On symmetric quivers and their degenerations

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(Or: *Just pick the right quiver*)

Structure

1 Algebraic Lie-theoretic motivation

$n \in \{2l, 2l+1\}$

CONJUGATION

$Gl_n(\mathbb{C})$ (1) Jordan (2) Gerstenhaber $\circlearrowright$ $\mathcal{N} := \{N \in \mathbb{C}^{n\times n} \mid N^n = 0\}$

nilp. cone

$\begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \in B$

Borel

(1) B.-Reineke (2) B.-Reineke $\circlearrowright$ $\mathcal{N}^{(2)} := \{N \mid N^2 = 0\}$

2-nilpotent

Questions: (1) Orbits $B.N$

(2) Orbit closures $\overline{B.N}$

1 Algebraic Lie-theoretic motivation

Define

$$Q = \begin{tikzcd} \underset{1}{\bullet} \arrow[r, "\alpha_1"] & \underset{2}{\bullet} \arrow[r, "\alpha_2"] & \cdots & \underset{l}{\bullet} \arrow[r, "\alpha_l"] & \underset{\omega}{\bullet} \arrow[loop above, "\xi"] \arrow[r, "\alpha_l^*"] & \underset{l^*}{\bullet} \arrow[r] & \cdots \arrow[r] & \underset{2^*}{\bullet} \arrow[r, "\alpha_1^*"] & \underset{1^*}{\bullet} \end{tikzcd}$$

$$\underline{d} = (1, 2, \ldots, l, n, l, \ldots, 2, 1)$$

$$\left.\begin{matrix} \mathbb{C}Q \\ \cup\!\!| \\ I = (\alpha_l^* \circ \alpha_l, \ \xi^2) \end{matrix}\right\} \ A := \mathbb{C}Q / I$$

$$\underset{\text{rep variety}}{R_{\underline{d}}} \supseteq R_{\underline{d}}^{\circ} := \{ (M_\alpha)_\alpha \mid M_{\alpha_i} \text{ injective}, \ M_{\alpha_i^*} \text{ surjective } \forall i \}$$

$$\circlearrowleft \ G_{\underline{d}} = \prod_i Gl_{d_i}(k)$$

__Lemma__

$$\{ B\text{-orbits in } \mathcal{N}^{(2)} \} \xleftrightarrow{\text{bij.}} \{ G_{\underline{d}}\text{-orbits in } R_{\underline{d}}^{\circ} \}$$

↑ preserves orbit closure relations!

1 Algebraic Lie-theoretic motivation

$n \in \{2\ell, 2\ell+1\}$

TYPE A ??? induced ??? CONJUGATION

$Gl_n(\mathbb{C})$ (1) Jordan (2) Gerstenhaber $\circlearrowright$ $\mathcal{N} := \{N \in \mathbb{C}^{n \times n} \mid N^n = 0\}$ nilp. cone

TYPES B, C, D

$\cup \qquad \cup$

$Lie\, G = \mathfrak{g}$ $\qquad$ G classical $\circlearrowright$ $\mathcal{N} \cap \mathfrak{g}$

(1) Springer–Steinberg
(2) Hesselink

$\cup \qquad \cup \qquad \cup \qquad \cup$

(1) B.- Cerulli Irelli - Esposito
(2) In progress

$B \cap G = B_G$ $\circlearrowright$ $\mathcal{N}_{\mathfrak{g}}^{(2)} := \mathcal{N}^{(2)} \cap \mathfrak{g}$

$\cap \qquad \cap$

$\begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \in B$ Borel $\qquad$ (1) B.- Reineke (2) B.- Reineke $\circlearrowright$ $\mathcal{N}^{(2)} := \{N \mid N^2 = 0\}$ 2-nilpotent

Questions: (1) Orbits $B.N$, $B_G.N$ $\qquad N \in \mathcal{N}_{\mathfrak{g}}^{(2)}$

