Algebras of amenable representation type and (dimensional) expansion

Sebastian Eckert

Bielefeld University

FD seminar

25th March 2021

Hyperfiniteness and Amenability

Definition (based on [Ele17])

Let k be a field, A be a finite dimensional k-algebra and let $\mathcal{M}$ be a set of A-modules. $\mathcal{M}$ is called **hyperfinite** provided for every $\varepsilon > 0$ there exists $L_\varepsilon > 0$ such that for every $M \in \mathcal{M}$ there exists a submodule $P \subseteq M$ such that

$$\dim_k P \geq (1 - \varepsilon) \dim_k M, \tag{1}$$

and modules $N_1, N_2, \ldots N_t \in \operatorname{mod} A$, with $\dim_k N_i \leq L_\varepsilon$, such that $P \cong \bigoplus_{i=1}^t N_i$.

The k-algebra A is said to be of **amenable representation type** provided the set of all finite dimensional A-modules (or more specific, a set which meets any isomorphism class of finite dimensional A-modules) is hyperfinite.

Motivation

Conjecture (Elek '17)

Let k be a countable algebraically closed field and A be a finite dimensional algebra of infinite representation type over k. Then A is of tame representation type if and only if A is of amenable representation type.

Some (non-)examples

Example (finite representation type)

An algebra A of finite representation type is amenable.

Theorem (Elek '17)

Let k be a countable field. Any string algebra R is of amenable representation type.

Theorem (Elek '17)

The wild Kronecker quiver algebras are not of amenable representation type.

Some observations

Remark

It is enough to check for hyperfiniteness on indecomposable modules.

Proposition

A family of modules having submodules of globally bounded codimension in a hyperfinite family is hyperfinite.

Proposition

Left-exact functors with bounds on dimensions of the image preserve hyperfiniteness.

The 2-Kronecker quiver

Example

$$1 \overset{a}{\underset{b}{\rightrightarrows}} 2$$

Let k be any field. Then the path algebra of the 2-Kronecker quiver is of amenable representation type.

Representations of the Kronecker quiver

Question

Given any ε, can we find L_ε such that for all finite dimensional Kronecker-modules M there is a submodule P with $\dim P \geq (1-\varepsilon) \dim M$ which decomposes into summands of dimension bounded by L_ε?

Well-known classification of indecomposable Kronecker-modules:

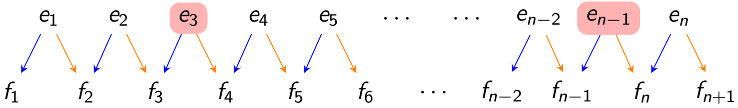
$$P_n \colon k^n \underset{\left[\begin{smallmatrix} 0 \\ \mathrm{id} \end{smallmatrix}\right]}{\overset{\left[\begin{smallmatrix} \mathrm{id} \\ 0 \end{smallmatrix}\right]}{\rightrightarrows}} k^{n+1}, \qquad Q_n \colon k^{n+1} \underset{[\,0 \ \mathrm{id}\,]}{\overset{[\,\mathrm{id} \ 0\,]}{\rightrightarrows}} k^n, \qquad R_n(\phi, \psi) \colon k^n \underset{\psi}{\overset{\phi}{\rightrightarrows}} k^n,$$

where $\forall\, n \in \mathbb{N}$ either

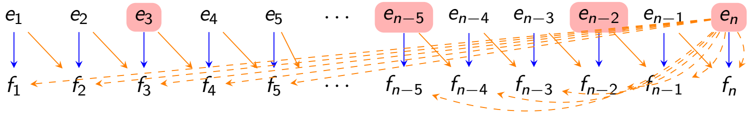
- $\phi = \mathrm{id}$ and ψ is companion matrix of power of monic irreducible over k, or
- $\psi = \mathrm{id}$ and ϕ is given by companion matrix of polynomial λ^m.

Finding a large submodule

- For preprojective P_n:



- for regular indecomposables:



- For the postinjective indecomposables, use the surjective map to the simple injective to find a submodule without postinjective summands.

Tame hereditary path algebras

Proposition

Let Q be a quiver of tubular type (p, q, r), where $p > 1$. Let all extended Dynkin quivers of type $(p-1, q, r)$ be amenable. If T is an inhomogeneous simple regular module belonging to a tube of rank p in Γ_{kQ}, then $T^{\perp}$ is hyperfinite.

