Cluster Categories & Rational Curves

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Outline

$$\begin{tikzcd}
\text{Rational curves} \arrow[rr, bend left, "\text{contraction}"] \arrow[dr] & & \text{singularities} \arrow[ll, bend left, "\text{resolution}"] \arrow[dl] \\
& \text{Non comm. algebra / category} &
\end{tikzcd}$$

§ Contraction of rational curves

$\mathcal{C} = \{C_1, \dots C_t\}$ a collection of rational curves in a space Y. A contraction of $\mathcal{C}$ is a map $f: Y \to X$ s.t

1) f is an isom outside $\bigcup_{i=1}^{t} C_i$
2) f maps $\bigcup_{i=1}^{t} C_i$ to a point in X
3) X is a "reasonable" space.

Question

1) When is $\mathcal{C}$ contractible?
2) If $\mathcal{C}$ is contractibe what can we say about singularity of X?

Def'n Let Y be a smooth quasi-proj 3fold $/\mathbb{C}$

A contraction is a birational proj morphism

$f : Y \to X$ s.t

$Ex(f)$ = exceptional fiber of f

1) X is normal

2) f is an isom. in codim 1

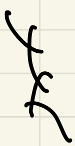
f is called a flopping contraction if

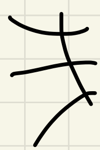
3) K_Y is f-trivial.

Rmk if f is a flopping contraction then

- X has isolated Gorenstein terminal singularities (in dim 3 are hypersurface)

- $Ex(f) = \bigcup_{i=1}^{t} C_i$ is a tree of rational curves with normal crossings





formal flopping contraction

$p \in X$ singularity

$$R = \widehat{\mathcal{O}}_{X,p} \cong \mathbb{C}[[x,y,z,w]]/(g)$$

$$\begin{tikzcd} Y \arrow[d, "f"'] & \hat{Y} := Y \times_X \operatorname{Spec} R \arrow[l] \arrow[d, "\hat{f}"] \arrow[dl, phantom, "\lrcorner" very near start] \\ X & \operatorname{Spec} R \arrow[l] \end{tikzcd}$$

formal contraction

§ Deformation algebra

$E_1, \cdots E_t \in \operatorname{coh} Y$ is called a

<u>semi-simple</u> collection if

$$\operatorname{Hom}(E_i, E_j) = \begin{cases} \mathbb{C} & i = j \\ 0 & i \neq j \end{cases}$$

$$E := \bigoplus_{i=1}^{t} E_i \qquad \ell := \operatorname{End}(E) = \mathbb{C}e_1 \times \cdots \times \mathbb{C}e_t$$

$\overline{\mathrm{Def}}_E : \underbrace{\mathrm{dgArt}}_{}{}^{-}_{\ell} \longrightarrow \mathrm{Gpd}$

non comm. dg Artinian negatively graded.

$R \longrightarrow R\text{-family} \in D(R^{op} \otimes \mathrm{Coh}\, Y)$

[Efimov-Lunts-Orlov]

Thm Let $E_1, \cdots E_t$ be a s.s collection of coh. sheaves with compact support.

Then $\overline{\mathrm{Def}}_E$ is pro-represented by

$$\Gamma := (\hat{T}_\ell V, d)$$

determined by A_∞-str on V.

$$V = \sum \mathrm{Ext}_Y^{\geq 1}(E, E)$$

Γ : deformation alg of $E_1, \cdots E_t$.

properties of Γ

- Γ is negatively graded
- homologically smooth
- if Y is Calabi-Yau i.e. $\omega_Y \cong \mathcal{O}_Y$

then Γ is a bimodule CY alg.

i.e. $R\mathrm{Hom}_{\Gamma^e}(\Gamma, \Gamma^e) \cong \Sigma^{-\dim Y} \Gamma$

Cluster category

$$\mathcal{C}_\Gamma = \mathrm{per}\,\Gamma / D_{fd}\Gamma$$ [Amiot]

- $\mathrm{Hom}_{\mathcal{C}_\Gamma}(\Gamma, \Gamma) \cong H^0 \Gamma$
- $\mathcal{C}_\Gamma$ is 2CY if interpreted appropriately

Example

if $C_1, \dots C_t$ is a collection of rational curves on Y with normal crossings then $\mathcal{O}_{C_1}, \dots, \mathcal{O}_{C_t}$ is semi-simple.