(2) Orbit closures $\overline{B.N}$, $\overline{B_G.N}$

2 Symmetric Representation Theory

Symmetric quivers and algebras

Let Q be a finite quiver

$Q_0 = \{\text{vertices}\}$

$Q_1 = \{\text{arrows}\} \ni \alpha : s(\alpha) \to t(\alpha)$

If $\sigma : Q_0 \cup Q_1 \to Q_0 \cup Q_1$ is
$x \mapsto \sigma(x)$

an involution on Q_0

a reversing involution on Q_1

Then $Q := (Q, \sigma)$ is called a <u>symmetric quiver</u>

<u>Examples</u>:

2 Symmetric Representation Theory

Symmetric quivers and algebras

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Then $Q := (Q, \sigma)$ is called a <u>symmetric quiver</u>

<u>Examples</u>:

$$\begin{tikzcd}
 & 2 \arrow[dl, "\alpha"'] \arrow[dr, "\xi"] & & \sigma(1) \arrow[dl, "\sigma(\alpha)"'] \\
1 & & \sigma(2) &
\end{tikzcd}$$

$$\begin{tikzcd}
 & & & & \sigma(2) \arrow[dl, "\sigma(\beta)"'] \arrow[dr, "\sigma(\alpha)"] & \\
1 \arrow[dr, "\alpha"] & & & \omega \arrow[dll, "\beta"'] & & \sigma(1) \\
 & 2 & & & &
\end{tikzcd}$$

$$\begin{tikzcd}
1 \arrow[r, "\alpha"] & 2 \arrow[r, "\beta"] & \omega \arrow[loop above, "\xi"] \arrow[r, "\sigma(\beta)"] & \sigma(2) \arrow[r, "\sigma(\alpha)"] & \sigma(1)
\end{tikzcd}$$

2 Symmetric Representation Theory

Symmetric quivers and algebras

Let $\mathbb{C}Q$ = path algebra of Q $\rightsquigarrow$ σ extends to paths

$I \subseteq \mathbb{C}Q$ admissible ideal

with $\sigma(I) \subseteq I$

Then

$A := \mathbb{C}Q / I$ finite-dim., associative algebra with 1 (not commutative)

σ induces $\rightsquigarrow$ iso $\sigma : A \longrightarrow A^{op} \simeq A$

self-duality $\nabla : \operatorname{Rep} A \longrightarrow \operatorname{Rep} A$

2 Symmetric Representation Theory

Symmetric representations

Fix $\underline{d} = (d_i)_{i \in Q_0}$ dim vector with $d_i = d_{\sigma(i)}$ $\forall i$

$V = \bigoplus_{i \in Q_0} V_i$ $\quad \dim V_i = d_i$

$\varepsilon \in \{\pm 1\}$

$\langle \, , \rangle : V \to V$ non-degenerate bilin. form s.th.

- $\langle \, , \rangle_{V_i \times V_j} = 0$ unless $i = \sigma(j)$
- $\langle v, w \rangle = \varepsilon \langle w, v \rangle$

2 Symmetric Representation Theory

Symmetric representations

$\xleftarrow{\text{?induced?}}$

$\boxed{\varepsilon\text{-REPRESENTATIONS}}$ $\boxed{\text{REPRESENTATIONS}}$

$$\{(M_\alpha)_\alpha \mid M_\alpha^* = -M_{\sigma(\alpha)}\} =: R_{\underline{d}}^{\varepsilon} \subseteq R_{\underline{d}} \subseteq \bigoplus_{\alpha \in Q_1} \operatorname{Hom}(V_{s(\alpha)}, V_{t(\alpha)})$$

adjoint wrt $\langle , \rangle$

$\circlearrowleft$ $\qquad$ $\circlearrowleft$ base change

$$\{(g_i)_i \mid g_i = g_{\sigma(i)}^{*\,-1}\} =: G_{\underline{d}}^{\varepsilon} \subseteq G_{\underline{d}} = \prod_{i \in Q_0} GL_{d_i}(\mathbb{C})$$

$\varepsilon = 1$: orthogonal

$\varepsilon = -1$ symplectic

orbits $\equiv$ iso classes in rep cat

denote both M

self-dual

$M \underset{\varepsilon}{\simeq} \nabla M$

2 Symmetric Representation Theory

Symmetric representations

First answer: Orbits are induced!

