On higher torsion classes

Hipolito Treffinger

Universität Bonn

February 18, 2021

- ***On higher torsion classes***
  jt. J. Asadollahi, P. Jørgenesen and S. Schroll.
  `https://arxiv.org/abs/2101.01402`

- ***On higher torsion classes***
  jt. J. Asadollahi, P. Jørgenesen and S. Schroll.
  `https://arxiv.org/abs/2101.01402`
- ***TBD***
  jt. J. August, J. Haugland, K. Jacobsen, S. Kvamme and Y. Palu.
  In preparation.

Plan of the talk

Overview

- Cluster algebras [FZ]

Overview

- Cluster algebras [FZ]
- Cluster categories [CCS][BMRRT][A]

Overview

- Cluster algebras [FZ]
- Cluster categories [CCS][BMRRT][A]
  n-cluster tilting subcategories

Overview

- Cluster algebras [FZ]
- Cluster categories [CCS][BMRRT][A]
  n-cluster tilting subcategories
- Higher Auslander correspondence [I]

Overview

- Cluster algebras [FZ]
- Cluster categories [CCS][BMRRT][A]

  n-cluster tilting subcategories
- Higher Auslander correspondence [I]

  Higher homological algebra

Overview

- Cluster algebras [FZ]
- Cluster categories [CCS][BMRRT][A]
  n-cluster tilting subcategories
- Higher Auslander correspondence [I]
  Higher homological algebra
- n-angulated categories [GKO]

Overview

- Cluster algebras [FZ]
- Cluster categories [CCS][BMRRT][A]
  n-cluster tilting subcategories
- Higher Auslander correspondence [I]
  Higher homological algebra
- n-angulated categories [GKO]
- n-abelian categories [Ja]

Overview

- Cluster algebras [FZ]
- Cluster categories [CCS][BMRRT][A]
  n-cluster tilting subcategories
- Higher Auslander correspondence [I]
  Higher homological algebra
- n-angulated categories [GKO]
- n-abelian categories [Ja]
- n-torsion classes and n-t-structures [Jø]

Overview

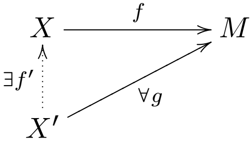
- Cluster algebras [FZ]
- Cluster categories [CCS][BMRRT][A]
  n-cluster tilting subcategories
- Higher Auslander correspondence [I]
  Higher homological algebra
- n-angulated categories [GKO]
- n-abelian categories [Ja]
- n-torsion classes and n-t-structures [Jø]

Today: Interplay between higher and classical homological algebra

- $\mathcal{A}$ is a small length abelian category

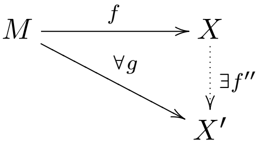
- $\mathcal{A}$ is a small length abelian category
  - A full subcategory $\mathcal{X}$ of $\mathcal{A}$ is *contravariantly finite* if for all $M \in \mathcal{A}$ there exists a *right $\mathcal{X}$-approximation*, that is a map $f : X \to M$ such that $\mathrm{Hom}_{\mathcal{A}}(X', f) : \mathrm{Hom}_{\mathcal{A}}(X', X) \twoheadrightarrow \mathrm{Hom}_{\mathcal{A}}(X', M)$ for all $X' \in \mathcal{X}$.

- $\mathcal{A}$ is a small length abelian category
    - A full subcategory $\mathcal{X}$ of $\mathcal{A}$ is *contravariantly finite* if for all $M \in \mathcal{A}$ there exists a *right $\mathcal{X}$-approximation*, that is a map $f : X \to M$ such that $\mathrm{Hom}_{\mathcal{A}}(X', f) : \mathrm{Hom}_{\mathcal{A}}(X', X) \twoheadrightarrow \mathrm{Hom}_{\mathcal{A}}(X', M)$ for all $X' \in \mathcal{X}$.



- $\mathcal{A}$ is a small length abelian category
  - A full subcategory $\mathcal{X}$ of $\mathcal{A}$ is *contravariantly finite* if for all $M \in \mathcal{A}$ there exists a *right $\mathcal{X}$-approximation*, that is a map $f : X \to M$ such that $\mathrm{Hom}_{\mathcal{A}}(X', f) : \mathrm{Hom}_{\mathcal{A}}(X', X) \twoheadrightarrow \mathrm{Hom}_{\mathcal{A}}(X', M)$ for all $X' \in \mathcal{X}$.
  - A full subcategory $\mathcal{X}$ of $\mathcal{A}$ is *covariantly finite* if for all $M \in \mathcal{A}$ there exists a exists a *left $\mathcal{X}$-approximation*, that is a map $f : M \to X$ such that $\mathrm{Hom}_{\mathcal{A}}(f, X') : \mathrm{Hom}_{\mathcal{A}}(M, X') \hookrightarrow \mathrm{Hom}_{\mathcal{A}}(X, X')$ for all $X' \in \mathcal{X}$.

- $\mathcal{A}$ is a small length abelian category
  - A full subcategory $\mathcal{X}$ of $\mathcal{A}$ is *contravariantly finite* if for all $M \in \mathcal{A}$ there exists a *right $\mathcal{X}$-approximation*, that is a map $f : X \to M$ such that $\mathrm{Hom}_{\mathcal{A}}(X', f) : \mathrm{Hom}_{\mathcal{A}}(X', X) \twoheadrightarrow \mathrm{Hom}_{\mathcal{A}}(X', M)$ for all $X' \in \mathcal{X}$.
  - A full subcategory $\mathcal{X}$ of $\mathcal{A}$ is *covariantly finite* if for all $M \in \mathcal{A}$ there exists a exists a *left $\mathcal{X}$-approximation*, that is a map $f : M \to X$ such that $\mathrm{Hom}_{\mathcal{A}}(f, X') : \mathrm{Hom}_{\mathcal{A}}(M, X') \hookrightarrow \mathrm{Hom}_{\mathcal{A}}(X, X')$ for all $X' \in \mathcal{X}$.



