Higher Dimensional Ideal Approximation Theory

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Higher Dimensional Ideal Approximation Theory

Contents:

- Approximation Theory (background and history)
- Ideal Approximation Theory (in an exact category)
- Higher Homological Algebra (n-exact categories)
- Higher Ideal Approximation Theory

Throughout

- $\mathscr{A}$ is an abelian category;
- $\mathscr{M}$ is an additive full subcategory of $\mathscr{A}$;
- Λ is an Artin algebra;
- mod-Λ is the category of finitely generated (right) Λ-modules.

The main idea of the classical Approximation Theory is to select suitable subcategories $\mathcal{C}$, and then approximate arbitrary objects by the ones from $\mathcal{C}$.

The starting point of approximation theory is the discovery of the existence of injective envelopes by Baer in 1940.

Independent researches by Auslander, Reiten and Smalø, Solberg, etc, for Artin algebras, and by Enochs, Jenda, Xu, etc, for arbitrary rings, created a general theory of right and left approximations, or precovers and preenvelopes of modules.

[X] J. Xu, *Flat Cover of Modules*, Lecture Notes in Math. 1634, Springer, New York, 1996.

[EJ] E.E. Enochs and O.M.G. Jenda, Relative Homological Algebra, de Gruyter Exp. Math. 30, Walter de Gruyter Co., 2000.

[GT] R. Göbel and J. Trlifaj, Approximations and Endomorphism Algebras of Modules: de Gruyter Expositions in Mathematics, 41, Walter de Gruyter, Berlin (2006).

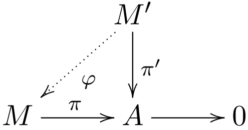
A

right $\mathscr{M}$-approximation of $A \in \mathscr{A}$ (= a $\mathscr{M}$-precover of $A \in \mathscr{A}$)

is a morphism $\pi : M \to A$ with $M \in \mathscr{M}$ such that

$$\mathscr{A}(-, M)|_{\mathscr{M}} \longrightarrow \mathscr{A}(-, A)|_{\mathscr{M}} \longrightarrow 0$$

is exact, i.e.



$\mathscr{M}$ is called a

contravariantly finite (= a precovering)

subcategory of $\mathscr{A}$ if every object of $\mathscr{A}$ admits a right $\mathscr{M}$-approximation.

Examples. Projective precover, flat precover, Gorenstein projective precover, etc.

Dually, the notion of

left $\mathscr{M}$-approximations (= $\mathscr{M}$-preenvelopes)

and

covariantly finite (= preenveloping)

subcategories are defined.

Example. Injective preenvelope, pure-injective preenvelope, cotorsion preenvelope, Gorenstein injective preenvelope, etc.

Let $\pi : M \to A$ be an epimorphism with $M \in \mathscr{M}$. Set $C := \mathrm{Ker}\pi$.

If $\mathrm{Ext}^1_{\mathscr{A}}(\mathscr{M}, C) = 0$, then clearly π is a $\mathscr{M}$-precover of A,

$$0 \longrightarrow C \longrightarrow M \longrightarrow A \longrightarrow 0.$$

This leads us to the following definition.

Definition

An epimorphism $\pi : M \to A$, with $M \in \mathscr{M}$, is called a special $\mathscr{M}$-precover of A if $C \in \mathscr{M}^{\perp}$.

Dually the notion of special $\mathscr{M}$-preenvelope is defined.

One of the basic tools in classical approximation theory is the notion of Cotorsion pairs.

Let $\mathscr{A}$ be an abelian category. A cotorsion pair is a pair $(\mathscr{C}, \mathscr{D})$ of full subcategories of $\mathscr{A}$ such that

$$\mathscr{C}^{\perp_1} = \mathscr{D} \quad \text{and} \quad \mathscr{C} = {}^{\perp_1}\mathscr{D},$$

where

$$\mathscr{C}^{\perp_1} = \{A \in \mathscr{A} : \mathrm{Ext}^1_{\mathscr{A}}(C, A) = 0 \text{ for all } C \in \mathscr{C}\}.$$

$\mathscr{C}$ is called a cotorsion class and $\mathscr{D}$ is called a cotorsion-free class.

Salce's Lemma and Wakamatsu's Lemma are two pillars of the classical approximation theory:

Salce's Lemma states that in a cotorsion pair $(\mathscr{C}, \mathscr{D})$ in $\mathscr{A}$, $\mathscr{C}$ is special precovering if and only if $\mathscr{D}$ is special preenveloping.