<u>Theorem</u> (MWZ 2000, DW 2002)

$M, N \in R_{\underline{d}}^{\varepsilon}$ $\quad G_{\underline{d}}^{\varepsilon} M = G_{\underline{d}}^{\varepsilon} N \iff G_{\underline{d}} M = G_{\underline{d}} N$

2 Symmetric Representation Theory

Our motivating example

Define

$$\sigma(\xi) = \xi$$

$$Q = \begin{tikzcd} \underset{1}{\bullet} \arrow[r, "\alpha_1"] & \underset{2}{\bullet} \arrow[r, "\alpha_2"] & \cdots & \underset{\ell}{\bullet} \arrow[r, "\alpha_\ell"] & \underset{\omega = \sigma(\omega)}{\bullet} \arrow[loop above, "\xi"] \arrow[r, "\sigma(\alpha_\ell)"] & \underset{\sigma(\ell)}{\bullet} \arrow[r] & \cdots \arrow[r] & \underset{\sigma(2)}{\bullet} \arrow[r, "\sigma(\alpha_1)"] & \underset{\sigma(1)}{\bullet} \end{tikzcd}$$

$$\underline{d} = (1, 2, \ldots, \ell, n, \ell, \ldots, 2, 1)$$

$$\left.\begin{array}{l} \mathbb{C}Q \\ \cup \! \mid \\ I = (\sigma(\alpha_\ell) \circ \alpha_\ell, \xi^2) \end{array}\right\} A := \mathbb{C}Q/I$$

$$R_{\underline{d}} \supseteq R_{\underline{d}}^{o} := \{ (M_\alpha)_\alpha \mid M_{\alpha_i} \text{ injective}, M_{\sigma(\alpha_i)} \text{ surjective } \forall i \}$$

<u>Lemma</u>

$$\{ B\text{-orbits in } \mathcal{N}^{(2)} \} \xleftrightarrow{\text{bij.}} \{ G_{\underline{d}}\text{-orbits in } R_{\underline{d}}^{o} \}$$

$$\{ B_\sigma\text{-orbits in } \mathcal{N}^{\omega} \cap \mathfrak{g} \} \xleftrightarrow{\text{bij}} \{ G_{\underline{d}}^{\varepsilon}\text{-orbits in } R_{\underline{d}}^{o} \cap R_{\underline{d}}^{\varepsilon} \}$$

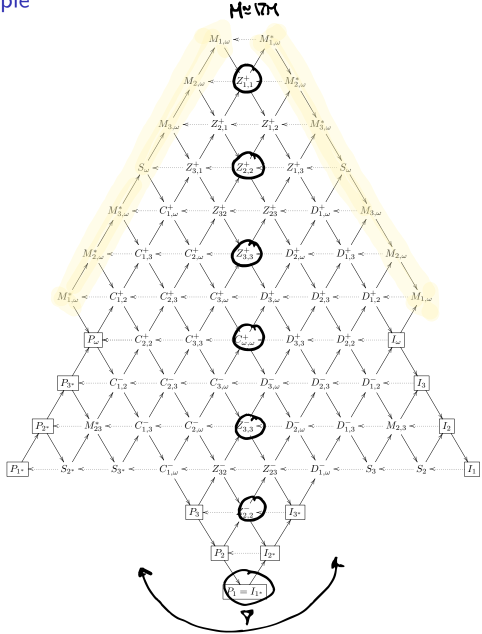
preserves orbit closure relations!

orbits $\equiv$ iso classes

closures $\equiv$ (ε-)degenerations

2 Symmetric Representation Theory

Our motivating example



3 Degenerations

Let $M, N \in R_{\underline{d}}$

$$M \le_{deg} N \quad :\iff \quad G_{\underline{d}} N \subseteq \overline{G_{\underline{d}} M} \subseteq R_{\underline{d}} \qquad \text{"degeneration"}$$

$$M \le_{hom} N \quad :\iff \quad \dim \mathrm{Hom}(U, M) \le \dim \mathrm{Hom}(U, N) \quad \forall U \in \mathrm{Rep}\, A \qquad \text{"hom order"}$$

$$M \le_{Ext} N \quad :\iff \quad \exists M_1, \dots, M_k \in \mathrm{Rep}\, A \text{ and} \qquad \text{"Ext order"}$$

short ex. seq.

