Infinite friezes and triangulations of annuli

joint Baur - Jacobsen - Kulkarni - Todorov

FD Seminar, 25 February 2021

Outline

1. Finite frieze patterns
2. Infinite friezes
3. Representation theoretic aspects

① Frieze Patterns

Coxeter '71

Infinite arrays of numbers where

$$\boxed{\begin{matrix} & b & \\ a & & d \\ & c & \end{matrix} \iff ad-bc=1}$$

$$\begin{array}{l|l}
\cdots\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ \cdots & \text{row } 0\text{'s} \\
\cdots\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ \cdots & \text{row } 1\text{'s} \\
\cdots\ 4\ 1\ 2\ 2\ 2\ 1\ 4\ 1\ 2\ 2\ 2\ 1\ 4\ 1\ \cdots & \\
\cdots\ 3\ 1\ 3\ 3\ 1\ 3\ 3\ 1\ 3\ 3\ 1\ 3\ 3\ 1\ \cdots & \text{width} = 3 \\
\cdots\ 2\ 2\ 1\ 4\ 1\ 2\ 2\ 2\ 1\ 4\ 1\ 2\ 2\ 2\ \cdots & \\
\cdots\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ \cdots & \text{row } 1\text{'s} \\
0\ 0\ 0\ 0\ 0\ \cdots\ 0 & \text{row } 0\text{'s}
\end{array}$$

Coxeter '71

- Finite friezes of width m are periodic with period dividing $m+3$
- Friezes are invariant under glide symmetry

Conway-Coxeter '73

$$\left\{ \begin{array}{c} \text{finite integral} \\ \text{friezes with} \\ \text{width } n \end{array} \right\} \xleftrightarrow{1:1} \left\{ \begin{array}{c} \text{triangulations} \\ \text{of convex} \\ (n+3)\text{-gon} \end{array} \right\}$$

$$\cdots\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ \cdots$$

$$\cdots\ 4\ \boxed{1\ 2\ 2\ 2\ 1\ 4}\ 1\ 2\ 2\ 2\ 1\ 4\ 1\ \cdots$$

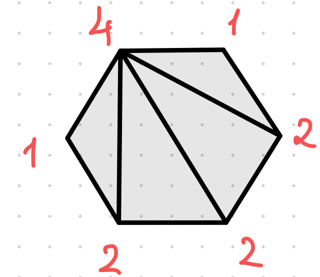
$$\cdots\ 3\ 1\ 3\ 3\ 1\ 3\ 3\ 1\ 3\ 3\ 1\ 3\ 3\ 1\ \cdots$$

$$\cdots\ 2\ 2\ 1\ 4\ 1\ 2\ 2\ 2\ 1\ 4\ 1\ 2\ 2\ 2\ \cdots$$

$$\cdots\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ \cdots$$

quiddity row

quiddity segm



2 Infinite friezes

Baur-Parsons-Tschabold '16

Periodic integral friezes

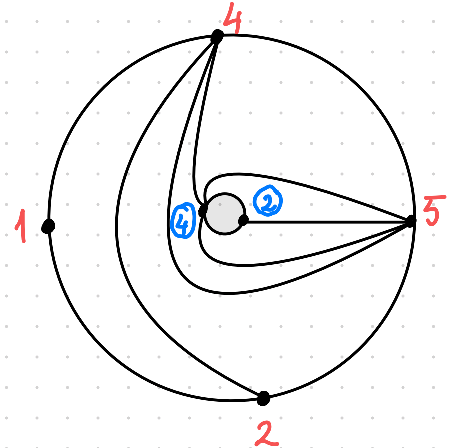
row of 0's & 1's ; infinite width

$a \, {}^{b}_{c} \, d \iff ad-bc=1$

...	0	0	0	0	0	0	0	...
...	1	1	1	1	1	1		...
...	2	$\boxed{5 \quad 4 \quad 1 \quad 2}$				5	4	...
...	9	19	3	1	9	19		...
...	4	34	14	2	4	34	14	...
...	15	25	9	7	15	25		...
...	26	11	16	31	26	11	16	...
...	19	7	55	115	19	7		...
	⋮	⋮	⋮	⋮	⋮	⋮		