Wakamatsu's Lemma proves that if a class $\mathscr{C}$ is closed under extension, then $\mathscr{C}$-envelopes and $\mathscr{C}$-covers are special.

Ideal Approximation Theory

Ideal Approximation Theory

In classical approximation theory, approximation is done by objects from a subcategory.

But a nice generalization of the classical approximation theory, known as ideal approximation theory, is studied systematically in [FGHT] that give morphisms and ideals of categories equal importance as objects and subcategories.

In this theory, the role of the objects and subcategories in classical approximation theory is replaced by morphisms and ideals of the category.

[FGHT] X. H. Fu, P. A. Guil Asensio, I. Herzog and B. Torrecillas, *Ideal approximation theory*, Adv. Math., (2013).

Let $\mathscr{A}$ be an additive category. A two sided ideal $\mathscr{I}$ of $\mathscr{A}$ is a subfunctor

$$\mathscr{I}(-,-) : \mathscr{A}^{\mathrm{op}} \times \mathscr{A} \longrightarrow \mathscr{A}b$$

of the bifunctor $\mathscr{A}(-,-)$ that associates to every pair A and A' of objects in $\mathscr{A}$ a subgroup $\mathscr{I}(A, A') \subseteq \mathscr{A}(A, A')$ such that

- If $f \in \mathscr{I}(A, A')$ and $g \in \mathscr{A}(A', C)$, then $gf \in \mathscr{I}(A, C)$,
- If $f \in \mathscr{I}(A, A')$ and $h \in \mathscr{A}(D, A)$, then $fh \in \mathscr{I}(D, A')$.

$$D \xrightarrow{h} A \xrightarrow{\quad f \quad} A' \xrightarrow{g} C$$

Thus, an ideal of a category is an additive subfunctor of the Hom functor, which is closed under compositions by morphisms from left and right.

Let $\mathscr{I}$ be an ideal of $\mathscr{A}$ and $A \in \mathscr{A}$ be an object of $\mathscr{A}$.

Definition

An $\mathscr{I}$-precover of A is a morphism $C \xrightarrow{\varphi} A$ in $\mathscr{I}$ such that any other morphism $C' \xrightarrow{\varphi'} A$ in $\mathscr{I}$ factors through φ, i.e.

$$\begin{array}{ccccc} & & C' & & \\ & \overset{\psi}{\swarrow} & \downarrow{\scriptstyle \varphi'} & & \\ C & \xrightarrow{\varphi} & A & \longrightarrow & 0 \end{array}$$

$\mathscr{I}$ is called a precovering ideal if every object $A \in \mathscr{A}$ admits an $\mathscr{I}$-precover.

The notions of $\mathscr{I}$-preenvelope and preenveloping ideals are defined dually.

Auslander and Reiten [AR] shows that every object $M \in \text{mod-}\Lambda$ has a cover (resp., envelope) with respect to the ideal $\text{Jac}(\text{mod-}\Lambda)$ given by the Jacobson radical of $\text{mod-}\Lambda$.

[AR] M. Auslander and I. Reiten, *Representation theory of artin algebras III: almost split sequences*, Comm. Algebra (1975).

Another important example, is given by the ideal of phantom morphisms in $\text{Mod-}R$ that studied extensively by Herzog [H].

[H] I. Herzog, *The phantom cover of a module*, Adv. Math. (2007).

Note that this point of view is more general, because the module M is represented by the unit morphism $1_M : M \longrightarrow M$, and an additive subcategory $\mathscr{C}$ by the ideal $\mathscr{I}(\mathscr{C})$ of morphisms that factor through $\mathscr{C}$.

As another references, Fu and Herzog in [FH] studied ideal versions of some notions of classical approximation theory, such as cotorsion pairs, Salce's Lemma and Wakamatsu's Lemma.

[FH] X. Fu, and I. Herzog, *Powers of the phantom ideal*, Proc. London Math. Soc. (2016).

and very recently, Breaz and Modoi [BM] studied ideal cotorsion pairs in extension closed subcategories of triangulated categories.

[BM] S. Breaz and G. C. Modoi, *Ideal cotorsion theories in triangulated categories*, J. Algebra, 2021, in press.

For more research in this direction see [EAO].