Theorem

Let Q be an acyclic quiver of extended Dynkin type. Let k be any field. Then the path algebra kQ of Q is of amenable representation type.

Sketch of the proof

Pick a tube $\mathbb{T}$ of rank $p \geq 2$ (or maximal rank)

- Preprojective X either is in $S^{\perp}$ for regular simple $S \in \mathbb{T}$ or $\exists Y$ with $0 \to Y \to X \to T \to 0$ exact and $Y \in S^{\perp}$ for regular simples $S, T \in \mathbb{T}$.
- Indecomposable regular modules: either in $S^{\perp}$ (via orthogonality) or have submodule in $T^{\perp}$ for some regular-simple $T \in \mathbb{T}$.
- For indecomposable postinjectives: induction on the defect, showing hyperfiniteness of $\mathcal{N}_d :=$ {indecomposable modules of defect $\leq d$}.

Going further

With similar methods, we show the analogue result for all finite dimensional, tame hereditary algebras.

- Tame concealed works okay.
- There are partial results for tubular canonical algebras: preprojective, postinjective and integral slope modules (using classification of [DMM14])
- One might do it for clannish algebras, as Elek did it for string algebras.

Input from graph theory

Problem

How to approach the wild/non-amenable part of the conjecture?

Hyperfiniteness for modules based on notion from graph theory:

Definition (Elek)

Collection $\mathcal{G}$ of finite graphs is **hyperfinite** if $\forall \varepsilon > 0\ \exists K_\varepsilon$ finite s.t. $\forall G \in \mathcal{G}\ \exists S \subset E(G)$ s.t. $|S| \leq \varepsilon |V(G)|$ and every connected component of $G \setminus S$ has at most K_ε vertices.

Remark

Related notion of fragmentability ([EM94]) can be used to show that preprojective and postinjective component of wild Kronecker quivers are hyperfinite.

Expander Graphs

Definition

$G = (V, E)$, k-regular is an ε-**expander** if $\forall A \subset V$ with $|A| \leq \frac{|V|}{2}$,

$|N(A)| \geq (1+\varepsilon)|A|$, where $N(A) = \{y \in V : distance(y, A) \leq 1\}$.

Given a group G and S a finite, symmetric set of generators of G, the *Cayley graph* $Cay(G, S)$ is the graph with vertex set G and edges connecting x to sx for $s \in S$, thus each vertex $x \in G$ is connected to the $|S|$ elements sx, so $Cay(G, S)$ is a regular graph. Now, the above condition becomes

$$|N(A)| = |A \cup \bigcup_{i=1}^{k} s_i A| \geq (1+\varepsilon)|A|.$$

Dimension expanders and non-hyperfinite families

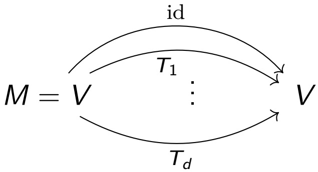
Definition (Barak-Impagliazzo-Shpilka-Wigderson)

k a field, $d \in \mathbb{N}$, $\alpha > 0$, V k-vector space, and $T_1, \ldots, T_d$ k-linear endomorphisms of V. The pair $(V, \{T_i\}_{i=1}^d)$ is an α-**dimension expander of degree** d if $\forall W \subset V$ with $\dim W \leq \frac{\dim_k V}{2}$, we have $\dim_k \left(W + \sum_{i=1}^d T_i(W) \right) \geq (1+\alpha) \dim_k W$.

Proposition

k be a field, $d \in \mathbb{N}$ and $\alpha > 0$. If $\{(V_i, \{T_l^{(i)}\}_{l=1}^d)\}_{i \in I}$ is a sequence of α-dimension expanders of degree d s.t. $\dim V_i$ is unbounded, then the induced family of $k\Theta(d+1)$-modules $M_i = \left((V_i, V_i), \left(\mathrm{id}, T_1^{(i)}, \ldots, T_d^{(i)} \right) \right)$ is not hyperfinite.

Sketch of proof



All small summands of M, say $W_I \overset{\rightarrow}{\underset{\rightarrow}{\cdots}} Z_I$, must have $\dim Z_I \leq (1+\alpha) \dim W_I$. But in the source vertex, we also need $\sum_I W_I \geq (1-2\varepsilon) \dim V$. A contradiction.

