FD seminar – Bonn

Attractors of torus actions on quiver moduli

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Families of indecomposable representations

Let

- Q connected quiver
- $\langle \_, \_ \rangle_Q : \mathbb{Z}^{Q_0} \times \mathbb{Z}^{Q_0} \to \mathbb{Z}$ Euler form of Q
- $d \in \mathbb{N}^{Q_0}$ dimension vector

Theorem (Kac)

If $\langle d, d \rangle_Q \leq 1$, then exists family of pairwise non-isomorphic indecomposable representations parametrized by $1 - \langle d, d \rangle_Q$ many continuous parameters.

Families of indecomposable representations

Question

Can we find (in special cases) *explicitly given* families of pairwise non-isomorphic indecomposables parametrized by $1 - \langle d, d \rangle_Q$ *independent* continuous parameters?

Call this a "generic normal form".

Approach

Use torus actions on quiver moduli.

Setup

Fix the following data:

- Q connected quiver;
  $Q_0 = \{\text{vertices of } Q\}$ and $Q_1 = \{\text{arrows of } Q\}$
- $d \in \mathbb{N}^{Q_0}$ dimension vector
- $\theta : \mathbb{Z}^{Q_0} \to \mathbb{Z}$ homomorphism such that $\theta(d) = 0$

Assumptions

1. Q is acyclic
2. d is θ-coprime, i.e. $\theta(d') \neq 0$ for all $0 \leq d' \leq d$, unless $d' = 0$ or $d' = d$

Setup

Fix complex vector spaces V_i of dimension d_i. Define

$$R(Q,d) = \bigoplus_{a \in Q_1} \mathrm{Hom}(V_{s(a)}, V_{t(a)})$$
$$G_d = \prod_{i \in Q_0} \mathrm{GL}(V_i)$$
$$PG_d = G_d/\Delta$$

where $\Delta = \{(t\,\mathrm{id}_{V_i})_i \mid t \in \mathbb{C}^\times\}$. Action $G_d \curvearrowright R(Q,d)$ by

$$g \cdot M = (g_{t(a)} M_a g_{s(a)}^{-1})_a$$

descends to action $PG_d \curvearrowright R(Q,d)$.

Group action vs. isomorphism

Lemma

Let $M, N \in R(Q, d)$, viewed as representations of Q. Then

$$M \cong N \Leftrightarrow M \text{ and } N \text{ lie in same } PG_d\text{-orbit.}$$

Generic normal form

Definition

Let $Z \subseteq R(Q, d)$ locally closed. Call Z a generic normal form (for indecomposable representations of dimension vector d) if

- all $M \in Z$ are indecomposable
- $PG_d \cdot M \cap PG_d \cdot N = \emptyset$ for all $M, N \in Z$ with $M \neq N$
- $Z \cong \mathbb{A}^n$ where $n = 1 - \langle d, d \rangle_Q$.

Semi-stable representations

Definition

Let $M \in R(Q, d)$.

- M is θ-semi-stable if $\theta(\underline{\dim}\, M') \leq 0$ for every subrepresentation $0 \neq M' \subsetneqq M$
- M is θ-stable if $\theta(\underline{\dim}\, M') < 0$ or every subrepresentation $0 \neq M' \subsetneqq M$

Remark

If M is θ-stable then M indecomposable

Define

$$R(Q, d)^{\theta-\text{sst}} = \{M \in R(Q, d) \mid M \text{ is } \theta\text{-semi-stable}\}$$
$$R(Q, d)^{\theta-\text{st}} = \{M \in R(Q, d) \mid M \text{ is } \theta\text{-stable}\}$$

Two PG_d-invariant Zariski open subsets.

Quiver moduli

Theorem (King)

There exists PG_d-linearized ample line bundle $L(\theta)$ on $R(Q,d)$ such that for all $M \in R(Q,d)$:

$$M \text{ is } \theta\text{-(semi-)stable} \Leftrightarrow M \text{ is (semi-)stable w.r.t. } L(\theta).$$

Can therefore define GIT quotients

Definition

- $M^{\theta-\mathrm{sst}}(Q,d) = R(Q,d)^{\theta-\mathrm{sst}} /\!\!/ PG_d$ called θ-semi-stable quiver moduli space
- $M^{\theta-\mathrm{st}}(Q,d) = R(Q,d)^{\theta-\mathrm{st}} / PG_d$ called θ-stable quiver moduli space