[EAO] S. Estrada, P.A. Guil Asensio and F. Ozbek, *Covering ideals of morphisms and module representations of the quiver* $\mathbb{A}_2$, J. Pure Appl. Algebra (2014).

Example. Let $(\mathscr{A}, \mathscr{E})$ be an exact category and $\mathscr{F} \subseteq \mathrm{Ext}^1$ be a subfunctor. A morphism φ in $\mathscr{A}$ is an $\mathscr{F}$-phantom morphism if the pullback of any $\mathscr{E}$-conflation along φ is a conflation in $\mathscr{F}$. The collection of all $\mathscr{F}$-phantom morphisms is an ideal of $\mathscr{A}$, usually denoted by $\Phi(\mathscr{F})$.

For instance, if you consider the subfunctor Pext, consisting of pure extensions, then a morphism $\varphi : X \to M$ is called a phantom morphism if the pull back of any short exact sequence, ending at M along φ is a pure exact sequence:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N & \longrightarrow & U & \longrightarrow & X & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow{\scriptstyle\varphi} & & \\
0 & \longrightarrow & N & \longrightarrow & K & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

[H] I. Herzog, *The phantom cover of a module*, Adv. Math. (2007).

The main result of [FGHT]:

If the exact category $(\mathscr{A}, \mathscr{E})$ has enough injective objects and projective morphisms, then an ideal $\mathscr{I}$ of $\mathscr{A}$ is special precovering if and only if there is a subfunctor $\mathscr{F}$ of Ext^1 with enough injective morphisms such that $\mathscr{I}$ is the ideal of $\mathscr{F}$-phantom morphisms, i.e. $\mathscr{I} = \Phi(\mathscr{F})$.

The crucial step in the proof is a generalization of Salces Lemma for ideal cotorsion pairs.

Resolutions, ...

Higher Homological Algebra

Higher Homological Algebra

Auslander in the late sixties - early seventies proved that there is a bijective correspondence, known as Auslander's correspondence, between

- Morita equivalence classes of artin algebras Λ of finite representation type,
- Morita equivalence classes of Auslander algebras.

An Artin algebra Γ is called an Auslander algebra if it satisfies the following homological conditions

$$\text{gl.dim}\Gamma \leq 2 \leq \text{dom.dim}\Gamma.$$

Iyama developed a higher version of Auslander's correspondence, by introducing the notion of n-cluster tilting subcategories.

In fact, he studied finite-dimensional algebras Γ satisfying the inequalities

$$\text{gl.dim}\Gamma \leq n+1 \leq \text{dom.dim}\Gamma.$$

$$\{\text{Rep-finite algebras}\} \overset{\text{Auslander}}{\longleftrightarrow} \{\Gamma \mid \text{gl.dim}\Gamma \leq 2 \leq \text{dom.dim}\Gamma\}$$

$$??? \overset{\text{Iyama}}{\longleftrightarrow} \{\Gamma \mid \text{gl.dim}\Gamma \leq n+1 \leq \text{dom.dim}\Gamma\}$$

n-cluster tilting subcategories

[I1] O. Iyama, *Higher-dimensional Auslander-Reiten theory on maximal orthogonal subcategories*, Adv. Math. (2007).

[I2] O. Iyama, *Auslander correspondence*, Adv. Math. (2007).

[I3] O. Iyama, *Cluster tilting for higher Auslander algebras*, Adv. Math. (2011).

Later Jasso [J] showed that every n-cluster tilting subcategory of an abelian, resp. exact, category $\mathscr{A}$ has an n-abelian, resp. n-exact, structure.

[J] G. Jasso, *n-Abelian and n-exact categories*, Math. Z, (2016).

Let us recall briefly the notion of n-exact categories.

n-exact categories

Let $(\mathscr{A}, \mathscr{E})$ be an exact category, that is, $\mathscr{A}$ is an additive category and $\mathscr{E}$ is a class of composable pairs (also called kernel-cokernel pairs) of morphisms in $\mathscr{A}$ which is closed under isomorphisms and satisfies axioms of Definition 2.1 of [B]. See also [FS].

The composable pair (i, p) in $\mathscr{E}$ is denoted by

$$A' \rightarrowtail A \twoheadrightarrow A'',$$

$i : A' \to A$ is called an $\mathscr{E}$-admissible monic and $p : A \to A''$ is called an $\mathscr{E}$-admissible epic (conflations, inflations and deflations).