$$0 \to U_i \to M_{i+1} \to V_i \to 0 \quad \forall i$$

$$M_1 \cong M, \; M_k \cong N, \; M_i \cong U_i \oplus V_i$$

GENERAL: $\le_{Ext} \overset{\text{Bongartz}}{\Longrightarrow} \le_{deg} \overset{\text{Abeasis-Del Fra, Riedtmann}}{\Longrightarrow} \le_{hom}$

A REP-FINITE: $\le_{deg} \underset{\text{Zwara}}{\iff} \le_{hom}$

DYNKIN: $\le_{Ext} \underset{\text{Bongartz}}{\iff} \le_{deg} \iff \le_{hom}$

3 Symmetric Degenerations

Let $M, N \in R_{\underline{d}}^{\varepsilon}$

$M \leq_{deg}^{\varepsilon} N \quad :\Longleftrightarrow \quad G_{\underline{d}}^{\varepsilon} M \subseteq \overline{G_{\underline{d}}^{\varepsilon} N}$ "ε-degeneration"

$M \leq_{Ext}^{\varepsilon} N \quad :\Longleftrightarrow \quad \exists$ ε-reps $M_1, \dots, M_k$ and short ex. seq. "ε-Ext"

$0 \to L_i \hookrightarrow M_{i-1} \to V_i \to 0 \quad \forall i$

$M_1 \cong M, \; M_k \cong N, \; M_i \cong L_i \oplus \nabla L_i \oplus L_i^{\perp}/L_i$

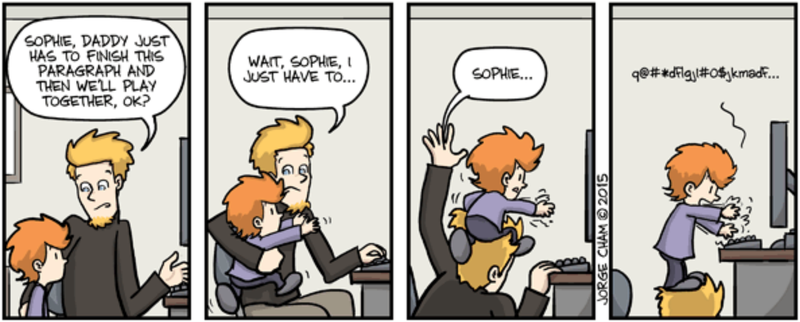
L_i isotropic in M_{i-1}

GENERAL $\quad \leq_{Ext}^{\varepsilon} \overset{\text{B.-Cerulli Irelli}}{\Longrightarrow} \leq_{deg}^{\varepsilon} \Longrightarrow \leq_{deg} \; (\Longrightarrow \; \leq_{hom})$

REP FINITE, DYNKIN $\Big\}$ Don't want to spoil it now ☺

4 Results

? $\leq_{dg}$ $\Rightarrow$ $\leq^{\varepsilon}_{deg}$?



"Piled Higher and Deeper" by Jorge Cham
www.phdcomics.de

4 Dynkin case

$Q = (Q, \sigma)$ connected symm. quiver of finite type

$A = \mathbb{C}Q$

$\varepsilon \in \{\pm 1\}$ $\quad$ $\underline{d}$ symm. dim vector

<u>Theorem</u> (B. – Cerulli-Irelli)

Let $M, N \in R_{\underline{d}}^{\varepsilon}$

Then $M \leq_{Ext}^{\varepsilon} N \iff M \leq_{deg}^{\varepsilon} N \iff M \leq_{deg} N$

<u>Strategy of proof</u> $\quad \leq_{deg} \Rightarrow \leq_{Ext}^{\varepsilon}$

L indec sth $[L, N]^1 = 0 \rightsquigarrow L \hookrightarrow M$
can be chosen isotropically!