Constructing an example

Problem (Wigderson '04)

For fixed field k, fixed d, fixed α, find α-dim. expanders of degree d of arbitrarily large dimension.

Solutions

- Lubotzky–Zelmanov '08 for char $k = 0$
- for general k, reduction of Dvir–Shpilka '08/'11 shows that result of Bourgain '09/'13 on "monotone transformations with expansion property" solves it

Corollary

Let k a field, char $k = 0$*. Then the wild Kronecker algebra* $K\Theta(3)$ *is not of amenable representation type.*

A construction

Proposition ([LZ08])

If $\rho\colon \Gamma \to U_n(\mathbb{C})$ is an irreducible unitary representation, then $(\mathbb{C}^n, \rho(S))$ is an α-dimension expander of degree $|S|$ where $\alpha = \frac{\kappa^2}{12}$, $\kappa = K_\Gamma^S(S\ell_n(\mathbb{C}), \operatorname{adj}\rho)$, where $S\ell_n(\mathbb{C})$ denotes the subspace of all linear transformations of zero trace, and $\operatorname{adj}\rho$ is the adjoint representation on $\mathrm{End}(\mathbb{C}^n)$ induced by conjugation.

Now,

- find representations of $\mathrm{SL}(2, p)$ of arbitrarily large dimension (Steinberg)
- $SL(2, \mathbb{Z})$ has property (τ) (inspired by property (T)), this is proved via an application of Selberg's $\frac{3}{16}$ Theorem

An example

$\{((k^p, k^p), (\mathrm{id}, T_p, S_p))\}_{p \in \mathbb{P}}$, where

$$T_p = \begin{pmatrix} 0 & \dots & 0 & -1 & -1 \\ 1 & & & -1 & -1 \\ & \ddots & & \vdots & \vdots \\ & & 1 & -1 & -1 \\ 0 & \dots & 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}_p(\mathbb{Q}),$$

$$S_3 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, S_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 1 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$S_7 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \dots$$

Strictly wild algebras are not amenable

Definition

A f.d. k-algebra. A is **strictly wild** if $\exists$ orthogonal pair (X, Y) of f.d., f.p. modules, s.t. $\mathrm{End}(X)$, $\mathrm{End}(Y)$ are division rings and

$$p = \dim_{\mathrm{End}_A(Y)} \mathrm{Ext}^1_A(X, Y) \cdot \dim_{\mathrm{End}_A(X)} \mathrm{Ext}^1_A(X, Y) \geq 5.$$

Theorem

Let A be a finite dimensional k-algebra. If A is strictly wild, then A is not of amenable representation type.

Tools

Proposition

$\{M_i\}_{i\in I} \subseteq \operatorname{mod} A$ non-hyperfinite family of modules. Let $K_1, K_2 > 0$. Functors $F_i\colon \operatorname{mod} A \to \operatorname{mod} B$, $G_i\colon \operatorname{mod} B \to \operatorname{mod} A$ s.t.

- *$G_iF_i(M_i) \cong M_i$ for all $i \in I$,*
- *all G_i are left exact,*
- *$K_1 \dim_k F_i(M_i) \leq \dim_L G_iF_i(M_i)$ for all $i \in I$,*
- *$\dim_L G_i(X) \leq K_2 \dim_k X$ for all $X \in \operatorname{mod} B$ and $i \in I$,*

preserve these counterexamples to hyperfiniteness.

Idea

Use suitable tensor product functor $\operatorname{mod} L\Theta(d) \to \operatorname{mod} A$ for F_is.

A locally wild example

Theorem

The local wild algebra $A = k\langle x_1, x_2, x_3\rangle / M_2$, where M_2 is the ideal generated by all monomials of degree two, is not of amenable representation type.

Proof.

The functor $F\colon \operatorname{mod} A \to \operatorname{mod} k\Theta(3)$, with

$F(M) = \operatorname{top} M \overset{x_1\cdot -}{\underset{x_3\cdot -}{\overset{x_2\cdot -}{\rightrightarrows}}} \operatorname{rad} M,$ is exact and preserves monomorphisms if we ignore simple modules. □

A problem?

Here, we use that A is a radical square zero algebra.
What functor should one use in general?
If the (restricted) functor is not left exact, can we preserve submodules?