- $\mathcal{A}$ is a small length abelian category
  - A full subcategory $\mathcal{X}$ of $\mathcal{A}$ is *contravariantly finite* if for all $M \in \mathcal{A}$ there exists a *right $\mathcal{X}$-approximation*, that is a map $f : X \to M$ such that $\operatorname{Hom}_{\mathcal{A}}(X', f) : \operatorname{Hom}_{\mathcal{A}}(X', X) \twoheadrightarrow \operatorname{Hom}_{\mathcal{A}}(X', M)$ for all $X' \in \mathcal{X}$.
  - A full subcategory $\mathcal{X}$ of $\mathcal{A}$ is *covariantly finite* if for all $M \in \mathcal{A}$ there exists a exists a *left $\mathcal{X}$-approximation*, that is a map $f : M \to X$ such that $\operatorname{Hom}_{\mathcal{A}}(f, X') : \operatorname{Hom}_{\mathcal{A}}(M, X') \hookrightarrow \operatorname{Hom}_{\mathcal{A}}(X, X')$ for all $X' \in \mathcal{X}$.
  - A full subcategory $\mathcal{X}$ of $\mathcal{A}$ is *functorially finite* if $\mathcal{X}$ is covariantly and contravariantly finite.

- Let $\mathcal{X}$ be a full subcategory of $\mathcal{A}$.

- Let $\mathcal{X}$ be a full subcategory of $\mathcal{A}$.
  - $\mathrm{Fac}(\mathcal{X}) = \{Y \in \mathcal{A} : \exists$ exact sequence $X \to Y \to 0,$ for some $X \in \mathcal{X}\}$

- Let $\mathcal{X}$ be a full subcategory of $\mathcal{A}$.
  - $\mathrm{Fac}(\mathcal{X}) = \{Y \in \mathcal{A} : \exists$ exact sequence $X \to Y \to 0,$ for some $X \in \mathcal{X}\}$
    We say that $\mathcal{X}$ *generates* $\mathcal{A}$ if $\mathrm{Fac}(\mathcal{X}) = \mathcal{A}$.

- Let $\mathcal{X}$ be a full subcategory of $\mathcal{A}$.
  - $\mathrm{Fac}(\mathcal{X}) = \{Y \in \mathcal{A} : \exists$ exact sequence $X \to Y \to 0,$ for some $X \in \mathcal{X}\}$
    We say that $\mathcal{X}$ *generates* $\mathcal{A}$ if $\mathrm{Fac}(\mathcal{X}) = \mathcal{A}$.
  - $\mathrm{Sub}(\mathcal{X}) = \{Y \in \mathcal{A} : \exists$ exact sequence $0 \to Y \to X,$ for some $X \in \mathcal{X}\}$

- Let $\mathcal{X}$ be a full subcategory of $\mathcal{A}$.
  - $\mathrm{Fac}(\mathcal{X}) = \{Y \in \mathcal{A} : \exists$ exact sequence $X \to Y \to 0,$ for some $X \in \mathcal{X}\}$
    We say that $\mathcal{X}$ *generates* $\mathcal{A}$ if $\mathrm{Fac}(\mathcal{X}) = \mathcal{A}$.
  - $\mathrm{Sub}(\mathcal{X}) = \{Y \in \mathcal{A} : \exists$ exact sequence $0 \to Y \to X,$ for some $X \in \mathcal{X}\}$
    We say that $\mathcal{X}$ *cogenerates* $\mathcal{A}$ if $\mathrm{Sub}(\mathcal{X}) = \mathcal{A}$.

- Let $\mathcal{X}$ be a full subcategory of $\mathcal{A}$.
  - $\mathrm{Fac}(\mathcal{X}) = \{Y \in \mathcal{A} : \exists$ exact sequence $X \to Y \to 0,$ for some $X \in \mathcal{X}\}$
    We say that $\mathcal{X}$ *generates* $\mathcal{A}$ if $\mathrm{Fac}(\mathcal{X}) = \mathcal{A}$.
  - $\mathrm{Sub}(\mathcal{X}) = \{Y \in \mathcal{A} : \exists$ exact sequence $0 \to Y \to X,$ for some $X \in \mathcal{X}\}$
    We say that $\mathcal{X}$ *cogenerates* $\mathcal{A}$ if $\mathrm{Sub}(\mathcal{X}) = \mathcal{A}$.

Remark

In classical homological algebra results are true up to isomorphism.
In higher homological algebra the results are true up to homotopy.

Definition

Let $\mathcal{A}$ be an abelian category. A functorially finite generating-cogenerating subcategory $\mathcal{M}$ of $\mathcal{A}$ is n-cluster tilting if

$$\begin{aligned}\mathcal{M} &= \{X \in \mathcal{A} : \operatorname{Ext}^i_{\mathcal{A}}(X, M) = 0 \text{ for all } M \in \mathcal{M} \text{ and all } 1 \leq i \leq n-1\} \\ &= \{Y \in \mathcal{A} : \operatorname{Ext}^i_{\mathcal{A}}(M, Y) = 0 \text{ for all } M \in \mathcal{M} \text{ and all } 1 \leq i \leq n-1\}.\end{aligned}$$

n-cluster tilting subcategories and n-abelian categories

n-cluster tilting subcategories and n-abelian categories

- The notion of n-abelian categories was introduced by Jasso.

n-cluster tilting subcategories and n-abelian categories

- The notion of n-abelian categories was introduced by Jasso.

Theorem (Jasso)

Let $\mathcal{A}$ be an abelian category having an n-cluster tilting subcategory $\mathcal{M}$. Then $\mathcal{M}$ is an n-abelian category.

n-cluster tilting subcategories and n-abelian categories

- The notion of n-abelian categories was introduced by Jasso.

Theorem (Jasso)

Let $\mathcal{A}$ be an abelian category having an n-cluster tilting subcategory $\mathcal{M}$. Then $\mathcal{M}$ is an n-abelian category.

Theorem (Kvamme, Ebrahimi–Nasr-Isfahani)

Let $\mathcal{M}$ be a small n-abelian category. Then there exists an abelian category $\mathcal{A}$ and a fully faithful functor $F : \mathcal{M} \to \mathcal{A}$ such that the essential image $F(\mathcal{M})$ of F is an n-cluster tilting subcategory of $\mathcal{A}$.

n-cluster tilting subcategories and n-abelian categories

- The notion of n-abelian categories was introduced by Jasso.