$\rightsquigarrow$ different cases $\quad L \cong \nabla L$
$\quad L \oplus \nabla L$

$\rightsquigarrow$ symmetry of ARQ, in part. generic quotients
$\to$ inductive argument

4 Dynkin case

$$A_2 := \underset{1}{\bullet} \xrightarrow{\overset{\sigma(\alpha)}{\overset{\shortparallel}{\alpha}}} \underset{\sigma(1)}{\bullet}$$

$$\underline{d} = (n \ , \ n) \qquad\qquad \mathbb{C}^{n\times n}/GL_n \times GL_n \quad \text{only inv.} = rk$$

$\varepsilon = -1$

$\langle \, , \rangle$ symplectic form on $\mathbb{C}^n \oplus \mathbb{C}^n$

$$\varepsilon\text{-reps} : \{ M \in \mathbb{C}^{n\times n} \mid M_\alpha^* = -M_\alpha \} = R_{\underline{d}}^{\varepsilon} \circlearrowleft$$

$$\{ (g_1, g_2) \mid g_1 = (g_2^*)^{-1} \} = G_{\underline{d}}^{\varepsilon}$$

orbits = symplectic matrices / sympl. congruences

$\xRightarrow{\text{Thm}}$ closure ordering by rk

4 Algebraic Lie-theoretic (counter)example

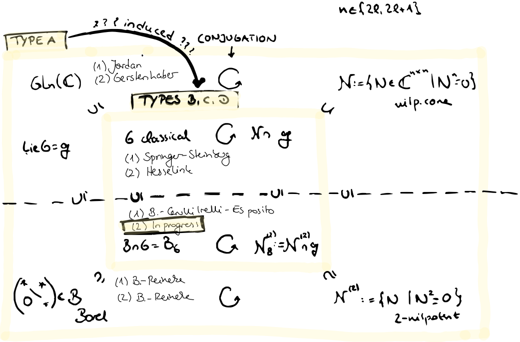
Orthogonal types B and D

$\varepsilon = 1$

$$Q = \begin{tikzcd} \underset{1}{\bullet} \arrow[r, "\alpha_1"] & \underset{2}{\bullet} \arrow[r, "\alpha_2"] & \underset{\omega}{\bullet} \arrow[loop above, "\xi"] \arrow[r, "\sigma(\alpha_2)"] & \underset{\sigma(2)}{\bullet} \arrow[r, "\sigma(\alpha_1)"] & \underset{\sigma(1)}{\bullet} \end{tikzcd} \qquad A := \mathbb{C}Q / (\xi^2, \sigma(\alpha_2) \circ \alpha_2)$$

$\underline{d} = (1, 2, 5, 2, 1)$ TYPE B

$\underline{d} = (1, 2, 4, 2, 1)$ TYPE D



Question: $\leq_{\deg} \Leftrightarrow \leq_{\deg}^{\varepsilon}$?

4 Algebraic Lie-theoretic (counter)example

Orthogonal types

Type B

+20

$$11 \quad \left(\begin{array}{cc|c|cc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & \mathbf{-1} & 0 & 0 & 0 \end{array}\right)$$

$$8 \quad \left(\begin{array}{cc|c|cc} 0 & 0 & 0 & 0 & 0 \\ \mathbf{1} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{-1} & 0 \end{array}\right)$$

$$6 \quad \left(\begin{array}{cc|c|cc} 0 & \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \mathbf{-1} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

ORBIT DIMENSION IN $\mathcal{N}_{\mathfrak{g}}$ (TYPE A)

$$3 \quad \left(\begin{array}{cc|c|cc} 0 & 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & \mathbf{-1} \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & & 0 & 0 & 0 \end{array}\right)$$

$$0 \quad \left(\begin{array}{cc|c|cc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Type D

+16

$$7 \quad \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline \mathbf{1} & 0 & 0 & 0 \\ 0 & \mathbf{-1} & 0 & 0 \end{array}\right)$$

$$6 \quad \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ \mathbf{1} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{-1} & 0 \end{array}\right)$$

$$4 \quad \left(\begin{array}{cc|cc} 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \mathbf{-1} \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$3 \quad \left(\begin{array}{cc|cc} 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & \mathbf{-1} \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$0 \quad \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

calculate $\le_{deg}$ by $\le_{hom}$

4 Algebraic Lie-theoretic (counter)example

Orthogonal types

Type B

+9

$$11 \quad \left(\begin{array}{cc|c|cc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & \mathbf{-1} & 0 & 0 & 0 \end{array}\right) \qquad 4$$

$$8 \quad \left(\begin{array}{cc|c|cc} 0 & 0 & 0 & 0 & 0 \\ \mathbf{1} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{-1} & 0 \end{array}\right) \qquad 3$$

$$6 \quad \left(\begin{array}{cc|c|cc} 0 & \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \mathbf{-1} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \qquad 2$$