[B] Th. Bühler, *Exact categories*, Expo. Math. (2010).

[FS] L. Frerick and D. Sieg, Exact categories in Functional Analysis, (2010).

Let $\mathscr{C}$ be an additive category. Let $f^0 : X^0 \longrightarrow X^1$ be a morphism in $\mathscr{C}$. An n-cokernel of f^0 is a sequence

$$X^1 \xrightarrow{f^1} X^2 \longrightarrow \cdots \longrightarrow X^n \xrightarrow{f^n} X^{n+1}$$

of morphisms in $\mathscr{C}$ such that for every $X \in \mathscr{C}$ the induced sequence

$$0 \longrightarrow \mathscr{C}(X^{n+1}, X) \xrightarrow{f^n_*} \cdots \xrightarrow{f^1_*} \mathscr{C}(X^1, X) \xrightarrow{f^0_*} \mathscr{C}(X^0, X)$$

of abelian groups is exact. We denote the n-cokernel of f^0 by

$$(f^1, f^2, \cdots, f^n).$$

The notion of n-kernel of a morphism $f^n : X^n \longrightarrow X^{n+1}$ is defined similarly, or rather dually.

A sequence

$$X^0 \xrightarrow{f^0} X^1 \longrightarrow \cdots \longrightarrow X^n \xrightarrow{f^n} X^{n+1}$$

of objects and morphisms in $\mathscr{C}$, is called n-exact if

- $(f^0, f^1, \cdots, f^{n-1})$ is an n-kernel of f^n and
- $(f^1, f^2, \cdots, f^n)$ is an n-cokernel of f^0.

An n-exact sequence like the above one, usually will be denoted by

$$X^0 \overset{f^0}{\rightarrowtail} X^1 \xrightarrow{f^1} X^2 \longrightarrow \cdots \longrightarrow X^n \overset{f^n}{\twoheadrightarrow} X^{n+1}.$$

Consider the complex

$$\mathbf{X}: \ X^0 \xrightarrow{f^0} X^1 \longrightarrow \cdots \longrightarrow X^{n-1} \xrightarrow{f^{n-1}} X^n$$

and morphism $g^0 : X^0 \longrightarrow Y^0$ in $\mathscr{C}$. An n-pushout diagram of $\mathbf{X}$ along g^0 is a morphism

$$\begin{array}{ccccccccc}
\mathbf{X}: & & X^0 & \longrightarrow & X^1 & \longrightarrow & \cdots & \longrightarrow & X^n \\
\downarrow \mathbf{g} & & \downarrow g^0 & & \downarrow g^1 & & & & \downarrow g^n \\
\mathbf{Y}: & & Y^0 & \longrightarrow & Y^1 & \longrightarrow & \cdots & \longrightarrow & Y^n
\end{array}$$

of complexes such that in the mapping cone $\mathbf{C} = \mathrm{cone}(\mathbf{g})$

$$X^0 \xrightarrow{d_{\mathbf{C}}^{-1}} X^1 \oplus Y^0 \xrightarrow{d_{\mathbf{C}}^{0}} \cdots \xrightarrow{d_{\mathbf{C}}^{n-2}} X^n \oplus Y^{n-1} \xrightarrow{d_{\mathbf{C}}^{n-1}} Y^n$$

the sequence $(d_{\mathbf{C}}^0, d_{\mathbf{C}}^1, \cdots, d_{\mathbf{C}}^{n-1})$ is an n-cokernel of $d_{\mathbf{C}}^{-1}$.

A morphism $f : \mathbf{X} \longrightarrow \mathbf{Y}$ of n-exact sequences is called a weak isomorphism if f^k and f^{k+1} are isomorphisms for some $k \in \{0, 1, \cdots, n+1\}$, where we set $n+2 := 0$.

Definition

Let $\mathscr{C}$ be an additive category. An n-exact structure on $\mathscr{C}$ is a class $\mathscr{X}$ of n-exact sequences

$$\eta : \ X^0 \xrightarrow{f^0} X^1 \longrightarrow \cdots \longrightarrow X^n \xrightarrow{f^n} X^{n+1}$$

in $\mathscr{C}$, called $\mathscr{X}$-admissible n-exact sequences, that is closed under weak isomorphisms and satisfies the following axioms.