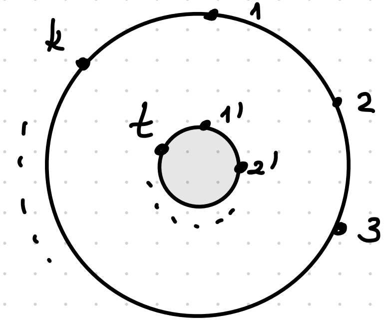
quiddity sequence

[BPT] Infinite periodic friezes come from triangulations of annuli



0	0	0
1	1	
4	2	4
7	7	
24	24	
	⋮	

$C_{k,t}$ annulus, $\mathcal{T}$ triangulation

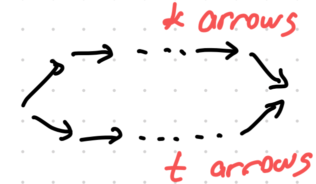


$\rightsquigarrow$ $(\mathcal{F}_k, \mathcal{F}_t)$ pair of infinite friezes associated with $(C_{k,t}; \mathcal{T})$

Question How are these two friezes related?

Definition Skeletal frieze: no 1's in its quiddity seqn $\boxed{q^s}$

Skeletal triangulations: no arcs connecting marked points on the same boundary component $\boxed{\mathcal{T}^s}$

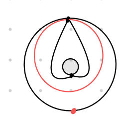
$\boxed{\text{Note}}$ $\mathcal{T} = \mathcal{T}^s$ iff $Q_{\mathcal{T}}$ is a non-oriented cycle 

Proposition $\mathcal{T}$ triangulation of $C_{k,t}$ annulus $\rightsquigarrow$ $\mathcal{T}^s$ its skeletal triangulation

$(\mathcal{F}_k, \mathcal{F}_t)$ pair of friezes assoc. $(C_{k,t} ; \mathcal{T})$

$(\mathcal{F}_k^s, \mathcal{F}_t^s)$ ———— '' ———— $(C^s_{k^s, t^s} ; \mathcal{T}^s)$

$$\begin{tikzcd}
\mathcal{T} \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & (\mathcal{F}_k, \mathcal{F}_t) \arrow[d, squiggly] \\
\mathcal{T}^s \arrow[r] & (\mathcal{F}_k^s, \mathcal{F}_t^s)
\end{tikzcd}$$



Theorem [BGJKT]

$$\left\{\begin{array}{c}\text{skeletal quiddity}\\ \text{sequence}\\ q=(a_1\ a_2\ \dots\ a_k)\\ n = k + \underbrace{\sum_{i=1}^{k}(a_i-2)}_{t}\end{array}\right\} \overset{1:1}{\longleftrightarrow} \left\{\begin{array}{c}\text{skeletal}\\ \text{triangulations}\\ \text{of } C_{k,t}\end{array}\right\}$$