Theorem (Jasso)

Let $\mathcal{A}$ be an abelian category having an n-cluster tilting subcategory $\mathcal{M}$. Then $\mathcal{M}$ is an n-abelian category.

Theorem (Kvamme, Ebrahimi–Nasr-Isfahani)

Let $\mathcal{M}$ be a small n-abelian category. Then there exists an abelian category $\mathcal{A}$ and a fully faithful functor $F : \mathcal{M} \to \mathcal{A}$ such that the essential image $F(\mathcal{M})$ of F is an n-cluster tilting subcategory of $\mathcal{A}$.

From now on we assume that $\mathcal{M}$ is an n-cluster tilting subcategory of $\mathcal{A}$.

Torsion pairs

Torsion pairs

Definition (Dickson)

Let $\mathcal{A}$ be an abelian category. Then the pair $(\mathcal{T}, \mathcal{F})$ of full subcategories of $\mathcal{A}$ is a torsion pair if the following conditions are satisfied:

Torsion pairs

Definition (Dickson)

Let $\mathcal{A}$ be an abelian category. Then the pair $(\mathcal{T}, \mathcal{F})$ of full subcategories of $\mathcal{A}$ is a torsion pair if the following conditions are satisfied:

- For every module M in $\mathcal{A}$ there exists a short exact sequence

$$0 \to tM \xrightarrow{\iota_M} M \xrightarrow{\pi_M} fM \to 0$$

  where $tM \in \mathcal{T}$ and $fM \in \mathcal{F}$.

Torsion pairs

Definition (Dickson)

Let $\mathcal{A}$ be an abelian category. Then the pair $(\mathcal{T}, \mathcal{F})$ of full subcategories of $\mathcal{A}$ is a torsion pair if the following conditions are satisfied:

- For every module M in $\mathcal{A}$ there exists a short exact sequence

$$0 \to tM \xrightarrow{\iota_M} M \xrightarrow{\pi_M} fM \to 0$$

  where $tM \in \mathcal{T}$ and $fM \in \mathcal{F}$.
- $\mathrm{Hom}_{\mathcal{A}}(X, Y) = 0$ for all $X \in \mathcal{T}$ and $Y \in \mathcal{F}$.

Torsion pairs

Definition (Dickson)

Let $\mathcal{A}$ be an abelian category. Then the pair $(\mathcal{T}, \mathcal{F})$ of full subcategories of $\mathcal{A}$ is a torsion pair if the following conditions are satisfied:

- For every module M in $\mathcal{A}$ there exists a short exact sequence

$$0 \to tM \xrightarrow{\iota_M} M \xrightarrow{\pi_M} fM \to 0$$

  where $tM \in \mathcal{T}$ and $fM \in \mathcal{F}$.
- $\mathrm{Hom}_{\mathcal{A}}(X, Y) = 0$ for all $X \in \mathcal{T}$ and $Y \in \mathcal{F}$.

If $(\mathcal{T}, \mathcal{F})$ is a torsion pair we say that $\mathcal{T}$ is a torsion class and that $\mathcal{F}$ is a torsion free class.

Torsion classes revisited

Definition

Let $\mathcal{A}$ be an abelian category. A full subcategory $\mathcal{T}$ of $\mathcal{A}$ is an torsion class in $\mathcal{A}$ if for every $M \in \mathcal{A}$ there exists a short exact sequence

$$0 \to tM \to M \to fM \to 0,$$

where tM is an object of $\mathcal{T}$ and the sequence

$$0 \to \operatorname{Hom}_{\mathcal{A}}(T, fM) \to 0$$

is exact, for all objects T in $\mathcal{T}$.

Torsion classes revisited

Definition

Let $\mathcal{A}$ be an abelian category. A full subcategory $\mathcal{T}$ of $\mathcal{A}$ is an torsion class in $\mathcal{A}$ if for every $M \in \mathcal{A}$ there exists a short exact sequence

$$0 \to tM \to M \to fM \to 0,$$

where tM is an object of $\mathcal{T}$ and the sequence

$$0 \to \operatorname{Hom}_{\mathcal{A}}(T, fM) \to 0$$

is exact, for all objects T in $\mathcal{T}$. In this case tM is called the torsion subobject of M with respect to $\mathcal{T}$.

n-torsion classes

Definition (Jørgensen)

Let $\mathcal{M}$ be an n-abelian category. A full subcategory $\mathcal{U}$ of $\mathcal{M}$ is an n-torsion class if for every $M \in \mathcal{M}$ there exists an n-exact sequence

$$0 \to U^M \to M \to V^1 \xrightarrow{v^1} \cdots \xrightarrow{v^{n-1}} V^n \to 0,$$

where U^M is an object of $\mathcal{U}$ and the sequence

$$0 \to \operatorname{Hom}_{\mathcal{M}}(U, V^1) \to \operatorname{Hom}_{\mathcal{M}}(U, V^2) \to \cdots \to \operatorname{Hom}_{\mathcal{M}}(U, V^n) \to 0$$

is exact, for all objects U in $\mathcal{U}$.

n-torsion classes

Definition (Jørgensen)

Let $\mathcal{M}$ be an n-abelian category. A full subcategory $\mathcal{U}$ of $\mathcal{M}$ is an n-torsion class if for every $M \in \mathcal{M}$ there exists an n-exact sequence

$$0 \to U^M \to M \to V^1 \xrightarrow{v^1} \cdots \xrightarrow{v^{n-1}} V^n \to 0,$$

where U^M is an object of $\mathcal{U}$ and the sequence

$$0 \to \mathrm{Hom}_{\mathcal{M}}(U, V^1) \to \mathrm{Hom}_{\mathcal{M}}(U, V^2) \to \cdots \to \mathrm{Hom}_{\mathcal{M}}(U, V^n) \to 0$$

is exact, for all objects U in $\mathcal{U}$. In this case U^M is called the n-torsion subobject of M with respect to $\mathcal{U}$.

n-torsion classes

Definition (Jørgensen)

Let $\mathcal{M}$ be an n-abelian category. A full subcategory $\mathcal{U}$ of $\mathcal{M}$ is an n-torsion class if for every $M \in \mathcal{M}$ there exists an n-exact sequence

$$0 \to U^M \to M \to V^1 \xrightarrow{v^1} \cdots \xrightarrow{v^{n-1}} V^n \to 0,$$

where U^M is an object of $\mathcal{U}$ and the sequence

$$0 \to \operatorname{Hom}_{\mathcal{M}}(U, V^1) \to \operatorname{Hom}_{\mathcal{M}}(U, V^2) \to \cdots \to \operatorname{Hom}_{\mathcal{M}}(U, V^n) \to 0$$

is exact, for all objects U in $\mathcal{U}$. In this case U^M is called the n-torsion subobject of M with respect to $\mathcal{U}$.