(E_0) The sequence $0 \rightarrowtail 0 \to \cdots \to 0 \twoheadrightarrow 0$ is an admissible n-exact sequence.

(E_1) The class of admissible monomorphisms is closed under composition.

(E_2) The n-pushout of an admissible n-exact sequence $(d_X^0, d_X^1, ..., d_X^n)$ along morphism $f : X^0 \to Y^0$ exists.

(E_1^{op}) The class of admissible epimorphisms is closed under composition.

(E_2^{op}) The n-pullback of an admissible n-exact sequence $(d_X^0, d_X^1, ..., d_X^n)$ along morphism $g : Y^{n+1} \to X^{n+1}$ exists.

Definition

An n-exact category is a pair $(\mathscr{C}, \mathscr{X})$ where $\mathscr{C}$ is an additive category and $\mathscr{X}$ is an n-exact structure on $\mathscr{C}$.

The most known examples of n-exact categories are n-cluster tilting subcategories of exact categories, introduced by Iyama.

Definition

Let $(\mathscr{A}, \mathscr{E})$ be an exact category. A subcategory $\mathscr{C}$ of $\mathscr{A}$ is called an n-cluster tilting subcategory if it satisfies the following conditions.

(i) For every object $A \in \mathscr{A}$, there exists an admissible monomorphism $A \rightarrowtail C$, which is also a left $\mathscr{C}$-approximation of A.

(ii) For every object $A \in \mathscr{A}$, there exists an admissible epimorphism $C' \twoheadrightarrow A$, which is also a right $\mathscr{C}$-approximation of A.

(iii) There exists equalities $\mathscr{C}^{\perp_n} = \mathscr{C} = {}^{\perp_n}\mathscr{C}$, where

$$\mathscr{C}^{\perp_n} = \{A \in \mathscr{A} : \mathrm{Ext}^i_{\mathscr{E}}(C, A) = 0 \text{ for all } C \in \mathscr{C} \text{ and all } 1 \leq i \leq n-1\},$$

$${}^{\perp_n}\mathscr{C} = \{A \in \mathscr{A} : \mathrm{Ext}^i_{\mathscr{E}}(A, C) = 0 \text{ for all } C \in \mathscr{C} \text{ and all } 1 \leq i \leq n-1\}.$$

Theorem

Let $(\mathscr{A}, \mathscr{E})$ be an exact category and $\mathscr{C}$ be an n-cluster tilting subcategory of $\mathscr{A}$. Set

$$\mathscr{X} = \{\mathbf{C} : C^0 \rightarrowtail C^1 \to \cdots \to C^n \twoheadrightarrow C^{n+1}\}$$

such that

$$\mathbf{C} \text{ is } \mathscr{E}\text{-acyclic and } C^i \in \mathscr{C},\ \forall\ i \in \{0, 1, \ldots, n+1\}.$$

Then $(\mathscr{C}, \mathscr{X})$ is an n-exact category.

Higher Ideal Approximation Theory

Higher Ideal Approximation Theory

We have investigated:

- Higher ideal cotorsion pairs;
- Higher phantom ideals;
- Salce's Lemma;
- Wakamatsu's Lemma.

Throughout,

- $\mathscr{C}$ is an n-cluster tilting subcategory of an exact category $(\mathscr{A}, \mathscr{E})$,
- $\mathscr{X}$ denotes the n-exact structure of $\mathscr{C}$, i.e. $(\mathscr{C}, \mathscr{X})$,
- $\mathscr{F}$ denotes an additive subfunctor of $\mathrm{Ext}^n_{\mathscr{X}}$,
- admissible n-exact sequences in $\mathscr{F}$ are called $\mathscr{F}$-admissible n-exact sequences.

Moreover, we denote $\mathrm{Ext}^n_{\mathscr{X}}$ by Ext^n, for simplicity.

Definition

Let $(\mathscr{C}, \mathscr{X})$ be an n-cluster tilting subcategory of an exact category $(\mathscr{A}, \mathscr{E})$. A class $\mathscr{F}$ of $\mathscr{E}$-acyclic complexes of length n in $\mathscr{C}$ is called an n-proper class if it contains all contractible $\mathscr{E}$-acyclic complexes, is closed under isomorphisms and finite direct sums and is closed under n-pullbacks and n-pushouts along any morphism in $\mathscr{A}$.

