QUANTUM SYMMETRIES THROUGH THE LENS OF LINEAR ALGEBRA

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Color Codes:
+ Known results / Recall
+ Today's heroes
+ Things to be done / obstructions
+ Else - to get your attention!

Outline:

(I) History & Motivation

(II) Main Results & Examples

(III) Questions & Answers (?)

(I) Motivation:

1) McKay Correspondence (1980)

- Finite group $G \leq SU_2$ (2x2 unitary complex matrices of det 1)
- Simple G-modules $S_1, S_2, \dots, S_m$
- $V = \mathbb{C}^2$
- $S_i \otimes V = \sum_{j=1}^{m} M_{i,j} S_j$

$\rightsquigarrow$ $M_V = (M_{ij})_{m\times m}$ McKay matrix for tensoring with V.

$\rightsquigarrow$ McKay quiver has nodes $1, 2, \dots, m$

M_{ij} arrows from node i to node j

[McKay]: $\hat{C} = 2I - M_V$ is an affine Cartan matrix of type A, D, E

$$\underline{s} = \begin{bmatrix} \dim S_1 \\ \dim S_2 \\ \vdots \\ \dim S_m \end{bmatrix}, \quad M_V \cdot \underline{s} = \underbrace{2}_{\dim V} \cdot \underline{s} \quad \text{and} \quad \hat{C}\underline{s} = (2I - M_V).\underline{s} = \underline{0}.$$

[McKay '1980]: McKay quivers over $G \leq SU_2$ $\xleftrightarrow[1\text{-}1]{\text{McKay corr.}}$ affine Dynkin diagrams of type A, D, E

$$\mathbb{Z}_n \longleftrightarrow \tilde{A}_{n-1}$$
$$\mathbb{D}_n \longleftrightarrow \tilde{D}_{n+2}$$
$$\mathbb{T} \longleftrightarrow \tilde{E}_6$$
$$\mathbb{O} \longleftrightarrow \tilde{E}_7$$
$$\mathbb{I} \longleftrightarrow \tilde{E}_8$$

2) [Steinberg '1985]: G any group over $\mathbb{C}$ (char. $\mathbb{k} = 0$)

V any f.d. faithful complex G-module with character χ_V

the only element of G that acts as the identity on V is the identity of the group

$\chi_1, \chi_2, \dots, \chi_m$ characters of the simple G-modules $S_1, \dots, S_m$

$S_i \otimes V = \sum_{j=1}^{m} M_{ij} S_j$ $\rightsquigarrow$ McKay matrix $M_V = (M_{ij})$

(adjacency matrix of McKay quiver).

$$M_V \cdot \underline{s} = (\dim V) . \underline{s} = \chi_V(1) . \underline{s} \qquad \text{for } \underline{s} = \begin{bmatrix} \dim S_1 \\ \dim S_2 \\ \vdots \\ \dim S_m \end{bmatrix}$$

Moreover, $M_V \cdot \begin{bmatrix} \chi_1(g) \\ \chi_2(g) \\ \vdots \\ \chi_m(g) \end{bmatrix} = \chi_V(g) . \begin{bmatrix} \chi_1(g) \\ \chi_2(g) \\ \vdots \\ \chi_m(g) \end{bmatrix} \quad \forall g \in G$

$\rightsquigarrow$ • Columns of the character table are right eigenvectors of M_V.

• Conjugacy class representatives give a complete set of right eigenvectors.

$\rightsquigarrow$ Relationship b/ween the McKay matrix and the character table.

3) [Ginberg - Huang - Reiner '20] : in char. $p>0$

Columns of the Brauer character table of G are right eigenvectors of M_V, with eigenvalues $\chi_V(g_j)$ where g_j are conjugacy class representatives of G of order p' relatively prime to p.

• $M_V \cdot \underline{s} = (\dim V) . \underline{s} = tr_V(1) . \underline{s} \qquad$ for $\underline{s} = \begin{bmatrix} \dim S_1 \\ \dim S_2 \\ \vdots \\ \dim S_m \end{bmatrix}$

right eigenvector

• $\underline{p} . M_V = (\dim V) . \underline{p} = tr_V(1) . \underline{p} \qquad$ for $\underline{p} = [\dim P_1 \;\; \dim P_2 \; \dots \; \dim P_m]$

left eigenvector

4) [Benkart - Diaconis - Liebeck - Tiep' 20] : McKay matrices and quivers determine interesting Markov chains.