With this definition there is no n-torsion free class associated to an n-torsion class.

Introduction

- Overview
- Abelian and n-abelian categories
- Torsion and n-torsion classes

From n-torsion classes to torsion classes

- Classical torsion classes in disguise
- The poset of n-torsion classes
- Harder-Narasimhan filtrations in n-abelian categories

Functorially finite n-torsion classes

- Generating functorially finite n-torsion classes
- From τ-tilting theory to τ_n-tilting theory

Classical torsion classes in disguise

Question: What is the relation between the torsion classes of $\mathcal{A}$ and the n-torsion classes of $\mathcal{M}$?

Classical torsion classes in disguise

Question: What is the relation between the torsion classes of $\mathcal{A}$ and the n-torsion classes of $\mathcal{M}$?

- $T(\mathcal{X})$ is the minimal torsion class of $\mathcal{A}$ containing $\mathcal{X}$.

Classical torsion classes in disguise

Question: What is the relation between the torsion classes of $\mathcal{A}$ and the n-torsion classes of $\mathcal{M}$?

- $T(\mathcal{X})$ is the minimal torsion class of $\mathcal{A}$ containing $\mathcal{X}$.

Theorem (Asadollahi-Jørgensen-Schroll-T.)

A torsion class $\mathcal{T}$ in $\mathcal{A}$ is of the form $T(\mathcal{U})$ for some n-torsion class $\mathcal{U}$ in $\mathcal{M}$ if and only if the following hold:

Classical torsion classes in disguise

Question: What is the relation between the torsion classes of $\mathcal{A}$ and the n-torsion classes of $\mathcal{M}$?

- $T(\mathcal{X})$ is the minimal torsion class of $\mathcal{A}$ containing $\mathcal{X}$.

Theorem (Asadollahi-Jørgensen-Schroll-T.)

A torsion class $\mathcal{T}$ in $\mathcal{A}$ is of the form $T(\mathcal{U})$ for some n-torsion class $\mathcal{U}$ in $\mathcal{M}$ if and only if the following hold:

1. $tM \in \mathcal{U}$ for all $M \in \mathcal{M}$ where t is the torsion functor associated to the torsion class $\mathcal{T}$;

Classical torsion classes in disguise

Question: What is the relation between the torsion classes of $\mathcal{A}$ and the n-torsion classes of $\mathcal{M}$?

- $T(\mathcal{X})$ is the minimal torsion class of $\mathcal{A}$ containing $\mathcal{X}$.

Theorem (Asadollahi-Jørgensen-Schroll-T.)

A torsion class $\mathcal{T}$ in $\mathcal{A}$ is of the form $T(\mathcal{U})$ for some n-torsion class $\mathcal{U}$ in $\mathcal{M}$ if and only if the following hold:

1. $tM \in \mathcal{U}$ for all $M \in \mathcal{M}$ where t is the torsion functor associated to the torsion class $\mathcal{T}$;
2. $\mathcal{T}$ is the minimal torsion class in $\mathcal{A}$ containing $\{tM : M \in \mathcal{M}\}$;

Classical torsion classes in disguise

Question: What is the relation between the torsion classes of $\mathcal{A}$ and the n-torsion classes of $\mathcal{M}$?

- $T(\mathcal{X})$ is the minimal torsion class of $\mathcal{A}$ containing $\mathcal{X}$.

Theorem (Asadollahi-Jørgensen-Schroll-T.)

A torsion class $\mathcal{T}$ in $\mathcal{A}$ is of the form $T(\mathcal{U})$ for some n-torsion class $\mathcal{U}$ in $\mathcal{M}$ if and only if the following hold:

1. $tM \in \mathcal{U}$ for all $M \in \mathcal{M}$ where t is the torsion functor associated to the torsion class $\mathcal{T}$;
2. $\mathcal{T}$ is the minimal torsion class in $\mathcal{A}$ containing $\{tM : M \in \mathcal{M}\}$;
3. $\operatorname{Ext}^{n-1}_{\mathcal{A}}(X, Y) = 0$, for all $X \in \{tM : M \in \mathcal{M}\}$ and $Y \in \{fM' : M' \in \mathcal{M}\}$.

Classical torsion classes in disguise

Question: What is the relation between the torsion classes of $\mathcal{A}$ and the n-torsion classes of $\mathcal{M}$?

- $T(\mathcal{X})$ is the minimal torsion class of $\mathcal{A}$ containing $\mathcal{X}$.

Theorem (Asadollahi-Jørgensen-Schroll-T.)

A torsion class $\mathcal{T}$ in $\mathcal{A}$ is of the form $T(\mathcal{U})$ for some n-torsion class $\mathcal{U}$ in $\mathcal{M}$ if and only if the following hold:

1. $tM \in \mathcal{U}$ for all $M \in \mathcal{M}$ where t is the torsion functor associated to the torsion class $\mathcal{T}$;
2. $\mathcal{T}$ is the minimal torsion class in $\mathcal{A}$ containing $\{tM : M \in \mathcal{M}\}$;
3. $\operatorname{Ext}^{n-1}_{\mathcal{A}}(X, Y) = 0$, for all $X \in \{tM : M \in \mathcal{M}\}$ and $Y \in \{fM' : M' \in \mathcal{M}\}$.

In this case $\mathcal{U} = \mathcal{T} \cap \mathcal{M} = \{tM : M \in \mathcal{M}\}$.