Every n-proper class of $\mathscr{E}$-acyclic complexes of length n gives rise to an additive subfunctor $\mathrm{Ext}^n_{\mathscr{F}}$ of $\mathrm{Ext}^n_{\mathscr{X}}$. On the other hand, every additive subfunctor of $\mathrm{Ext}^n_{\mathscr{X}}$ induces an n-proper class of $\mathscr{E}$-acyclic complexes of length n.

[FS] L. Frerick and D. Sieg, Exact categories in Functional Analysis, (2010).

Higher ideal cotorsion pairs

and

Salce's Lemma

Notation.

Let $\mathscr{M}$ be a collection of morphisms in $\mathscr{C}$. We define the right and left orthogonals of $\mathscr{M}$, respectively, as follows

$$\mathscr{M}^{\perp} := \{g : \quad \mathrm{Ext}^n_{\mathscr{X}}(m, g) = 0 \quad for \quad all \quad m \in \mathscr{M}\},$$

$$^{\perp}\mathscr{M} := \{f : \quad \mathrm{Ext}^n_{\mathscr{X}}(f, m) = 0 \quad for \quad all \quad m \in \mathscr{M}\}.$$

Theorem

Let $\mathscr{M}$ be a collection of morphisms in $\mathscr{C}$. Then both $\mathscr{M}^{\perp}$ and $^{\perp}\mathscr{M}$ are ideals of $\mathscr{C}$.

Definition

Let $\mathscr{I}$ and $\mathscr{J}$ be ideals of $\mathscr{C}$. A pair $(\mathscr{I}, \mathscr{J})$ is called an n-orthogonal pair of ideals if for every $f \in \mathscr{I}$ and every $g \in \mathscr{J}$, the pair (f, g) is an n-orthogonal pair of morphisms, that is, the morphism

$$\mathrm{Ext}^n(f, g) : \mathrm{Ext}^n(A, B) \longrightarrow \mathrm{Ext}^n(X, Y)$$

of abelian groups is zero.

Definition

The n-orthogonal pair $(\mathscr{I}, \mathscr{J})$ of ideals in $\mathscr{C}$ is called an n-ideal cotorsion pair if $\mathscr{I} = {}^{\perp}\mathscr{J}$ and $\mathscr{J} = \mathscr{I}^{\perp}$.

Definition

Let $\mathscr{I}$ be an ideal of $\mathscr{C}$ and $A \in \mathscr{C}$ be an arbitrary object. A morphism $i : X^n \to A$ in $\mathscr{I}$ is called a special $\mathscr{I}$-precover of A if it obtained as the rightmost morphism in an n-pushout of an admissible n-exact sequence η along a morphism $j : Y \longrightarrow A' \in \mathscr{I}^{\perp}$. It is depicted by the following diagram

$$\begin{array}{cccccccccc}
\eta : & Y & \longrightarrow & Y^1 & \longrightarrow & \cdots & \longrightarrow & Y^n & \longrightarrow & A \\
& \downarrow{\scriptstyle j} & & \downarrow & & & & \downarrow & & \downarrow \\
\eta' : & A' & \longrightarrow & X^1 & \longrightarrow & \cdots & \longrightarrow & X^n & \xrightarrow{i} & A.
\end{array}$$

The ideal $\mathscr{I}$ is called a special precovering ideal if every object $A \in \mathscr{C}$ has a special $\mathscr{I}$-precover. The notions of special $\mathscr{I}$-preenvelopes and special preenveloping ideals are defined dually.

In the following we present a source of n-ideal cotorsion pairs.

Theorem

Let $\mathscr{I}$ be a special precovering ideal of $\mathscr{C}$. Then the n-orthogonal pair of ideals $(\mathscr{I}, \mathscr{I}^{\perp})$ is an n-ideal cotorsion pair.

Definition

Let $\mathscr{F}$ be a subfunctor of $\mathrm{Ext}^n_{\mathscr{X}}$. A morphism φ in $\mathscr{C}$ is called an n-$\mathscr{F}$-phantom morphism if the n-pullback of every $\mathscr{X}$-admissible n-exact sequence along φ is an $\mathscr{F}$-admissible n-exact sequence.

The collection of all n-$\mathscr{F}$-phantom morphisms by $\Phi(\mathscr{F})$. Note that $\Phi(\mathscr{F})$ forms an ideal of $\mathscr{C}$.