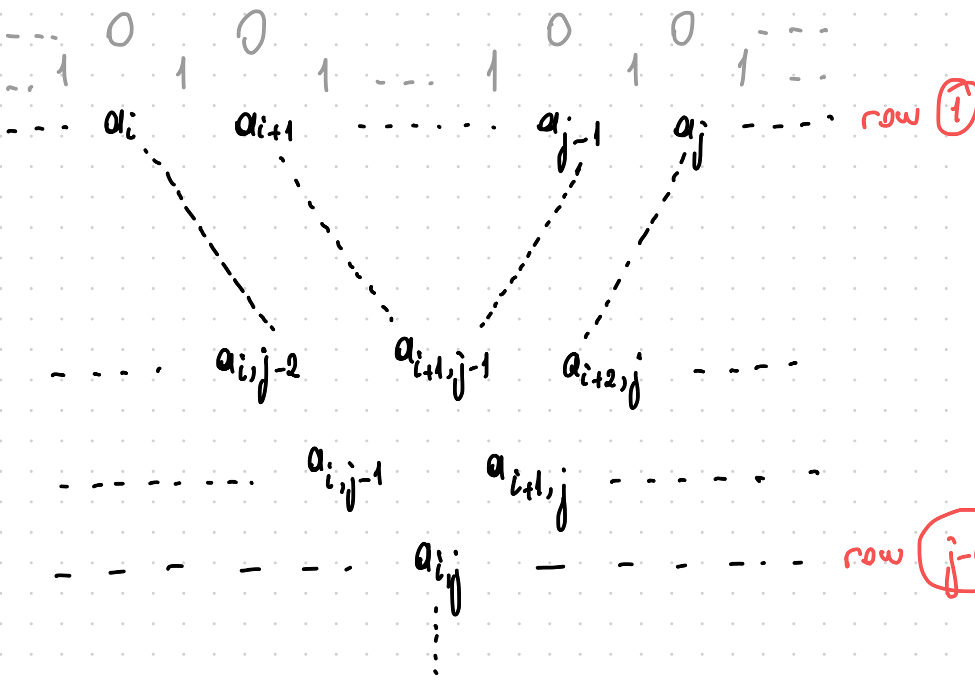
Theorem [BGJKT] Let $\mathcal{F}_k$ be an infinite frieze. Then

(i) $\mathcal{F}_k^s$ uniquely determines $\mathcal{F}_t^s$ such that $(\mathcal{F}_k^s, \mathcal{F}_t^s)$ pair

(ii) $\mathcal{F}_k$ gives rise to an infinide of infinite friezes $\mathcal{F}_t$ such that $(\mathcal{F}_k, \mathcal{F}_t)$ pair.

Growth coefficient

$\mathcal{F}$ infinite frieze of period n,
i.e. $q = (a_1\ a_2 \dots a_n)$



The growth coefficient of $\mathcal{F}$ is defined by

$$\boxed{s_{\mathcal{F}} := a_{i,n+i-1} - a_{i-1,n+i-2}}$$

→ invariant for $\mathcal{F}$ [BPT]

Theorem BGJKT

$$a_{i,n} = \sum_{\substack{I \subseteq \{i,\dots,j\} \\ \text{pair\_excluding}}} (-1)^{\ell_I} \prod_{k \in I} a_k$$

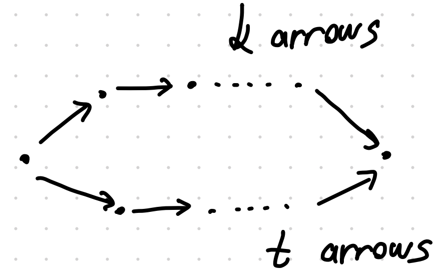
ℓ_I : # pairs excluded from $\{i,\dots,j\}$

$$S_F = \left(\sum_{\substack{I \subseteq \{1,\dots,n\} \\ \text{cyclical pair-excluding}}} (-1)^{\ell_I} \prod_{k \in I} a_k \right) + \delta_n$$

$$\delta_n = \begin{cases} 0 & n \text{ odd} \\ 1 & 4 \mid n \\ -1 & \text{otherwise} \end{cases}$$

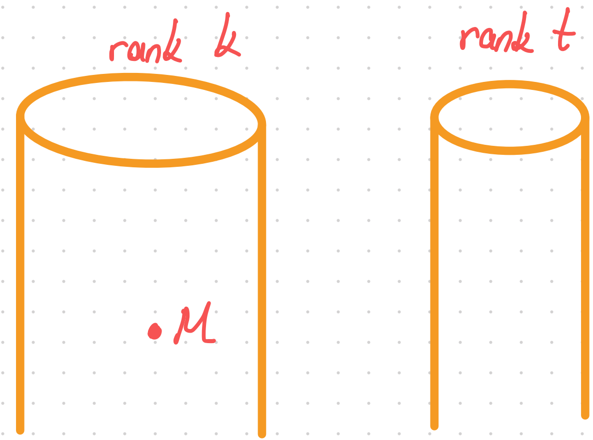
③ Module theoretic interpretation

$C_{k,t}$ annulus , T^s triangulation $\rightsquigarrow Q^s$



$\rightsquigarrow \Lambda = kQ^s$ cluster-tilted algebra of type $\tilde{A}_{k,t}$