$\rightsquigarrow$ McKay matrices in other worlds ?

$\boxed{\text{GOAL:}}$ Study McKay matrices for any f.d. Hopf algebras.

↳ H $\Bbbk$-vector space

bialgebra $\begin{cases} \cdot \text{ algebra } (m: H\otimes H \to H,\ u: \Bbbk \to H) \\ \cdot \text{ coalgebra } (\Delta: H \to H\otimes H,\ \varepsilon: H \to \Bbbk) \end{cases}$

• antipode $S: H \to H$

ex: $\Bbbk G$, $\Bbbk[x_1, x_2, \dots, x_n]$, $U(\mathfrak{g})$

↖ Lie algebra $\mathfrak{g}$

II) Main Findings: [BBKNZ '21]

- Let H be a f.d. Hopf algebra over a field $\Bbbk$ ($\Bbbk = \overline{\Bbbk}$, char. $\Bbbk = 0$)
- V any f.d. H-module with character tr_V (trace) $tr_V(1) = \dim V$
- $S_1, S_2, \dots, S_m$ simple H-modules
- P_i = projective cover of S_i

⇝ Problem: $S_i \otimes V$ may NOT be completely reducible (H is not nec. semisimple)!

- Let $M_{ij} = [S_i \otimes V : S_j]$ multiplicity of S_j as a composition factor of $S_i \otimes V$.

  In the Grothendieck ring: $[S_i].[V] = \sum_{j=1}^{m} M_{ij} [S_j]$

- McKay matrix $M_V = (M_{ij})_{m\times n}$ ⇝ McKay quiver

1) McKay matrices for Hopf algebras:

- coproduct of H: $\Delta(h) = \sum_h h_{(1)} \otimes h_{(2)} \in H \otimes H$, $\forall h \in H$
- U, V are H-modules, $h\cdot(u\otimes v) = \sum_h h_{(1)}\cdot u \otimes h_{(2)}\cdot v$ ⇝ $U\otimes V$ is again an H-module

⇝ take traces: $\sum_h tr_{S_i}(h_{(1)}) . tr_V(h_{(2)}) = tr_{S_i\otimes V}(h) = \sum_{j=1}^{m} M_{ij} . tr_{S_j}(h)$

- Define $\mathrm{Tr}_S(h) := [\mathrm{tr}_{S_1}(h) \quad \mathrm{tr}_{S_2}(h) \quad \dots \quad \mathrm{tr}_{S_m}(h)]^T$

$$\Rightarrow \quad M_V.\mathrm{Tr}_S(h) = \sum_h \mathrm{tr}_V(h_{(2)}).\mathrm{Tr}_S(h_{(1)})$$

right eig.vectors

- If $h = \underline{\text{grouplike element}}\ g$: ($\Delta(g) = g \otimes g$) $\quad M_V.\mathrm{Tr}_S(g) = \mathrm{tr}_V(g).\mathrm{Tr}_S(g)$ ⇝ $\mathrm{Tr}_S(g)$ is an eigenvector of M_V
- If $h = 1$: $\mathrm{Tr}_S(1) = \underline{s}$ and $M_V.\underline{s} = \mathrm{tr}_V(1).\underline{s} = (\dim V).\underline{s}$. ⇝ Recover [GHR'20]

- Projective and left eigenvectors:

Let $Q_V := (Q_{ij})$ where $Q_{ij} = [\overbrace{P_i \otimes V}^{\text{projective}} : P_j]$ projective McKay matrix

$$\Rightarrow \quad \sum_h \mathrm{tr}_{P_i}(h_{(1)}).\mathrm{tr}_V(h_{(2)}) = \mathrm{tr}_{P_i \otimes V}(h) = \sum_{j=1}^{m} Q_{ij}.\mathrm{tr}_{P_j}(h)$$

$$\sum_h \mathrm{tr}_V(h_{(2)}).\mathrm{tr}_{P_i}(h_{(1)}) = \sum_{j=1}^{m} \mathrm{tr}_{P_j}(x).(Q_V^T)_{ji}$$