The poset of n-torsion classes

- **tors($\mathcal{A}$)** $:= \{\mathcal{T} : \mathcal{T}$ is a torsion class of $\mathcal{A}\}$

The poset of n-torsion classes

- **tors($\mathcal{A}$)** $:= \{\mathcal{T} : \mathcal{T}$ is a torsion class of $\mathcal{A}\}$
- n**-tors($\mathcal{M}$)** $:= \{\mathcal{U} : \mathcal{U}$ is an n-torsion class of $\mathcal{M}\}$

The poset of n-torsion classes

- **tors($\mathcal{A}$)** $:= \{\mathcal{T} : \mathcal{T}$ is a torsion class of $\mathcal{A}\}$
- n**-tors($\mathcal{M}$)** $:= \{\mathcal{U} : \mathcal{U}$ is an n-torsion class of $\mathcal{M}\}$

Corollary (Asadollahi-Jørgensen-Schroll-T.)

Let $\mathcal{M}$ be the an n-cluster tilting subcategory of $\mathcal{A}$. Then the map $T(-) : n\text{-tors}(\mathcal{M}) \to \text{tors}(\mathcal{A})$ is a poset monomorphism.

Theorem (Demonet-Iyama-Reading-Reiten-Thomas)

Let $\mathcal{A} = \operatorname{mod} A$ be the module category of an artinian algebra A. Then $\operatorname{tors}(\mathcal{A})$ is a complete lattice.

Theorem (Demonet-Iyama-Reading-Reiten-Thomas)

Let $\mathcal{A} = \operatorname{mod} A$ be the module category of an artinian algebra A. Then $\operatorname{tors}(\mathcal{A})$ is a complete lattice.

- Conjecture: n-tors$(\mathcal{M})$ is a lattice.

Theorem (Demonet-Iyama-Reading-Reiten-Thomas)

Let $\mathcal{A} = \operatorname{mod} A$ be the module category of an artinian algebra A. Then $\operatorname{tors}(\mathcal{A})$ is a complete lattice.

- Conjecture: n-tors$(\mathcal{M})$ is a lattice.

$T(-)$: n-tors$(\mathcal{M}) \to$ tors$(\mathcal{A})$ is not a morphism of lattices.

Theorem (Demonet-Iyama-Reading-Reiten-Thomas)

Let $\mathcal{A} = \operatorname{mod} A$ be the module category of an artinian algebra A. Then $\operatorname{tors}(\mathcal{A})$ is a complete lattice.

- Conjecture: n-tors$(\mathcal{M})$ is a lattice.

$T(-)$: n-tors$(\mathcal{M}) \rightarrow$ tors$(\mathcal{A})$ is not a morphism of lattices.

Example

$$\mathcal{A} = \quad \operatorname{mod} KQ/I \qquad Q = \ 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \qquad I = < \alpha\beta >$$

Theorem (Demonet-Iyama-Reading-Reiten-Thomas)

Let $\mathcal{A} = \operatorname{mod} A$ be the module category of an artinian algebra A. Then $\operatorname{tors}(\mathcal{A})$ is a complete lattice.

- Conjecture: n-tors$(\mathcal{M})$ is a lattice.

$T(-)$: n-tors$(\mathcal{M}) \to$ tors$(\mathcal{A})$ is not a morphism of lattices.

Example

$$\mathcal{A} = \quad \operatorname{mod} KQ/I \qquad Q = \; 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \qquad I =< \alpha\beta >$$

$$\mathcal{M} = \operatorname{add}\{P(1) \oplus P(2) \oplus P(3) \oplus I(1)\} \subset \mathcal{A}$$

Theorem (Demonet-Iyama-Reading-Reiten-Thomas)

Let $\mathcal{A} = \operatorname{mod} A$ be the module category of an artinian algebra A. Then $\operatorname{tors}(\mathcal{A})$ is a complete lattice.

- Conjecture: n-tors$(\mathcal{M})$ is a lattice.

$T(-) : n$-tors$(\mathcal{M}) \to$ tors$(\mathcal{A})$ is not a morphism of lattices.

Example

$$\mathcal{A} = \quad \operatorname{mod} KQ/I \qquad Q = \ 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \qquad I = <\alpha\beta>$$

$$\mathcal{M} = \operatorname{add}\{P(1) \oplus P(2) \oplus P(3) \oplus I(1)\} \subset \mathcal{A}$$

- $\operatorname{add}\{P(3)\}$ and $\operatorname{add}\{I(1)\}$ are 2-torsion classes of $\mathcal{M}$.

Theorem (Demonet-Iyama-Reading-Reiten-Thomas)

Let $\mathcal{A} = \operatorname{mod} A$ be the module category of an artinian algebra A. Then $\operatorname{tors}(\mathcal{A})$ is a complete lattice.

- Conjecture: n-tors$(\mathcal{M})$ is a lattice.

$T(-)$: n-tors$(\mathcal{M}) \to$ tors$(\mathcal{A})$ is not a morphism of lattices.

Example

$$\mathcal{A} = \quad \operatorname{mod} KQ/I \qquad Q = \; 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \qquad I = <\alpha\beta>$$

$$\mathcal{M} = \operatorname{add}\{P(1) \oplus P(2) \oplus P(3) \oplus I(1)\} \subset \mathcal{A}$$

- $\operatorname{add}\{P(3)\}$ and $\operatorname{add}\{I(1)\}$ are 2-torsion classes of $\mathcal{M}$.
- $\operatorname{add}\{P(3)\}$ and $\operatorname{add}\{I(1)\}$ are torsion classes of $\mathcal{A}$.

Theorem (Demonet-Iyama-Reading-Reiten-Thomas)

Let $\mathcal{A} = \operatorname{mod} A$ be the module category of an artinian algebra A. Then $\operatorname{tors}(\mathcal{A})$ is a complete lattice.

- Conjecture: n-tors$(\mathcal{M})$ is a lattice.

$T(-)$: n-tors$(\mathcal{M}) \to$ tors$(\mathcal{A})$ is not a morphism of lattices.