§ singularity category

R : complete local hypersurface ring with isolated singularities

$$D_{sg}(R) := \frac{D^b(\mathrm{mod}\text{-}R)}{\mathrm{proj}\text{-}R}$$ singularity category

- [Buchweitz] $D_{sg}(R) \simeq \underline{CM}(R)$

- [Eisenbud] on $D_{sg}(R)$ $\Sigma^2 \cong \mathrm{Id}$

- $D_{sg}(R)$ is Hom-finite, CY

by isolatedness.

Question

1) $\bigcup_{i=1}^{t} C_i$: CY contractible?

1') Γ def alg. $\mathcal{C}_\Gamma$ is Hom-finite & 2-periodic?

2) singularity of X?

2') is R determined by $H^0\Gamma$?

3) $R = \mathbb{C}[\![x, y, z, w]\!]/(g)$

$\exists\ \hat{f} : \hat{Y} \longrightarrow R$ s.t $\hat{f}^{-1}(0) = \bigcup_{i=1}^{t} C_i$?

3') R: $D_{sg}(R) = \mathcal{C}_\Gamma$ for some CY alg Γ?

§ Results on flopping contractions

Thm $\hat{f} : \hat{Y} \longrightarrow \operatorname{Spec} R$ flopping contraction.

Γ : deformation alg of $C_1, \dots C_t$ where.

$\operatorname{Ex} \hat{f} = \bigcup_{i=1}^{t} C_i$.

1) [Donovan-Wemyss] $\dim_{\mathbb{C}} H^0 \Gamma < +\infty$

(we call such curves noncomm. rigid !)

2) [de Thanhoffer de Völcsey - Van den Bergh]

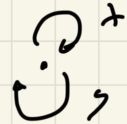
$$\mathcal{C}_\Gamma \simeq D_{sg}(R) \quad , \text{Hom-finite}$$

3) [Vd13] $\Gamma \simeq D(Q, w)$ Ginzburg

alg of Quiver Q and wt $\dfrac{\widehat{\mathbb{C}Q}}{[\widehat{\mathbb{C}Q}, \widehat{\mathbb{C}Q}]^{cl}}$

potential

$$H^0\Gamma = \frac{\widehat{\mathbb{C}Q}}{\langle D_a w \mid a \in Q_1 \rangle}$$ Jacobi alg.



w word

$$D_a w = \sum_{w = uav} vu \qquad D_x(x^2y) = xy + yx$$

$$w \in HH_0(\widehat{\mathbb{C}Q}) \longrightarrow [w] \in HH_0(H^0\Gamma)$$

[VdB] The right eq-class of $[w]$ is determined by the CY structure on Γ.

DG $\qquad\qquad$ Classical

$$\begin{tikzcd} \text{CY alg } \Gamma \arrow[r] & (H^0\Gamma, [w]) \arrow[l, dashed, bend left, "?"] \end{tikzcd}$$

Thm [H-Zhou]

Fix Q, $w, w' \in \dfrac{\widehat{\mathbb{C}Q}}{[\widehat{\mathbb{C}Q}, \widehat{\mathbb{C}Q}]^{cl}}$ with

finite dimensional Jacobi algebras

$\Gamma = D(Q, w)$ $\Gamma' = D(Q, w')$

Let $H^0\gamma : H^0\Gamma \longrightarrow H^0\Gamma'$ be an

$(\mathbb{C}Q_0\text{-})$ alg isom. s.t. $(H^0\gamma)_*[w] = [w']$

Then w is right equivalent to w'

As a consequence,

$H^0\gamma$ lifts to an isomorphism

$\gamma : \Gamma \xrightarrow{\simeq} \Gamma'$ as dg-algs

Thm [H-Keller]

R, R' complete local hypersurface rings with isolated singularity of dim n.

TFAE

1) $D_{sg} R \simeq D_{sg} R'$ as $\mathbb{Z}$-graded dg-cats.

2) $R \cong R'$

Rmk $D_{sg} R$ also admits a $\mathbb{Z}/2$-dg enhancement. It will become clear why we need the $\mathbb{Z}$-graded one!