Quiver moduli

Facts

- $R(Q,d)^{\theta-\mathrm{st}} \xrightarrow{\pi} M^{\theta-\mathrm{st}}(Q,d)$ is a principal PG_d-bundle
  - $PG_d \curvearrowright R(Q,d)^{\theta-\mathrm{st}}$ freely
  - points of $M^{\theta-\mathrm{st}}(Q,d) =$ isoclasses of θ-stable representations of $\underline{\dim}\ d$
- If $R(Q,d)^{\theta-\mathrm{st}} \neq \emptyset$ then $\dim M^{\theta-\mathrm{st}}(Q,d) = 1 - \langle d,d\rangle_Q$

Recall our assumptions:

1. Q acyclic
2. d is θ-coprime

Facts

Under our assumptions, $M^{\theta-\mathrm{st}}(Q,d) = M^{\theta-\mathrm{sst}}(Q,d)$ is smooth and projective

An example

Let

- $Q = K(5) : \bullet \xrightarrow{(5)} \bullet$
- $d = (2, 5)$
- $\theta = (5, -2)$

Let $A = (A_1, \ldots, A_5) \in R(Q, d) = M_{5\times 2}(\mathbb{C})^5$. Then

$$A \ \theta\text{-sst} \Leftrightarrow A \ \theta\text{-st} \Leftrightarrow \dim\langle A_1 x, \ldots, A_5 x\rangle \geq 3 \ (\text{all } x \in \mathbb{C}^2 \setminus \{0\}) \text{ and}$$
$$\text{im}(A_1) + \ldots + \text{im}(A_5) = \mathbb{C}^5$$

Fact

$M^{\theta-\text{st}}(K(5), (2, 5))$ smooth projective variety of dimension 22

Torus action

Let $T = \mathbb{C}^\times$.

- Choose weights $w_a \in \mathbb{Z}$ (all $a \in Q_1$)
- Define $T \curvearrowright R(Q, d)$ by $t.M = (t^{w_a} M_a)_a$

Remark

- $R(Q, d)^{\theta-\mathrm{st}}$ is T-invariant
- T-action and PG_d-action commute

Lemma

Obtain action $T \curvearrowright M^{\theta-\mathrm{st}}(Q, d)$

Fixed points

Let $M \in R(Q,d)^{\theta-\mathrm{st}}$ such that $[M] \in M^{\theta-\mathrm{st}}(Q,d)^T$.

- For all $t \in T$ exists unique $g \in PG_d$ such that $t.M = g \cdot M$
- Gives homomorphism $\rho = \rho_M : T \to PG_d$
- Choose lift $\dot{\rho} : T \to G_d = \prod_i \mathrm{GL}(V_i)$
- Induces weight space decompositions $V_i = \bigoplus_{m \in \mathbb{Z}} V_{i,m}$ such that

$$M_a(V_{s(a),m}) \subseteq V_{t(a),m+w_a}$$

Lemma

M defines representation $\dot{M}$ of (infinite) quiver $Q(w)$ (where $w = (w_a)_a$), given by

$$Q(w)_0 = Q_0 \times \mathbb{Z} \qquad Q(w)_1 = Q_1 \times \mathbb{Z}$$
$$s(a,m) = (s(a),m) \qquad t(a,m) = (t(a), m + w_a)$$

Fixed points

Remark

- Let $C_d = \{\beta \in \mathbb{N}^{Q(w)_0} \mid \sum_m \beta_{i,m} = d_i \text{ (all } i \in Q_0)\}$; then $\underline{\dim}\, \dot{M} \in C_d$
- For $n \in \mathbb{Z}$ have auto equivalence s_n on $\mathrm{Rep}_{\mathbb{C}}(Q(w))$ defined by $s_n(N)_{i,m} = N_{i,m+n}$ and $s_n(N)_{a,m} = N_{a,m+m}$
- Induces action $\mathbb{Z} \curvearrowright \mathbb{N}^{Q(w)_0}$ which leaves C_d invariant
- For two lifts $\dot{\rho}, \ddot{\rho}$ of ρ, we have $\ddot{M} \cong s_n(\dot{M})$ for a unique $n \in \mathbb{Z}$

Theorem (Weist)

$M^{\theta-\mathrm{st}}(Q,d)^T = \bigsqcup_{[\beta] \in C_d/\mathbb{Z}} F_\beta$, a finite disjoint union into connected components with

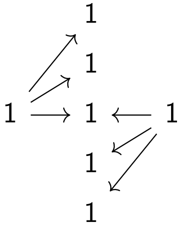
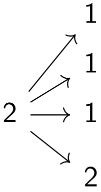
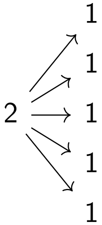
$$F_\beta \cong M^{\theta-\mathrm{st}}(Q(w),\beta)$$