Notation. Let $\mathscr{I}$ be an ideal of $\mathscr{C}$. The collection of all admissible n-exact sequences that are obtained from n-pullbacks of admissible n-exact sequences along morphisms in $\mathscr{I}$ is denoted by $\mathrm{PB}(\mathscr{I})$.

Theorem

Let $\mathscr{I}$ be an ideal of $\mathscr{C}$. Then $\mathrm{PB}(\mathscr{I})$ is an n-proper subclass of $\mathscr{X}$.

Corollary

Let $\mathscr{I}$ be a special precovering ideal of $\mathscr{C}$. Set $\mathscr{F} = \mathrm{PB}(\mathscr{I})$. Then $(\Phi(\mathscr{F}), \mathscr{F}\text{-inj})$ is an n-ideal cotorsion pair.

Recall that a morphism $f : X \longrightarrow A$ in $\mathscr{C}$ is called $\mathscr{F}$-projective if for every object B in $\mathscr{C}$, $\mathscr{F}(f, B) = 0$. In other words, $f : X \to A$ in $\mathscr{C}$ is $\mathscr{F}$-projective if the n-pullback of any $\mathscr{F}$-admissible n-exact sequence along f is contractible. An object A in $\mathscr{C}$ is called $\mathscr{F}$-projective if the identity morphism is an $\mathscr{F}$-projective morphism. The ideal of $\mathscr{F}$-projective morphisms is denoted by $\mathscr{F}$-proj. The notions of $\mathscr{F}$-injective morphisms and $\mathscr{F}$-injective objects are defined dually. The ideal of $\mathscr{F}$-injective morphisms is denoted by $\mathscr{F}$-inj.

Definition

Let $(\mathscr{C}, \mathscr{X})$ be an n-cluster tilting subcategory of $\mathscr{A}$ and

$$\begin{array}{ccccccccccc} X^0 & \longrightarrow & X^1 & \longrightarrow & X^2 & \longrightarrow & \cdots & \longrightarrow & X^n & \longrightarrow & X^{n+1} \\ \downarrow{\scriptstyle f^0} & & \downarrow{\scriptstyle f^1} & & \downarrow{\scriptstyle f^2} & & & & \downarrow{\scriptstyle f^n} & & \downarrow{\scriptstyle f^{n+1}} \\ Y^0 & \longrightarrow & Y^1 & \longrightarrow & Y^2 & \longrightarrow & \cdots & \longrightarrow & Y^n & \longrightarrow & Y^{n+1} \end{array}$$

be a morphism of admissible n-exact sequences in $\mathscr{X}$. Let $\mathscr{I}$ be an ideal of $\mathscr{C}$. We say that $\mathscr{I}$ is

closed under n-extensions by $\mathscr{X}$-injective objects,

if whenever $f^0 \in \mathscr{I}$ and X^{n+1} is an $\mathscr{X}$-injective object, then we can deduce that all the middle morphisms f^i, for $i \in \{1, 2, \ldots, n\}$, are in $\mathscr{I}$. Dually one can define the notion of an ideal closed under n-coextensions by $\mathscr{X}$-projective objects.

It is known for $n = 1$, see [FGHT].

Salce's Lemma

Theorem

Let $(\mathscr{C}, \mathscr{X})$ be an n-cluster tilting subcategory of an exact category $(\mathscr{A}, \mathscr{E})$. Let $(\mathscr{I}, \mathscr{J})$ be an n-ideal cotorsion pair such that $\mathscr{I}$ is closed under n-coextensions by $\mathscr{X}$-projective objects and $\mathscr{J}$ is closed under n-extensions by $\mathscr{X}$-injective objects. If $\mathscr{C}$ has enough $\mathscr{X}$-injective objects, then $\mathscr{I}$ is a special precovering ideal if and only if $\mathscr{J}$ is a special preenveloping ideal.

Definition

An n-ideal cotorsion pair $(\mathscr{I}, \mathscr{J})$ is called complete if every object in $\mathscr{C}$ admits a special $\mathscr{I}$-precover and a special $\mathscr{J}$-preenvelope.

Wakamatsu's Lemma

An ideal version of Wakamatsu's Lemma in an exact category is proved in [FH].

For a version in $(n+2)$-angulated categories see [Jor].

[FH] X. Fu, and I. Herzog, *Powers of the phantom ideal*, Proc. London Math. Soc. (2016).