$\rightsquigarrow$ The AR-quiver of Λ contains two non-homogeneous tubes of rank k and rank t



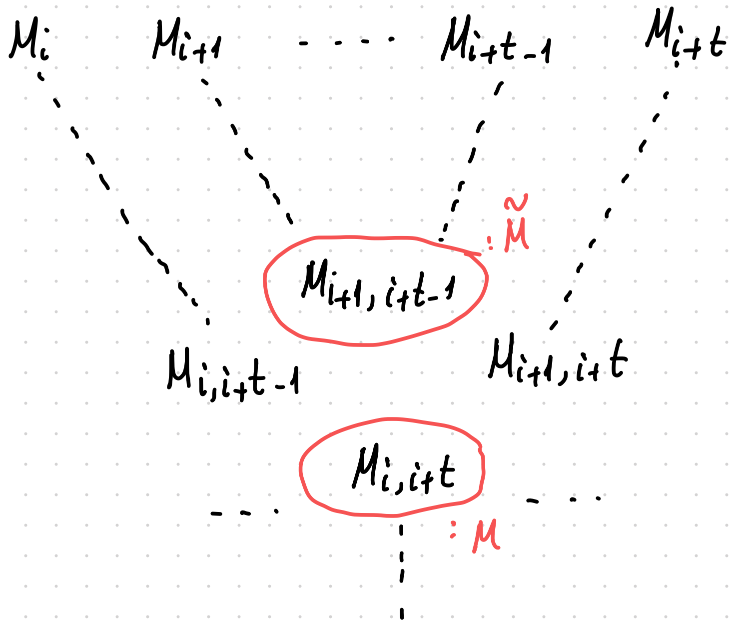
$\rightsquigarrow$ M indecomposable in these tubes

Consider a specialised CC-map

$$s(M) = \sum_{\substack{\underline{e}\,:\,\dim N \\ N \leq M}} \chi\left(Gr_{\underline{e}}\, M\right)$$

M rigid $\Longrightarrow$ $s(M) = \#$ submodules of M

M non-rigid $\Longrightarrow$ $s(M) = \#$ perfect matchings of the snake graph assoc. with M [G-Schroll '18]



$$s(M_i) = a_i$$

$$s(M_{i,i+t}) = a_{i;i+t}$$

for $t \geq 1$

Whenever $0 \to \tau M \to B \to M \to 0$ is an AR-sequence in mod Λ, then $s(\tau M)s(M) - s(B) = 1$

diamond rule

$$\boxed{\begin{aligned} M &= M_{i,i+t} \\ \tilde{M} &= M_{i+1,i+t-1} \end{aligned}}$$

$$N := M_i \oplus M_{i+1} \oplus \cdots \oplus M_{i+t}$$

$$N_j := N / (M_j \oplus M_{j+1})$$

$$N_{j_1,j_2} := N / (M_{j_1} \oplus M_{j_1+1} \oplus M_{j_2} \oplus M_{j_2+1})$$

$$\vdots$$

$$N_{j_1,j_2,\ldots,j_k} := N / (M_{j_1} \oplus M_{j_1+1} \oplus \cdots \oplus M_{j_k} \oplus M_{j_k+1})$$

Theorem [BGJKT]

$$s(M) - s(\tilde{M}) = s(N) - \sum_{j=i}^{i+t} s(N_j) + \sum_{\substack{j_1,j_2=i \\ j_1 < j_2 - 1}}^{i+t} s(N_{j_1,j_2}) + \cdots + R$$

$$R = \begin{cases} (-1)^{t/2} \cdot 2 & t \text{ even} \\ 0 & t \text{ odd} \end{cases}$$