An example (continued)

Let

- $Q = K(5)$, $d = (2,5)$, and $\theta = (5,-2)$
- weights for T-action such that $w_1 \gg w_2 \gg \ldots \gg w_5$

List of all $[\beta] \in C_d/\mathbb{Z}$ with $F_\beta \neq \emptyset$:

type			
$\#\beta$'s of this type	360	20	1
$M^{\theta-\mathrm{st}}(Q(w),\beta)$	$\{\mathrm{pt}\}$	$\{\mathrm{pt}\}$	$\mathrm{Bl}_4(\mathbb{P}^2)$

Corollary

$\chi(M^{\theta-\mathrm{st}}(K(5),(2,5))) = 380 + \chi(\mathrm{Bl}_4(\mathbb{P}^2)) = 387$

Białynicki-Birula decompositions

Let X smooth projective variety with action of $T = \mathbb{C}^\times$.

- Let $X^T = \bigsqcup_{\beta \in C} F_\beta$ decomposition into connected components
- Define attractor $X_\beta = \mathrm{Att}(F_\beta) = \{x \in X \mid \lim_{t \to 0} t.x \in F_\beta\}$
- For $x \in X^T$, obtain $T \curvearrowright T_x X$ by derivative of action map
- Gives weight space decomposition $T_x X = \bigoplus_{n \in \mathbb{Z}} (T_x X)_n$.

Theorem (Białynicki-Birula)

1. $X_\beta \subseteq X$ *locally closed, irreducible, and smooth.*
2. $X = \bigcup_{\beta \in C} X_\beta$, *a disjoint union.*
3. $\pi_\beta : X_\beta \to F_\beta$ *is Zariski locally trival fibration*
4. $\mathrm{Att}(x) := \pi_\beta^{-1}(x)$ *is affine space of dimension* $\sum_{n>0} \dim(T_x X)_n$

Tangent space of the moduli space

Let $M \in R(Q,d)^{\theta-\mathrm{st}}$ such that $[M] \in M^{\theta-\mathrm{st}}(Q,d)^T$. Obtain short exact sequence

$$
\begin{array}{ccccccccc}
 & & 0 & & & & & & \\
 & & \uparrow & & & & & & \\
0 & \longrightarrow & \mathfrak{pg}_d & \xrightarrow{d_e\,\mathrm{act}_{PG_d}(\_,M)} & T_M R(Q,d)^{\theta-\mathrm{st}} & \to & T_{[M]} M^{\theta-\mathrm{st}}(Q,d) & \to & 0 \\
 & & \uparrow & & \| & & \| & & \\
 & & \bigoplus_i \mathrm{End}(V_i) & \xrightarrow{[\_,M]} & \bigoplus_a \mathrm{Hom}(V_{s(a)}, V_{t(a)}) & \to & \mathrm{Ext}^1(M,M) & \longrightarrow & 0 \\
 & \nearrow & \uparrow & & & & & & \\
\mathrm{End}(M) & = & \mathbb{C} & & & & & & \\
\nearrow & & \uparrow & & & & & & \\
0 & & 0 & & & & & &
\end{array}
$$

and $[x, M] = (x_{t(a)} M_a - M_a x_{s(a)})_a$.

Weight spaces of the tangent space

Lemma

Exist linear actions $T \curvearrowright T_M R(Q,d)^{\theta-\mathrm{st}}$ *and* $T \curvearrowright \mathfrak{g}_d$ *such that the maps*

$$\mathfrak{g}_d \to T_M R(Q,d)^{\theta-\mathrm{st}} \to T_{[M]} M^{\theta-\mathrm{st}}(Q,d) \to 0$$

are T*-equivariant*

Lemma

With respect to above actions,

$$(\mathfrak{g}_d)_n = \bigoplus_{i\in Q_0} \bigoplus_{m\in\mathbb{Z}} \mathrm{Hom}(V_{i,m}, V_{i,m-n})$$

$$(T_M R(Q,d)^{\theta-\mathrm{st}})_n = \bigoplus_{a\in Q_1} \bigoplus_{m\in\mathbb{Z}} \mathrm{Hom}(V_{s(a),m}, V_{t(a),m+w_a-n})$$

Weight spaces of the tangent space

Theorem (Boos–F.)