Modify the definition

Definition

k a field, A f.d. k-algebra, $\mathcal{M} \subseteq \operatorname{mod} A$ a family of f.d. A-modules. $\mathcal{M}$ is **weakly hyperfinite** if $\forall \varepsilon > 0 \exists L_\varepsilon > 0$ s.t. $\forall M \in \mathcal{M}\ \exists \theta\colon N \to M$ for some $N \in \operatorname{mod} A$ s.t.

$$\dim_k \ker \theta \leq \varepsilon \dim M, \quad \dim_k \operatorname{coker} \theta \leq \varepsilon \dim M, \tag{2}$$

and $\exists N_1, \dots, N_t \in \operatorname{mod} A$ with $\dim_k N_i \leq L_\varepsilon$ s.t. $N \cong \bigoplus_{i=1}^t N_i$.
A k-algebra A has **weak amenable representation type** if $\operatorname{mod} A$ itself is a weakly hyperfinite family.

Remarks

- hyperfinite $\Rightarrow$ weakly hyperfinite
- Kronecker representations induced by dimension expanders are not even weakly hyperfinite

Finitely controlled wild algebras are not amenable

Let k be alg. closed.

Definition (Ringel)

An algebra A is **(finitely) controlled wild** if for any f.d. algebra B $\exists F : \mathrm{mod}\, B \to \mathrm{mod}\, A$ faithful exact and $C \in \mathrm{mod}\, A$ s.t.

1. $\mathrm{Hom}_A(FM, FN) = F(Hom_B(M, N)) \oplus \mathrm{Hom}_A(FM, FN)_{\mathrm{add}\, C}$, and
2. $\mathrm{Hom}_A(FM, FN)_{\mathrm{add}\, C} \subseteq \mathrm{rad}\, \mathrm{End}_A(FM)$.

Theorem

Let A be a finite dimensional k-algebra. If A is finitely controlled wild, then A is not of weakly amenable representation type.

Sketch of proof

Proof.

Use the functor $F\colon \operatorname{mod} k\Theta(d) \to A$ from the definition of controlled wildness. By [GP16, Theorem 4.2], $\exists G\colon \operatorname{mod} A \to \operatorname{mod} k\Theta(d)$ s.t. $(G \circ F)(M) \cong M$ for all $M \in \operatorname{mod} k\Theta(d)$. Indeed, on objects this functor is given by

$$G(X) = {}^{\mathrm{Hom}_A(F(K), X)}\!\big/_{\mathrm{Hom}_A(F(K), X)_{\mathcal{C}}},$$

where $\mathrm{Hom}_A(X, Y)_{\mathcal{C}} = \{A\text{-homs } X \to Y \text{ factoring through } \mathcal{C}\}$. Remains to check estimates on dimensions. □

Bibliography I

.*Tame hereditary path algebras and amenability*. Aug. 2018. arXiv: 1808.02092 [math.RT].

.*(Extended) Kronecker quivers and amenability*. 2020. arXiv: 2011.02040 [math.RT].

Piotr Dowbor, Hagen Meltzer and Andrzej Mróz. “Parametrizations for integral slope homogeneous modules over tubular canonical algebras”. In: *Algebr. Represent. Theory* 17.1 (2014), pp. 321–356. ISSN: 1386-923X. DOI: 10.1007/s10468-012-9386-7.

Gábor Elek. “Infinite dimensional representations of finite dimensional algebras and amenability”. In: *Math. Ann.* 369.1 (2017), pp. 397–439. ISSN: 0025-5831. DOI: 10.1007/s00208-017-1552-0.

Keith Edwards and Colin McDiarmid. “New upper bounds on harmonious colorings”. In: *J. Graph Theory* 18.3 (1994), pp. 257–267. ISSN: 0364-9024. DOI: 10.1002/jgt.3190180305.

Lorna Gregory and Mike Prest. “Representation embeddings, interpretation functors and controlled wild algebras”. In: *J. Lond. Math. Soc. (2)* 94.3 (2016), pp. 747–766. ISSN: 0024-6107. DOI: 10.1112/jlms/jdw055.

Alexander Lubotzky and Efim Zelmanov. “Dimension expanders”. In: *J. Algebra* 319.2 (2008), pp. 730–738. ISSN: 0021-8693. DOI: 10.1016/j.jalgebra.2005.12.033.