Example

$$\mathcal{A} = \quad \operatorname{mod} KQ/I \qquad Q = \ 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \qquad I = < \alpha\beta >$$

$$\mathcal{M} = \operatorname{add}\{P(1) \oplus P(2) \oplus P(3) \oplus I(1)\} \subset \mathcal{A}$$

- $\operatorname{add}\{P(3)\}$ and $\operatorname{add}\{I(1)\}$ are 2-torsion classes of $\mathcal{M}$.
- $\operatorname{add}\{P(3)\}$ and $\operatorname{add}\{I(1)\}$ are torsion classes of $\mathcal{A}$.
- $\operatorname{add}\{P(3) \oplus I(1)\}$ is a torsion class in $\mathcal{A}$ but is not a 2-torsion class in $\mathcal{M}$.

Properties of n-torsion classes

Theorem (Dickson)

Every torsion class $\mathcal{T}$ of an abelian category $\mathcal{A}$ is closed under extensions and quotients.

Properties of n-torsion classes

Theorem (Dickson)

Every torsion class $\mathcal{T}$ of an abelian category $\mathcal{A}$ is closed under extensions and quotients.

Theorem (August-Haugland-Jacobsen-Kvamme-Palu-T.)

Every n-torsion class $\mathcal{U}$ of an n-abelian category $\mathcal{M}$ is closed under n-extensions and n-quotients.

Theorem (August-Haugland-Jacobsen-Kvamme-Palu-T.)

Every n-torsion class $\mathcal{U}$ of an n-abelian category $\mathcal{M}$ is closed under n-extensions* and n-quotients**.

Theorem (August-Haugland-Jacobsen-Kvamme-Palu-T.)

Every n-torsion class $\mathcal{U}$ of an n-abelian category $\mathcal{M}$ is closed under n-extensions* and n-quotients**.

(*) If $U, U' \in \mathcal{U}$ then any n-exact sequence in $\mathcal{M}$ of the form

$$0 \to U \to V_1 \to \cdots \to V_n \to U' \to 0$$

is Yoneda equivalent to an n-exact sequence

$$0 \to U \to V_1' \to \cdots \to V_n' \to U' \to 0$$

where $V_i' \in \mathcal{U}$ for all $1 \leq i \leq n$.

Theorem (August-Haugland-Jacobsen-Kvamme-Palu-T.)

Every n-torsion class $\mathcal{U}$ of an n-abelian category $\mathcal{M}$ is closed under n-extensions* and n-quotients**.

(**) If $f : X \to U$ is a map in $\mathcal{M}$ where $U \in \mathcal{U}$. Then any n-cokernel

$$X \xrightarrow{f} U \xrightarrow{v_1} V_1 \xrightarrow{v_2} \dots \xrightarrow{v_n} V_n \to 0$$

of f is homotopicc to an n-cokernel

$$X \xrightarrow{f} U \xrightarrow{v'_1} V'_1 \xrightarrow{v'_2} \dots \xrightarrow{v'_n} V'_n \to 0$$

such that $V'_i \in \mathcal{U}$ for all $1 \leq i \leq n$.

Example

$$\mathcal{A} = \quad \text{mod } KQ/I \qquad Q = \; 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \qquad I = <\alpha\beta>$$

$$\mathcal{M} = \text{add}\{P(1) \oplus P(2) \oplus P(3) \oplus I(1)\} \subset \mathcal{A}$$

- $\text{add}\{P(3) \oplus I(1)\}$ is a torsion class in $\mathcal{A}$ but is not a 2-torsion class in $\mathcal{M}$.

Example

$$\mathcal{A} = \quad \text{mod } KQ/I \qquad Q = 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \qquad I =< \alpha\beta >$$

$$\mathcal{M} = \text{add}\{P(1) \oplus P(2) \oplus P(3) \oplus I(1)\} \subset \mathcal{A}$$

- add$\{P(3) \oplus I(1)\}$ is a torsion class in $\mathcal{A}$ but is not a 2-torsion class in $\mathcal{M}$.

The following is a 2-exact sequence in $\mathcal{M}$.

$$0 \to P(3) \to P(2) \to P(1) \to I(1) \to 0$$

Example

$$\mathcal{A} = \quad \text{mod } KQ/I \qquad Q = \ 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \qquad I = <\alpha\beta>$$

$$\mathcal{M} = \text{add}\{P(1) \oplus P(2) \oplus P(3) \oplus I(1)\} \subset \mathcal{A}$$

- $\text{add}\{P(3) \oplus I(1)\}$ is a torsion class in $\mathcal{A}$ but is not a 2-torsion class in $\mathcal{M}$.

The following is a 2-exact sequence in $\mathcal{M}$.

$$0 \to P(3) \to P(2) \to P(1) \to I(1) \to 0$$

Here $I(1), P(3) \in \text{add}\{P(3) \oplus I(1)\}$ but $P(1), P(2) \notin \text{add}\{P(3) \oplus I(1)\}$.

Towards stability conditions in higher homological algebra

- Aim: Obtain a notion of stability conditions for n-abelian categories.

Towards stability conditions in higher homological algebra

- Aim: Obtain a notion of stability conditions for n-abelian categories.
- Stability conditions in abelian categories [Rudakov, Bridgeland]:

Towards stability conditions in higher homological algebra

- Aim: Obtain a notion of stability conditions for n-abelian categories.
- Stability conditions in abelian categories [Rudakov, Bridgeland]:
  - Definition based on short exact sequences.

Towards stability conditions in higher homological algebra

- Aim: Obtain a notion of stability conditions for n-abelian categories.
- Stability conditions in abelian categories [Rudakov, Bridgeland]:
  - Definition based on short exact sequences.
    Not clear how to define them using n-exact sequences.

Towards stability conditions in higher homological algebra

- Aim: Obtain a notion of stability conditions for n-abelian categories.
- Stability conditions in abelian categories [Rudakov, Bridgeland]:
  - Definition based on short exact sequences.
    Not clear how to define them using n-exact sequences.
  - Stability conditions induce Harder-Narasimhan filtrations for every object in the category.