For $M \in R(Q,d)^{\theta-\mathrm{st}}$ such that $[M] \in M^{\theta-\mathrm{st}}(Q,d)^T$

$$(T_{[M]}M^{\theta-\mathrm{st}}(Q,d))_n \cong \mathrm{Ext}_{Q(w)}(\dot{M}, s_{-n}(\dot{M}))$$
$$\dim(T_{[M]}M^{\theta-\mathrm{st}}(Q,d))_n = \delta_{n,0} - \langle \beta, s_{-n}(\beta)\rangle_{Q(w)}$$

where $\dot{M}$ is lift of M to $Q(w)$ and $\beta := \underline{\dim}\,\dot{M}$.

Twisted filtrations

Let $N \in R(Q, d)$. Assume exist filtrations

$$\ldots \subseteq F_{i,n} \subseteq F_{i,n+1} \subseteq \ldots \subseteq V_i$$

(with $F_{i,-n} = 0$ and $F_{i,n} = V_i$ for $n \gg 0$) such that $N_a(F_{s(a),n}) \subseteq F_{t(a),n+w_a}$.

Definition

$F_* = (F_{i,*})_i$ is called a w-twisted filtration of N.

Remark

If N has w-twisted filtration F_*, then

$$F_{s(a),n}/F_{s(a),n-1} \to F_{t(a),n+w_a}/F_{t(a),n+w_a-1}$$

define representation of $Q(w)$. Call it $\mathrm{gr}^{F_*}(N)$.

Attractors

Proposition

Let $M, N \in R(Q,d)^{\theta-\mathrm{st}}$ *such that* $[M] \in M^{\theta-\mathrm{st}}(Q,d)^T$. *Then*

$$[N] \in \mathrm{Att}([M]) \Leftrightarrow \exists\ w\text{-twisted filtration } F_* \text{ of } N \text{ such that } \mathrm{gr}^{F_*}(N) \cong M \text{ as representations of } Q$$

Essentially a reformulation of a result of Kinser and Weist.

Attractors

Let $[M] \in M^{\theta-\mathrm{st}}(Q,d)^T$

- Let $\dot{M}$ lift of M to $Q(w)$
- $V_i = \bigoplus_n V_{i,n}$ the corresponding decompositions
- Define $F_{i,n} = \bigoplus_{m \leq n} V_{i,m}$

Define

$$R_{F_*} := \bigoplus_{k>0} \underbrace{\bigoplus_{a \in Q_1} \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}(V_{s(a),n}, V_{t(a),n+w_a-k})}_{=:R_{F_*,k}}$$

$$\mathfrak{u}_{F_*} := \bigoplus_{k>0} \underbrace{\bigoplus_{i \in Q_1} \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}(V_{i,n}, V_{i,n-k})}_{=:\mathfrak{u}_{F_*,k}}$$

$$[\mathfrak{u}_{F_*}, M] := \mathrm{im}\left(\mathfrak{u}_{F_*} \to R_{F_*},\ x \mapsto [x, M]\right)$$

$$\subseteq R_{F_*} \text{ is } \mathbb{Z}_{>0}\text{-graded subspace}$$

Attractors

Theorem (Boos–F.)

Let $[M] \in M^{\theta-\mathrm{st}}(Q,d)^T$*, let*

- $\dot{M}$ *lift of* M *to* $Q(w)$
- F_* *the corresponding filtration.*

Choose $\mathbb{Z}_{>0}$*-graded vector space complement* R' *of* $[\mathfrak{u}_{F_*}, M]$ *inside* R_{F_*}*. Then*

$$\begin{aligned} R' &\longrightarrow R(Q,d)^{\theta-\mathrm{st}} \xrightarrow{\ \pi\ } M^{\theta-\mathrm{st}}(Q,d) \\ N &\longmapsto M + N \end{aligned}$$

is well-defined and induces isomorphism $R' \xrightarrow{\cong} \mathrm{Att}([M])$*.*

Generic normal form

Corollary

Suppose that there exists $\beta \in C_d$ *such that*

1. $M^{\theta-\mathrm{st}}(Q(w), \beta) = \{[M]\}$ *and*
2. $\dim \mathrm{Att}([M]) = \dim M^{\theta-\mathrm{st}}(Q, d)$.