Main thm [H-K]

Let $\hat{f} : \hat{Y} \to \operatorname{Spec} R$ $\quad \hat{f}' : \hat{Y}' \to \operatorname{Spec} R'$

be (3d) flopping contractions

Γ, Γ' the associate deformation algs of

$\operatorname{Ex}(\hat{f})$ and $\operatorname{Ex}(\hat{f}')$ Then 1) $\Rightarrow$ 2)

1) $\exists$ derived eq

$$H^0\phi : D(H^0\Gamma) \xrightarrow{\ \sim\ } D(H^0\Gamma')$$

s.t. $\underbrace{(H^0\phi)_*[w] = [w']}_{\star} \in HH_0(H^0\Gamma')$

2) $R \cong R'$

Rmk Donovan-Wemyss conjectured that

thm holds without $(\star)$

Sketch of pf

- [VdB]

Let L_i be ample line bundles on $\hat{Y}$ s.t

$$\deg_{C_j} L_i = \delta_{ij}$$

$$r_i = \min\{\# \text{ of generator of } H^1(\hat{Y}, \mathcal{L}_i^{-1})\}$$

$$0 \to \mathcal{L}_i^{-1} \to \mathcal{N}_i \to \mathcal{O}_{\hat{Y}}^{\oplus r_i} \to 0$$ universal exentension

$$N_i = \hat{f}_* \mathcal{N}_i$$

$A = \operatorname{End}_R(R \oplus N_1 \oplus \cdots \oplus N_t)$ is a NCCR

i.e. - gl dim $A = 3$

- $A \in CM_R$

$R\mathrm{Hom}(\mathcal{O}_{\hat{Y}} \oplus \mathcal{N}_1 \oplus \cdots \mathcal{N}_t, -)$

$$: D(\mathrm{coh}\,\hat{Y}) \xrightarrow{\simeq} D(\mathrm{mod}\text{-}A)$$

$$\begin{array}{ccc} R \oplus N_1 \oplus \cdots & \xrightarrow{e_0} & R \\ \downarrow e_i & & \\ N_i & & \end{array}$$

$\bar{\ell} = \mathbb{C}e_0 + \cdots + \mathbb{C}e_t \qquad \ell = \bar{\ell}/e_0$

$\exists\ \bar{V}$ of dim $t+1$ s.t

$$A \xleftarrow{\sim} (T_{\bar{\ell}}\bar{V}_-, \bar{d}) =: \bar{\Gamma}$$

$\bar{\Gamma}$ (exact) 3CY

- [VdB - de Th d V]

Γ : deformation alg of $Ext(f)$

$$\Gamma \cong \bar{\Gamma} / \bar{\Gamma} e_0 \bar{\Gamma} \rightsquigarrow (H^0\Gamma, [w])$$

- $H^0\phi : D(H^0\Gamma) \xrightarrow{\sim} D(H^0\Gamma')$ preserving $[w]$.

mutation

$$\begin{tikzcd} D(H^0\Gamma) \arrow[r, "\sim"] & D(H^0\Gamma') \\ D\Gamma \arrow[u] \arrow[r, "\phi"] & D\Gamma' \arrow[u] \end{tikzcd}$$

by comparing tilting theory of $H^0\Gamma$ and Γ.

We may then assume

$$\Gamma = D(Q, w) \qquad \Gamma' = D(Q, w')$$

$$H^0\gamma : H^0\Gamma \longrightarrow H^0\Gamma'$$

$$H^0\gamma_*[w] = [w']$$

- [H-Zhou] $\gamma : \Gamma \xrightarrow{\cong} \Gamma'$

in general $D\Gamma \simeq D\Gamma'$

$$\begin{tikzcd} C_\Gamma \arrow[r, "\simeq"] \arrow[d] & C_{\Gamma'} \arrow[d] \\ D_{sg}R \arrow[r, "\simeq"] & D_{sg}R' \end{tikzcd}$$ as dg-cats

$$HH^0(D_{sg}R) \cong HH^0(D_{sg}R')$$

as comm $\mathbb{C}$-algs.

$R = \mathbb{C}[[x_1, \dots x_n]]/g$ with isolated sing.

$$HH^0(D_{sg}R) \cong \frac{R}{\left(\frac{\partial g}{\partial x_1} \dots \frac{\partial g}{\partial x_n}\right)} \leftarrow \text{Tyurina algebra } T_g$$

- [Mather–Yau]

$$T_g \cong T_{g'} \iff R \cong R'$$

<u>Some open problems</u> (related to finite dim'l algebras)

1) Q : n loop quiver if W is a Jacobi finite potential with no quadratic parts, then $n \le 2$?

2) Q arbitrary, W Jacobi finite

Suppose Jacobi alg is symmetric.

Is $\mathcal{C}_P$ necessarily 2-periodic ?

3) Classify Jacobi finite potentials of 2-loop quiver ?