Towards stability conditions in higher homological algebra

- Aim: Obtain a notion of stability conditions for n-abelian categories.
- Stability conditions in abelian categories [Rudakov, Bridgeland]:
  - Definition based on short exact sequences.
    Not clear how to define them using n-exact sequences.
  - Stability conditions induce Harder-Narasimhan filtrations for every object in the category.
    - Slicings [Bridgeland].

Towards stability conditions in higher homological algebra

- Aim: Obtain a notion of stability conditions for n-abelian categories.
- Stability conditions in abelian categories [Rudakov, Bridgeland]:
  - Definition based on short exact sequences.
    Not clear how to define them using n-exact sequences.
  - Stability conditions induce Harder-Narasimhan filtrations for every object in the category.
    - Slicings [Bridgeland].

Theorem (T.)

Let $\mathcal{A}$ be an abelian length category. Then every chain of torsion classes η induces a slicing $\mathcal{P}_\eta$ in $\mathcal{A}$. Moreover every slicing in $\mathcal{A}$ arises this way.

Harder-Narasimhan filtrations in abelian categories

Definition (T.)

A chain of torsion classes η in an abelian category $\mathcal{A}$ is a set of torsion classes

$$\eta := \{\mathcal{T}_s : s \in [0,1], \mathcal{T}_0 = \mathcal{A}, \mathcal{T}_1 = \{0\} \text{ and } \mathcal{T}_s \subseteq \mathcal{T}_r \text{ if } r \leq s\}.$$

Harder-Narasimhan filtrations in abelian categories

Definition (T.)

A chain of torsion classes η in an abelian category $\mathcal{A}$ is a set of torsion classes

$$\eta := \{\mathcal{T}_s : s \in [0,1], \mathcal{T}_0 = \mathcal{A}, \mathcal{T}_1 = \{0\} \text{ and } \mathcal{T}_s \subseteq \mathcal{T}_r \text{ if } r \leq s\}.$$

The slicing $\mathcal{P}_\eta$ induced by η is

$$\mathcal{P}_\eta := \{\mathcal{P}_s : \mathcal{P}_s \text{ is a subcategory of } \mathcal{A} \text{ and } s \in [0,1]\}.$$

Harder-Narasimhan filtrations in abelian categories

Theorem (T.)

Let $\mathcal{A}$ be an abelian category, η a chain of torsion classes in $\mathcal{A}$ and $\mathcal{P}_\eta$ be the slicing induced by η. Then every object $M \in \mathcal{A}$ admits a Harder-Narasimhan filtration with respect to η.

Harder-Narasimhan filtrations in abelian categories

Theorem (T.)

Let $\mathcal{A}$ be an abelian category, η a chain of torsion classes in $\mathcal{A}$ and $\mathcal{P}_\eta$ be the slicing induced by η. Then every object $M \in \mathcal{A}$ admits a Harder-Narasimhan filtration with respect to η. That is a filtration

$$M_0 \subset M_1 \subset \cdots \subset M_t$$

such that:

Harder-Narasimhan filtrations in abelian categories

Theorem (T.)

Let $\mathcal{A}$ be an abelian category, η a chain of torsion classes in $\mathcal{A}$ and $\mathcal{P}_\eta$ be the slicing induced by η. Then every object $M \in \mathcal{A}$ admits a Harder-Narasimhan filtration with respect to η. That is a filtration

$$M_0 \subset M_1 \subset \cdots \subset M_t$$

such that:

1. $0 = M_0$ and $M_t = M$;
2. there exists $r_k \in [0,1]$ such that $M_k/M_{k-1} \in \mathcal{P}_{r_k}$ for all $1 \leq k \leq t$;
3. $r_1 > r_2 > \cdots > r_t$.

Harder-Narasimhan filtrations in abelian categories

Theorem (T.)

Let $\mathcal{A}$ be an abelian category, η a chain of torsion classes in $\mathcal{A}$ and $\mathcal{P}_\eta$ be the slicing induced by η. Then every object $M \in \mathcal{A}$ admits a Harder-Narasimhan filtration with respect to η. That is a filtration

$$M_0 \subset M_1 \subset \cdots \subset M_t$$

such that:

1. $0 = M_0$ and $M_t = M$;
2. there exists $r_k \in [0,1]$ such that $M_k/M_{k-1} \in \mathcal{P}_{r_k}$ for all $1 \le k \le t$;
3. $r_1 > r_2 > \cdots > r_t$.

Moreover this filtration is unique up to isomorphism.

Harder-Narasimhan filtrations in n-abelian categories

Definition (Asadollahi-Jørgensen-Schroll-T.)

A chain of n-torsion classes δ in an n-abelian category $\mathcal{M}$ is a set of n-torsion classes

$$\delta := \{\mathcal{U}_s : s \in [0,1], \mathcal{U}_0 = \mathcal{M},\ \mathcal{U}_1 = \{0\} \text{ and } \mathcal{U}_s \subseteq \mathcal{U}_r \text{ if } r \leq s\}.$$

Harder-Narasimhan filtrations in n-abelian categories

Definition (Asadollahi-Jørgensen-Schroll-T.)

A chain of n-torsion classes δ in an n-abelian category $\mathcal{M}$ is a set of n-torsion classes

$$\delta := \{\mathcal{U}_s : s \in [0,1], \mathcal{U}_0 = \mathcal{M},\ \mathcal{U}_1 = \{0\} \text{ and } \mathcal{U}_s \subseteq \mathcal{U}_r \text{ if } r \leq s\}.$$

The slicing $\mathcal{Q}_\delta$ induced by δ is

$$\mathcal{Q}_\delta := \{\mathcal{Q}_s : \mathcal{Q}_s \text{ is a class of } n\text{-cokernels in } \mathcal{M} \text{ and } s \in [0,1]\}.$$

Harder-Narasimhan filtrations in n-abelian categories

Theorem (Asadollahi-Jørgensen-Schroll-T.)

Let $\mathcal{M}$ be an abelian length category, δ a chain of n-torsion classes in $\mathcal{M}$ and $\mathcal{Q}_\delta$ be the slicing induced by δ. Then every object $M \in \mathcal{M}$ admits a Harder-Narasimhan filtration with respect to δ.