- Define $\mathrm{Tr}_P(h) := [\mathrm{tr}_{P_1}(h) \quad \mathrm{tr}_{P_2}(h) \quad \dots \quad \mathrm{tr}_{P_m}(h)]$

$$\Rightarrow \quad \mathrm{Tr}_P(h).Q_V^T = \sum_h \mathrm{tr}_V(h_{(2)}).\mathrm{Tr}_P(h_{(1)}) \qquad (*)$$

left eig.vectors

- For $h = g$ grouplike: $\mathrm{Tr}_P(g).Q_V^T = \mathrm{tr}_V(g).\mathrm{Tr}_P(g)$
- For $h = 1$: $\underline{p}.Q_V^T = \mathrm{tr}_V(1)\,\underline{p}$. $\quad$ BUT Q_V^T is not quite M_V!

- Theorem (BBKNZ): $Q_V^T = M_{V^*}$. $\qquad V^* = \mathrm{Hom}_{\Bbbk}(V, \Bbbk)$

⇝ If $V \cong V^*$ then $(*)$ gives us left eigenvectors of the McKay matrix M_V.

⇝ When H is semisimple, $M_V = Q_V$. In addition, if V is self-dual, then M_V is orthogonally diagonalizable.

[BDLT'20]

Example : Small quantum group $u_q(\mathfrak{sl}_2) = \mathbb{C}\langle K^{\pm 1}, E, F \rangle$

n^3-dim'l

(BBKNZ: Done for Drinfeld double D_n of Taft algebra.)

$K^n = 1$, $KEK^{-1} = q^2 E$, $KFK^{-1} = q^{-2} F$,

$E^n = 0 = F^n$, $[E,F] = EF - FE = \dfrac{K - K^{-1}}{q - q^{-1}}$

q = primitive n^{th} root of 1 , n odd, $n \geq 3$

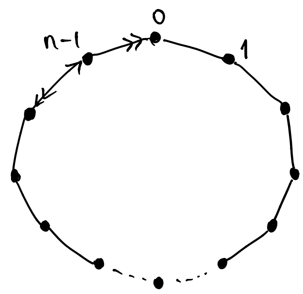
- n simple modules $V_0, V_1, \dots, V_{n-2}, V_{n-1}$ $\quad$ $\dim V_r = r+1$
- projective covers $\underbrace{P_0, P_1, \dots, P_{n-2}}_{\dim 2n}, V_{n-1}$ $\quad$ [Chari-Premet' 1994]

  both simple and projective

  $\dim V_{n-1} = n$
- $V = V_1$ $\quad$ $(\dim V = 2)$
- McKay matrix $\quad\quad\quad\quad$ McKay quiver

$$M_V = \begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ 1 & 0 & 1 & & & 0 \\ 0 & 1 & 0 & 1 & & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 & 1 \\ 2 & 0 & \cdots & 0 & 2 & 0 \end{pmatrix}_{n \times n}$$

2 copies of V_0 $\quad$ 2 copies of V_{n-2}



characteristic roots $q^j + q^{-j}$, $0 \leq j \leq n-1$ $\rightsquigarrow$ only $\frac{1}{2}(n+1)$ distinct eigenvalues

- **Eigenvectors for $u_q(\mathfrak{sl}_2)$:**

Chebyshev polynomials of 2^{nd} kind: $\left[\begin{array}{l} U_0(t) = 1, \quad U_1(t) = t, \\ U_r(t) = t \cdot U_{r-1}(t) - U_{r-2}(t), \quad r \geq 2 \end{array}\right]$

Let $e_j := [U_0 \ U_1 \ \dots \ U_{n-1}]^T$ where $U_r = U_r(q^j + q^{-j})$

$\Rightarrow$ $\boxed{e_j \text{ is a right eigenvector for } M_V,\ V = V_1 \text{ with eigenvalue } q^j + q^{-j},\ 0 \leq j \leq \frac{n-1}{2}}$