Let R' *as in Thm. Then the (closed) subset*

$$\{M\} + R' \subseteq R(Q, d)$$

is a generic normal form (for $M^{\theta-\mathrm{st}}(Q, d)$*)*

Remark

These conditions can be checked:

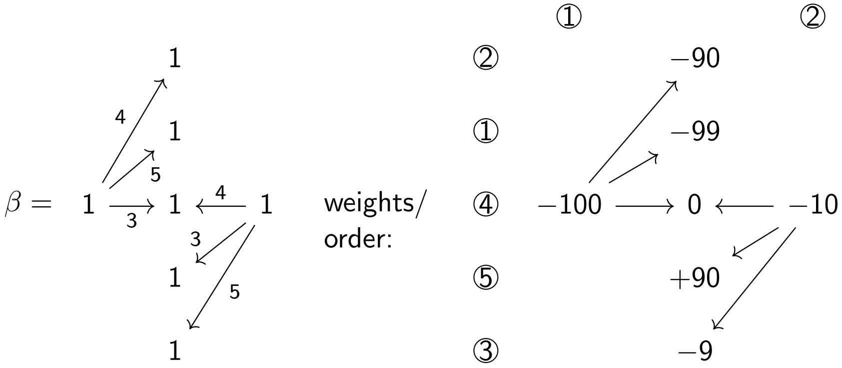
1. holds iff $\langle \beta, \beta \rangle_{Q(w)} = 1$ and $R(Q(w), \beta)^{\theta-\mathrm{st}} \neq \emptyset$
2. holds iff $-\sum_{m>0} \langle \beta, s_{-m}(\beta) \rangle_{Q(w)} = 1 - \langle d, d \rangle_Q$
   iff $\langle \beta, s_{-m}(\beta) \rangle_{Q(w)} = 0$ for all $m < 0$.

An example (continued)

Let

- $Q = K(5)$, $d = (2, 5)$, and $\theta = (5, -2)$
- $w_1 = 10,000$, $w_2 = 1,000$, $w_3 = 100$, $w_4 = 10$, and $w_5 = 1$

Conditions 1. and 2. of previous corollary hold for



An example (continued)

Let $M \in M^{\theta-\mathrm{st}}(Q(w), \beta)$. Then

$$M = \left(\begin{pmatrix} & \\ & \\ & \\ & \\ & \end{pmatrix}, \begin{pmatrix} & \\ & \\ & \\ & \\ & \end{pmatrix}, \begin{pmatrix} & \\ & \\ & \\ 1 & \\ & 1 \end{pmatrix}, \begin{pmatrix} & \\ 1 & \\ & \\ & 1 \\ & \end{pmatrix}, \begin{pmatrix} 1 & \\ & \\ & 1 \\ & \\ & \end{pmatrix} \right)$$

$$R_{F_*} = \left(\begin{pmatrix} * & * \\ * & * \\ * & * \\ * & * \\ * & * \end{pmatrix}, \begin{pmatrix} * & * \\ * & * \\ * & * \\ * & * \\ * & * \end{pmatrix}, \begin{pmatrix} * & * \\ * & * \\ * & * \\ & * \\ & \end{pmatrix}, \begin{pmatrix} * & * \\ & * \\ & * \\ & \\ & \end{pmatrix}, \begin{pmatrix} & * \\ & * \\ & \\ & \\ & \end{pmatrix} \right)$$

An example (continued)

$$\mathfrak{u}_{F_*} = \left(\begin{pmatrix} & b \\ & \end{pmatrix}, \begin{pmatrix} & a_{12} & a_{13} & a_{14} & a_{15} \\ & & a_{23} & a_{24} & a_{25} \\ & & & a_{34} & a_{35} \\ & & & & a_{45} \\ & & & & \end{pmatrix} \right)$$

$$[\mathfrak{u}_{F_*}, M] = \left((0), (0), \begin{pmatrix} {\color{blue}a_{14}} & a_{15} \\ {\color{blue}a_{24}} & a_{25} \\ {\color{blue}a_{34}} & a_{35} \\ & a_{45} - {\color{red}b} \\ & \end{pmatrix}, \begin{pmatrix} a_{12} & {\color{blue}a_{14}} \\ & {\color{blue}a_{24}} - {\color{red}b} \\ & {\color{blue}a_{34}} \\ & \\ & \end{pmatrix}, \begin{pmatrix} & a_{13} - {\color{red}b} \\ & a_{23} \\ & \\ & \\ & \end{pmatrix} \right)$$

Corollary

A generic normal form for $M^{\theta-\mathrm{st}}(K(5),(2,5))$ *is given by*

$$\{M\}+R'=\left(\begin{pmatrix}* & *\\ * & *\\ * & *\\ * & *\\ * & *\end{pmatrix},\begin{pmatrix}* & *\\ * & *\\ * & *\\ * & *\\ * & *\end{pmatrix},\begin{pmatrix} & \\ & \\ & \\ 1 & \\ & 1\end{pmatrix},\begin{pmatrix} & *\\ 1 & \\ & *\\ & 1\\ & \end{pmatrix},\begin{pmatrix}1 & \\ & \\ & 1\\ & \\ & \end{pmatrix}\right)$$

Thank you!