Rightarrow \text{(SP) for } T(A) \Rightarrow \text{(CI)}$$

(CI) $\Rightarrow$ $\mathrm{gl.dim}\,A < \infty$: Use a result on fin. dim. conj [Jensen-Lenzing 89]

$\quad\ \Rightarrow$ tw. frac. CY : Use Voigt's Lemma

$$\left(\begin{array}{l} A \in \mathrm{mod}^{\mathbb{Z}/2\mathbb{Z}}\, T(A) : \text{rigid} \overset{\text{[Voigt]}}{\Longrightarrow} \exists\, n \geq 1,\ \Omega^n_{T(A)}(A) \simeq A(\ell) \text{ in } \mathrm{mod}^{\overset{\mathbb{Z}}{\mathbb{Z}/2\mathbb{Z}}} A \\ \quad \Rightarrow A[-n] \simeq \nu^{\ell} A[\ell] \text{ in } D^b(A) \Rightarrow A : \text{tw. frac. CY} \end{array} \right)$$

$$\text{(tCY)} \underset{\text{[Darpö-I]}}{\Longrightarrow} \text{(CT)} \underset{\text{[Erdmann-Holm]}}{\Longrightarrow} \text{(CI)}$$

(CY) $\Rightarrow$ (tBP) is explained below $\square$

Cor ① periodicity conj. holds for $T(A)$ $\Longleftrightarrow$ HI question holds for A

② $\# \mathrm{Out}_{k}(A) < \infty \Rightarrow$ periodicity conj holds for $T(A)$

($\mathrm{Out}_{k}(T(A))$ is much bigger than $\mathrm{Out}_{k}(A)$)

Ex L : finite lattice $A = k[L]$: incidence alg

Period. conj. holds for $T(A)$

Cor Tw. frac CY. alg of fin. gl. dim. are closed under derived eq.

Answers another Question posed by [HI]

Ex ① A : d-RF ($\exists$ d-CT mod. $\mathrm{gl.dim}\,A \leq d$) $\overset{\text{[HI]}}{\Longrightarrow}$ A : tw. frac. CY

$$\Downarrow$$

② A : d-canonical alg. of wt $(p_1, \cdots, p_n)$ $\qquad$ $T(A)$: tw. periodic

s.t $n - d - 1 = \sum_{i=1}^{n} \frac{1}{p_i}$ $\Rightarrow$ A : $\frac{dp}{p}$-CY for $p = \mathrm{lcm}(p_1, \cdots, p_n)$

$\Rightarrow$ $T(A)$: $2(d+1)p$-periodic

③ A : d-RF and frac. CY $\Rightarrow$ d-Aus. alg B of A is frac. CY

$\Rightarrow$ $T(B)$: periodic

④ new examples of posets with frac. CY incidence alg A and periodic $T(A)$

Thm 2 A : fin. dim. k-alg, $k = \overline{k}$ (or more gen. $A/\mathrm{rad}A$ is a separable k-alg)

Assume $\mathrm{CYdim}\, A = (m, \ell)$ and $\mathrm{gl.dim}\, A < \infty$ $\quad B := T(A)$

Define $\varphi \in \mathrm{Aut}_k(T(A))$ by $\varphi(a, f) = (a, (-1)^{\ell+m} f)$ $\quad a \in A,\ f \in DA$

$\Rightarrow$ $\Omega^{\ell+m}_{B^e}(B) \simeq {}_\varphi B_1$ as B^e-mod.

$$\text{period } B = \begin{cases} \ell+m & \text{if } (-1)^{\ell+m} = 1 \text{ in } k \\ 2(\ell+m) & \text{else} \end{cases}$$

Ex Q : Dynkin quiver

$\mathrm{CYdim}\, kQ$ by [MY, CDIM] $\overset{\text{Thm 2}}{\Longrightarrow}$ period $T(kQ)$ by [BBK]

Sketch of proof of Thm 2 $\quad P$: proj. resol. of $DA \in \mathrm{mod}\, A^e$

$C := A \oplus P$: trivial ext. dg alg $\quad C \simeq B$: quasi-iso

Cofibrant resolution of dg C^e-module C is given by relative bar resolution [Keller]

$P^i := P \otimes_A \cdots \otimes_A P$ (i times)

$\exists$ sequence of dg C^e-modules whose total dg module is acyclic :

$$0 \to L \to C \otimes P^{\ell-1} \otimes C \to \cdots \to C \otimes P^2 \otimes C \to C \otimes P \otimes C \to C \otimes C \to C \to 0$$

$$L = P^\ell \oplus P^{\ell+1}$$

$$C \otimes P^{\ell-1} \otimes C = A \otimes P^{\ell-1} \otimes A \oplus A \otimes P^{\ell-1} \otimes P \oplus P \otimes P^{\ell-1} \otimes A \oplus P \otimes P^{\ell-1} \otimes P$$

$$C \otimes P^2 \otimes C = A \otimes P^2 \otimes A \oplus A \otimes P^2 \otimes P \oplus P \otimes P^2 \otimes A \oplus P \otimes P^2 \otimes P$$

$$C \otimes P \otimes C = A \otimes P \otimes A \oplus A \otimes P \otimes P \oplus P \otimes P \otimes A \oplus P \otimes P \otimes P$$

$$C \otimes C = A \otimes A \oplus A \otimes P \oplus P \otimes A \oplus P \otimes P$$

$$C = A \oplus P$$

$(-1)^\ell$ $\quad -1$ $\quad -1$ $\quad -1$ $\quad -1$

$\mathrm{per}\, C^e \Rightarrow C \simeq L[\ell]$ in $D_{sg}(C^e)$

$\frac{m}{\ell}$-CY $\Rightarrow$ $P^\ell \simeq A[m]$ in $D(A^e)$ $\qquad C \simeq B$, $\quad L[\ell] \simeq {}_\varphi B[\ell+m]$ in $D_{sg}(B^e) \simeq \underline{\mathrm{mod}}\, B^e$

$\Rightarrow L = \begin{pmatrix} P^\ell \\ P^{\ell+1} \end{pmatrix} \doteq B[m]$ in $D(B^e)$

Detailed calculation

$\square$