ORBIT DIMENSION IN $\mathcal{N}_\mathfrak{g}$ (TYPE A)

$$3 \quad \left(\begin{array}{cc|c|cc} 0 & 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & \mathbf{-1} \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & & 0 & 0 & 0 \end{array}\right) \qquad 1$$

$$0 \quad \left(\begin{array}{cc|c|cc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \qquad 0$$

Type D

+7

M

$$7 \quad \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline \mathbf{1} & 0 & 0 & 0 \\ 0 & \mathbf{-1} & 0 & 0 \end{array}\right) \qquad 2$$

ORBIT DIMENSION IN $\mathcal{N}_\mathfrak{g}^\varepsilon$ (TYPES B/D)

4

$$6 \quad \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ \mathbf{1} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{-1} & 0 \end{array}\right) \qquad 2$$

N

$M \le_{deg} N$

$M \not\le^{\varepsilon}_{deg} N$

$$4 \quad \left(\begin{array}{cc|cc} 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \mathbf{-1} \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad 1$$

$$3 \quad \left(\begin{array}{cc|cc} 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & \mathbf{-1} \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad 1$$

calculate ε-degs by hand

$$0 \quad \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad 0$$

4 Algebraic Lie-theoretic (counter)example

Orthogonal types

ORBIT CLOSURES IN $\mathcal{N}_{\mathfrak{g}^-}^{(2)}$

Type B

$$\left(\begin{array}{cc|c|cc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & \mathbf{-1} & 0 & 0 & 0 \end{array}\right) \quad 4$$

$$\left(\begin{array}{cc|c|cc} 0 & 0 & 0 & 0 & 0 \\ \mathbf{1} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{-1} & 0 \end{array}\right) \quad 3$$

$$\left(\begin{array}{cc|c|cc} 0 & \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \mathbf{-1} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \quad 2$$

$$\left(\begin{array}{cc|c|cc} 0 & 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & \mathbf{-1} \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & & 0 & 0 & 0 \end{array}\right) \quad 1$$

$$\left(\begin{array}{cc|c|cc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \quad 0$$

Type D

$$\left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ \mathbf{1} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{-1} & 0 \end{array}\right) \qquad \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline \mathbf{1} & 0 & 0 & 0 \\ 0 & \mathbf{-1} & 0 & 0 \end{array}\right) \quad 2$$

$$\left(\begin{array}{cc|cc} 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \mathbf{-1} \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \left(\begin{array}{cc|cc} 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & \mathbf{-1} \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad 1$$

$$\left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad 0$$

4 Algebraic Lie-theoretic (counter)example

Orthogonal types B and D

TYPE D:

$\leq_{deg} \iff \leq_{deg}^{\varepsilon}$ is in general not true.

TYPE B:

not known

(In single G-orbits, it seems to be true!)

TYPE C: $\leq_{deg} \iff \leq_{deg}^{\varepsilon}$ is true

(in preparation)

CONCLUSION

GENERAL: $\leq^{\varepsilon}_{Ext} \Rightarrow \leq^{\varepsilon}_{deg} \Rightarrow \leq_{hom}$

A REP-FINITE: $\leq^{\varepsilon}_{deg} \not\Leftrightarrow \leq_{hom}$

DYNKIN $\leq^{\varepsilon}_{Ext} \Leftrightarrow \leq^{\varepsilon}_{deg} \Leftrightarrow \leq_{hom}$

CONJECTURE

A rep-directed $\leq^{\varepsilon}_{Ext} \Leftrightarrow \leq^{\varepsilon}_{deg} \Leftrightarrow \leq_{hom}$

maybe even if all indecs are rigid ?!

THANK

YOU


THANK
YOU