Indeed, $\mathcal{U}_r(x+x^{-1}) = x^r + x^{r-2} + x^{r-4} + \dots + x^{-(r-2)} + x^{-r}$

- $\mathcal{U}_r(q^j+q^{-j}) = tr_{V_r}(K^j) \to e_j = Tr_S(K^j)$ trace vector of the grouplikes K^j

(j=0) • $\mathcal{U}_r(q^0+q^{-0}) = r+1 = \dim(V_r) \Rightarrow e_0 = Tr_S(1) = \underline{S}$

⇝ Not enough eigenvectors! (eigenvalues are repeated, q^j+q^{-j}: j and $-j$ have the same values)

q^j+q^{-j}, for $j\neq 0$, correspond to 2×2 Jordan blocks! (did NOT occur in group case)

– Let $\mathcal{L}_0(t) = 2$, $\mathcal{L}_1(t)=t$, $\mathcal{L}_r(t) = t\mathcal{L}_{r-1}(t) - \mathcal{L}_{r-2}(t)$, $r\geq 2$

modified Chebyshev

<u>Prop</u>: • $\mathcal{L}_r(t) = \mathcal{U}_r(t) - \mathcal{U}_{r-2}(t)$, $r\geq 2$,

and • $\mathcal{L}_r(x+x^{-1}) = x^r + x^{-r}$, $\forall r \geq 0$.

Let $f_j := [\mathcal{L}_{n-1}\ \ \mathcal{L}_{n-2}\ \dots\ \mathcal{L}_1\ \ 1]$ where $\mathcal{L}_r = \mathcal{L}_r(q^j+q^{-j})$, $r\geq 1$

⇒ $\boxed{f_j \text{ is a left eigenvector for } M_V,\ V=V_1 \text{ with eigenvalue } q^j+q^{-j},\ 0\leq j\leq \tfrac{n-1}{2}}$

$f_0 = [2\ \ 2\ \dots\ 2\ \ 1] = \frac{1}{n}[\underbrace{2n\ \ 2n\ \dots\ 2n}_{\dim P_i}\ \ \underset{\dim V_{n-1}}{n}] = \frac{1}{n}\cdot \underline{P}$

⇝ Since every simple module V_r is a polynomial in V_1 and V_0, these are eigenvectors for M_V for <u>any</u> finite dim'l $u_q(sl_2)$-module V.

// Example $u_q(sl_2)$

2) Fusion matrix N_V for f.d. Hopf algebra H:

- $S_1, S_2, \dots, S_m$ simples and P_i projective covers of S_i
- Cartan map: $[P_i] \xrightarrow{c} \sum_{j=1}^{m} [P_i : S_j][S_j] \in G_0$

  $[P_i] \in$ projective Grothendieck group K_0
- Cartan matrix $C = (c_{ij})$ where $c_{ij} = [P_i : S_j]$
- $r := \text{rank}(C)$
- [Cohen-Westreich 2008]: Assume H is a quasitriangular ribbon Hopf alg.

There is a subset $\mathcal{P} = \{\tilde{P}_1, \tilde{P}_2, \dots, \tilde{P}_r\}$ of the projective covers so that if

$$N_V = (N_{ij})_{r\times r} \quad \text{where } N_{ij} = [\tilde{P}_i \otimes V : \tilde{P}_j], \text{ then}$$

- N_V is diagonalizable for any simple H-module V.
- The eigenvectors don't depend on V.
- There is a "Verlinde formula", ie., $\exists$ matrix F and scalars $d_0, d_1, \dots, d_{r-1}$

so that $\boxed{F^{-1} N_V F} = \text{diag}\{\tilde{s}_1^{-1} d_1, \tilde{s}_2^{-1} d_2, \dots, \tilde{s}_r^{-1} d_r\}$, where $\tilde{s}_i = \dim \tilde{S}_i$

corresp. simples

(Motivation:)

- Verlinde's paper (1988) on diagonalizing fusion rules for 2D rational conformal field theory.

  N_V is related to the matrix Verlinde used.

