The Ziegler spectrum of tubular canonical algebras

FD Seminar

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The Ziegler spectrum

- Fact (Ziegler 1984): Associated with a ring A, we have a quasi-compact topological space $\mathrm{Zg}\, A$, which knows a lot about model theoretic properties of A-modules.
- We view modules as sets with operation, $(M, +, -, 0, - \cdot a (a \in A))$ and try to understand what we can say about modules in the first-order language.
- E.g.: Decidability problem (for $A = \mathbb{Z}$ goes back to Szmielew, 1955).
- Closed sets of $\mathrm{Zg}\, A$ bijectively correspond to subcategories of *Mod*-A, which are additive and first-order axiomatizable.
- Points of $\mathrm{Zg}\, A$ are the so-called indecomposable pure-injective (aka algebraically compact) modules.

Pure-injective modules

Definition (Jensen-Lenzing, 1989)
Let $\mathcal{C}$ be an additive category with products. An object $X \in \mathcal{C}$ is pure-injective if it has so-called infinite summing maps. I.e. for each set I there is a map $\Sigma_I \colon X^I \to X$ whose i-th component is the identity $1_X \colon X \to X$ for each $i \in I$.

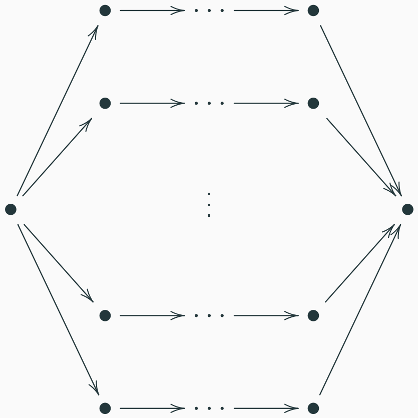
Example

1. If A is a finite-dimensional algebra over a field, all finite dimensional modules are pure-injective.
2. (Crawley-Boevey, Ringel) If A is the Kronecker algebra, then the indecomposable pure-injectives are
    - the indecomposable finite dimensional modules,
    - The Prüfer and adic modules corresponding to tubes,
    - the generic module.
3. $\operatorname{Zg} A$ is also known for domestic string algebras (Laking, Prest, Puninski, proving Ringel's conjecture).

Pure-injectives over tubular canonical algebras

Tubular canonical algebra

These are path algebras with quivers



with branches of lengths (2,2,2,2), (3,3,3), (4,4,2), (6,3,2), modulo suitable relations ($\text{gldim}\, A = 2$).

On module categories of tubular algebras

Large cotilting modules

- If A is a ring, a module $C \in \textit{Mod-}A$ is cotilting if
  (C1) inj. dim. $C < \infty$,
  (C2) $\mathrm{Ext}^{>0}(C^I, C) = 0$ for every set I,
  (C3) $\exists\, 0 \to C_n \to \cdots \to C_0 \to W \to 0$ for each injective cogenerator W, where all $C_i \in \mathrm{Prod}\, C$.
- Key facts:
  1. (Riccardo-Gregorio-Mantese 2007, Š. 2014): There is a Grothendieck category $\mathcal{C}$ and a derived equivalence between $\textit{Mod-}A$ and $\mathcal{C}$ identifying $\mathrm{Prod}\, C$ with the injective objects in $\mathcal{C}$.
  2. (Bazzoni 2004, Š. 2006): Cotilting modules are always pure-injective.
- Example: If A is a tubular canonical algebra, then $\textit{Mod-}A$ is derived equivalent to $\mathrm{Qcoh}\, \mathbb{X}$, where $\mathbb{X}$ is a tubular weighted projective line of the corresponding type.

The classification

Theorem (Angeleri Hügel-Kussin 2017)
Let A be a concealed canonical algebra of tubular type. The following is a complete list of the indecomposable pure-injective modules:

1. *the finite dimensional indecomposable modules,*
2. *the Prüfer modules, the adic modules, and the generic module associated with one of the tubular families,*
3. *the indecomposable modules in* $\operatorname{Prod} W_w$ *with* $w \in \mathbb{R}_+ \setminus \mathbb{Q}_+$,
4. *finitely many exceptional modules (these exceptions will disappear after tilting to a weighted projective line).*

Relation to elliptic curves

A rough classification

Theorem (Laking-Kussin 2020)
Let $\mathbb{X}$ be either a weighted projective line of tubular type (i.e. with weight types (2,2,2,2), (3,3,3), (4,4,2) or (6,3,2)) or an elliptic curve. The following is a complete list of the indecomposable pure-injective quasi-coherent sheaves:

1. *the indecomposable coherent sheaves,*
2. *the Prüfer modules, the adic modules, and the generic module associated with one of the tubular families,*
3. *the indecomposable sheaves in* $\mathrm{Prod}\, W_w$ *with* $w \in \mathbb{R} \setminus \mathbb{Q}$.