[Jor] P. Jøergensen, *Torsion classes and t-structures in higher homological algebra*, Int. Math. Res. Not. IMRN (2016).

Recall that an object A of $\mathscr{C}$ is said to be in $\mathscr{I}$ if the identity morphism 1_A is in $\mathscr{I}(A, A)$.

Definition

Let $\mathscr{I}$ be an ideal of $\mathscr{C}$. We say that $\mathscr{I}$ is

left closed under n-extensions by objects of $\mathscr{I}$

if for every morphism of $\mathscr{X}$-admissible n-exact sequences in $\mathscr{C}$ such as

$$\begin{array}{ccccccccccc} X^0 & \longrightarrow & X^1 & \longrightarrow & X^2 & \longrightarrow & \cdots & \longrightarrow & X^n & \longrightarrow & X^{n+1} \\ \downarrow{\scriptstyle f^0} & & \downarrow{\scriptstyle f^1} & & \downarrow{\scriptstyle f^2} & & & & \downarrow{\scriptstyle f^n} & & \downarrow \\ Y^0 & \longrightarrow & Y^1 & \longrightarrow & Y^2 & \longrightarrow & \cdots & \longrightarrow & Y^n & \longrightarrow & X^{n+1} \end{array}$$

X^1 is an object of $\mathscr{I}$ if $f^0 \in \mathscr{I}$ and X^{n+1} is an object of $\mathscr{I}$.

Definition

Let $\mathscr{I}$ be an ideal of $\mathscr{C}$ and A be an object of $\mathscr{C}$. An $\mathscr{I}$-precover $i : C \longrightarrow A$ of A is called an $\mathscr{I}$-cover if every endomorphism $f : C \longrightarrow C$ with the property that $if = i$ is necessarily an automorphism.

Wakamatsu's Lemma

Theorem

Let $(\mathscr{C}, \mathscr{X})$ be an n-cluster tilting subcategory of an exact category $(\mathscr{A}, \mathscr{E})$ with enough $\mathscr{X}$-injective objects. Let $\mathcal{I}$ be an ideal of $\mathscr{C}$ which is left closed under n-extensions by objects in $\mathscr{I}$. Let A be an object of $\mathscr{C}$ and $i : I \longrightarrow A$ be the $\mathscr{I}$-cover of A.

Then for every $X \in \mathscr{I}$, there exists the exact sequence

$$0 \to \mathrm{Ext}^n(X, K_n) \to \mathrm{Ext}^n(X, K_{n-1}) \to \cdots \to \mathrm{Ext}^n(X, K_1) \to 0,$$

of abelian groups, where $K_n \longrightarrow K_{n-1} \longrightarrow \cdots \longrightarrow K_1$ is an n-kernel of $i : I \longrightarrow A$.

Sketch of proof.

Consider n-exact sequence

$$K_n \rightarrowtail K_{n-1} \longrightarrow \cdots \longrightarrow K_1 \longrightarrow I \twoheadrightarrow A$$

in $\mathscr{X}$. Since $\mathscr{C}$ has enough $\mathscr{X}$-injective objects, it is closed under n-cosyzygies and hence we have the following long exact sequence

$$\cdots \longrightarrow \mathscr{C}(X, I) \xrightarrow{i^*} \mathscr{C}(X, A) \longrightarrow$$

$$\longrightarrow \mathrm{Ext}^n(X, K_n) \longrightarrow \cdots \longrightarrow \mathrm{Ext}^n(X, I) \xrightarrow{\widehat{i}} \mathrm{Ext}^n(X, A) \longrightarrow \cdots$$

of abelian groups. To prove the theorem, it is enough to show that i^* is surjective and $\widehat{i}$ is injective.

Since α is an $\mathscr{I}$-cover and $X \in \mathscr{I}$, i^* is surjective. To see $\widehat{i}$ is injective, let

$$\eta : I \rightarrowtail X^1 \longrightarrow X^2 \longrightarrow \cdots \longrightarrow X^n \twoheadrightarrow X$$

be an element of $\mathrm{Ext}^n(X, I)$ that maps to zero in $\mathrm{Ext}^n(X, A)$, that is, the n-pushout of η along i, say η', is a contractible n-exact sequence. We show that η itself should be a contractible n-exact sequence. This follows from a result of [Fe].

[Fe] F. Fedele, *d-Auslander-Reiten sequences in subcategories*, Proc. Edinburgh Math. Soc. (2020).

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