- Back to $u_q(\mathfrak{sl}_2)$: ($n \geq 3$ odd)

  $c(P_\ell) = c(P_{n-2-\ell})$

$$\begin{array}{ccccccc} \mathcal{P}: \{ V_{n-1}, & P_0, & P_1, & \dots, & P_{\frac{n-3}{2}} & \} \\ & \updownarrow & \updownarrow & \vdots & \updownarrow & \\ & P_{n-2}, & P_{n-3}, & \dots, & P_{\frac{n-1}{2}} & \end{array}$$

$V = V_1$

$$N_V = \begin{array}{c} \begin{array}{cccccccc} V_{n-1} & P_0 & P_1 & & & & & \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & & \cdots & & & 0 \\ 2 & 0 & 1 & & & & & 0 \\ 0 & 1 & 0 & 1 & & \cdots & & 0 \\ \vdots & & \ddots & \ddots & \ddots & & & \vdots \\ 0 & & \cdots & & 0 & 1 & 0 & 1 \\ 0 & 0 & & \cdots & & 0 & 1 & 1 \end{pmatrix}_{r\times r} \end{array}$$

eigenvalues $q^j + q^{-j}$, $0 \leq j \leq \frac{n-1}{2}$

same as for M_V

For each $q^j + q^{-j}$:

- Right eigenvectors of N_V : $[1, \mathcal{Y}_1, \dots, \mathcal{Y}_{\frac{n-1}{2}}]^T$, where $\mathcal{Y}_k = \mathcal{Y}_k(q^j + q^{-j})$,
- Left eigenvectors of N_V : $[\mathcal{V}_{\frac{n-1}{2}}, \dots, \mathcal{V}_1, 1]$, where $\mathcal{V}_k = \mathcal{V}_k(q^j+q^{-j})$,

where $\mathcal{V}_0(t) = 1$, $\quad \mathcal{V}_1(t) = t-1$, $\quad \mathcal{V}_k = t\mathcal{V}_{k-1}(t) - \mathcal{V}_{k-2}(t)$, $\quad k \geq 2$.

(Chebyshev polynomials of the 3rd kind)

$\rightsquigarrow$ "Chebyshev polynomials are everywhere dense in numerical analysis!"

[BBKNZ]: works all out for D_n, Drinfeld double of the Taft algebra H_n, $n \geq 3$, n odd, and V is any 2-dim'l simple H-mod.

– For D_n, the right eigenvectors of M_V are trace vectors of simple mods evaluated at grouplikes.

$\rightsquigarrow$ There exist Hopf algebras with <u>NO</u> grouplikes!

– For D_n, only <u>SOME</u> of the left eigenvectors of M_V are trace vectors of projective covers evaluated at grouplikes. $\rightsquigarrow$ do more to get more left eigenvectors!

- D_n, n odd, is a quasitriangular ribbon Hopf algebra

[BBKNZ2] uses the R-matrix and ribbon element to get a representation

$$\pi: TL_k(\xi) \longrightarrow \mathrm{End}_{D_n}(V^{\otimes k})$$

of the Temperley-Lieb algebra $TL_k(\xi)$ for $\xi = -(q^{\frac{1}{2}} + q^{-\frac{1}{2}})$ on $V^{\otimes k}$

for V = any 2-dim'l D_n-simple module.

- π is injective $\forall k$ for the unique self-dual 2-dim'l simple D_n-mod $V^{\dagger}$
- $TL_k(\xi) \cong \mathrm{End}_{D_n}((V^{\dagger})^{\otimes k})$ for $1 \leq k \leq 2n-2$.

⇝ proof use diagrammatics for $TL_k(\xi)$

III Questions: For a f.d. Hopf alg. H and V a f.d. H-module:

1) When is M_V diagonalizable?

↳ Not true for $u_q(sl_2)$ and $V = V_1$

2) When do M_V and N_V have the same eigenvalues?

↳ True for $u_q(sl_2)$ and $V = V_1$

3) What's a good notion of a "character table" for H?

↳ [Witherspoon]: did for f.d. semisimple almost cocommutative Hopf algebras.

4) Is the character table related to eigenvectors of McKay matrices?

↳ True for $H = \Bbbk G$ group algebra.

THANK YOU! ☺