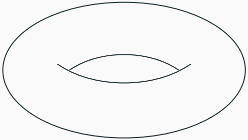
Complex elliptic curves

- A complex elliptic curve $\mathbb{E} \subseteq \mathbb{P}^2_{\mathbb{C}}$ is the projective closure of the zero set of

$$y^2 - x(x-1)(x-\lambda), \quad \lambda \in \mathbb{C} \setminus \{0, 1\} \quad \text{(Weierstrass normal form)}$$

- It is a Riemann surface, topologically a torus:

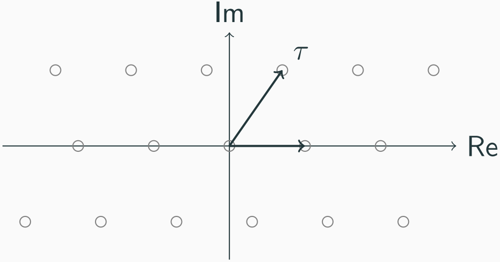
$\mathbb{E}$:



- As a Riemann surfaces, $\mathbb{E} \cong \mathbb{C}/\Lambda$, where $\Lambda \subseteq \mathbb{C}$ is a subgroup of $(\mathbb{C}, +)$ of rank 2 (using a Weierstrass elliptic functions).

Automorphisms of complex elliptic curves

$\Lambda \subseteq \mathbb{C}$:



- Fact: Each automorphism of $\mathbb{E} \cong \mathbb{C}/\Lambda$ is given by multiplication $z \in \mathbb{C}$ with $|z| = 1$ and $z\Lambda \in \Lambda$.
- So $\text{Aut}(\mathbb{E})$ is one of $\mathbb{Z}_2$ (generic case), $\mathbb{Z}_4$ (if $\tau = i$) or $\mathbb{Z}_6$ (if $\tau = e^{\frac{2\pi i}{6}}$).

Quotients with respect to a group action

- Let $\mathbb{E}$ be a complex elliptic curve and G a finite group acting on $\mathbb{E}$. Put $\mathbb{X} = \mathbb{E}/G$ (whatever it precisely means).
- Then G is one of $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$ or $\mathbb{Z}_6$, and the action (viewed as an action on $\mathbb{C}/\Lambda$) is rather explicit.
- In all the cases, $\mathbb{X}/G$ can be identified with $\mathbb{P}^1_{\mathbb{C}}$, and there are finitely many G-orbits with non-trivial stabilizers:

G	sizes of stabilizers
$\mathbb{Z}_2$	(2,2,2,2)
$\mathbb{Z}_3$	(3,3,3)
$\mathbb{Z}_4$	(4,4,2)
$\mathbb{Z}_6$	(6,3,2)

- Fact (Bundgaard-Nielsen 1951; Fox 1952): Complex tubular weighted projective lines are such quotients of elliptic curves, when viewed as orbifolds.

Categories of equivariant sheaves

- Let $\mathbb{E}$ be a complex elliptic curve and $G \circlearrowright \mathbb{E}$.
- This action induces actions $G \circlearrowright \operatorname{coh}\mathbb{E}$ and $G \circlearrowright \operatorname{Qcoh}\mathbb{E}$.
- If we put $\mathbb{X} := \mathbb{E}/G$ (whatever it precisely means), we may just define (thanks to Chen-Chen-Zhou 2015)

  $$\operatorname{coh}\mathbb{X} := (\operatorname{coh}\mathbb{E})^G \qquad \text{and} \qquad \operatorname{Qcoh}\mathbb{X} := (\operatorname{Qcoh}\mathbb{E})^G$$

  —categories of equivariant objects.
- Here: if $\mathcal{C}$ is a category and $G \circlearrowright \mathcal{C}$, then objects of $\mathbb{C}^G$ are of the form

  $$(X \in \mathcal{C}, \alpha_g \colon g * X \xrightarrow{\cong} X \ (g \in G)),$$

  where the α_g are subject to certain coherence relations.
- If G acts on an algebra, then $(Mod\text{-}A)^G \cong Mod\text{-}(G \ltimes A)$, where $G \ltimes A$ is the skew-group algebra.

Upshot

A strategy to understand indecomposable pure-injective sheaves on weighted projective lines:

- first understand pure-injective sheaves on elliptic curves,
- then inspect the possible equivariant structures on these.

Simple sheaves on non-commutative tori

(Quasi-)coherent sheaves on an elliptic curve

Continued fractions

Construction of simples on a non-commutative torus