Harder-Narasimhan filtrations in n-abelian categories

Theorem (Asadollahi-Jørgensen-Schroll-T.)

Let $\mathcal{M}$ be an abelian length category, δ a chain of n-torsion classes in $\mathcal{M}$ and $\mathcal{Q}_\delta$ be the slicing induced by δ. Then every object $M \in \mathcal{M}$ admits a Harder-Narasimhan filtration with respect to δ. That is a filtration

$$M_0 \subset M_1 \subset \cdots \subset M_t$$

such that:

Harder-Narasimhan filtrations in n-abelian categories

Theorem (Asadollahi-Jørgensen-Schroll-T.)

Let $\mathcal{M}$ be an abelian length category, δ a chain of n-torsion classes in $\mathcal{M}$ and $\mathcal{Q}_\delta$ be the slicing induced by δ. Then every object $M \in \mathcal{M}$ admits a Harder-Narasimhan filtration with respect to δ. That is a filtration

$$M_0 \subset M_1 \subset \cdots \subset M_t$$

such that:

1. $0 = M_0$ and $M_t = M$;
2. there exists $r_k \in [0,1]$ such that $coker(M_{k-1} \to M_k) \in \mathcal{Q}_{r_k}$ for all $1 \leq k \leq t$;
3. $r_1 > r_2 > \cdots > r_t$.

Harder-Narasimhan filtrations in n-abelian categories

Theorem (Asadollahi-Jørgensen-Schroll-T.)

Let $\mathcal{M}$ be an abelian length category, δ a chain of n-torsion classes in $\mathcal{M}$ and $\mathcal{Q}_\delta$ be the slicing induced by δ. Then every object $M \in \mathcal{M}$ admits a Harder-Narasimhan filtration with respect to δ. That is a filtration

$$M_0 \subset M_1 \subset \cdots \subset M_t$$

such that:

1. $0 = M_0$ and $M_t = M$;
2. there exists $r_k \in [0, 1]$ such that $coker(M_{k-1} \to M_k) \in \mathcal{Q}_{r_k}$ for all $1 \leq k \leq t$;
3. $r_1 > r_2 > \cdots > r_t$.

Moreover this filtration is unique up to isomorphism.

Compatibility of Harder-Narasimhan filtrations

- $T(-) : n\text{-tors}(\mathcal{M}) \to \text{tors}(\mathcal{A})$

Compatibility of Harder-Narasimhan filtrations

- $T(-) : n$-tors$(\mathcal{M}) \to$ tors$(\mathcal{A})$
- Let δ be a chain of n-torsion classes in $\mathcal{M}$ and consider the chain of torsion classes $T(\delta)$ in $\mathcal{A}$ as follows.

$$T(\delta) := \{T(\mathcal{U}_s) : s \in [0,1]\}.$$

Compatibility of Harder-Narasimhan filtrations

- $T(-) : n\text{-tors}(\mathcal{M}) \to \text{tors}(\mathcal{A})$
- Let δ be a chain of n-torsion classes in $\mathcal{M}$ and consider the chain of torsion classes $T(\delta)$ in $\mathcal{A}$ as follows.

$$T(\delta) := \{T(\mathcal{U}_s) : s \in [0,1]\}.$$

Theorem (Asadollahi-Jørgensen-Schroll-T.)

Let δ be a chain of n-torsion classes in $\mathcal{M} \subset \mathcal{A}$ and $M \in \mathcal{M}$. Then the Harder-Narasimhan filtration of M induced by δ is isomorphic to the Harder-Narasimhan filtration of M induced by $T(\delta)$.

Some more setting

Some more setting

- A is a finite dimensional algebra over a field K.

Some more setting

- A is a finite dimensional algebra over a field K.
- $\mathcal{A} = \operatorname{mod} A$.

Some more setting

- A is a finite dimensional algebra over a field K.
- $\mathcal{A} = \operatorname{mod} A$.
- $\mathcal{M}$ is an n-cluster tilting subcategory of $\mathcal{A}$

Some more setting

- A is a finite dimensional algebra over a field K.
- $\mathcal{A} = \operatorname{mod} A$.
- $\mathcal{M}$ is an n-cluster tilting subcategory of $\mathcal{A}$
- The n-th Auslander-Reiten translation in $\mathcal{A}$ is defined as $\tau_n M := \tau\Omega^{n-1}M$.

Some more setting

- A is a finite dimensional algebra over a field K.
- $\mathcal{A} = \operatorname{mod} A$.
- $\mathcal{M}$ is an n-cluster tilting subcategory of $\mathcal{A}$
- The n-th Auslander-Reiten translation in $\mathcal{A}$ is defined as $\tau_n M := \tau\Omega^{n-1}M$.
- We say that $M \in \mathcal{X} \subset \mathcal{A}$ is Ext^n-projective in $\mathcal{X}$ if $\operatorname{Ext}^n_{\mathcal{M}}(M, X) = 0$ for all $X \in \mathcal{X}$.

Some more setting

- A is a finite dimensional algebra over a field K.
- $\mathcal{A} = \operatorname{mod} A$.
- $\mathcal{M}$ is an n-cluster tilting subcategory of $\mathcal{A}$
- The n-th Auslander-Reiten translation in $\mathcal{A}$ is defined as $\tau_n M := \tau\Omega^{n-1} M$.
- We say that $M \in \mathcal{X} \subset \mathcal{A}$ is Ext^n-projective in $\mathcal{X}$ if $\operatorname{Ext}^n_{\mathcal{M}}(M, X) = 0$ for all $X \in \mathcal{X}$.
- An object $M \in \mathcal{M}$ is τ_n-rigid if $\operatorname{Hom}_A(M, \tau_n M) = 0$.

Some more setting

- A is a finite dimensional algebra over a field K.
- $\mathcal{A} = \operatorname{mod} A$.
- $\mathcal{M}$ is an n-cluster tilting subcategory of $\mathcal{A}$
- The n-th Auslander-Reiten translation in $\mathcal{A}$ is defined as $\tau_n M := \tau\Omega^{n-1} M$.
- We say that $M \in \mathcal{X} \subset \mathcal{A}$ is Ext^n-projective in $\mathcal{X}$ if $\operatorname{Ext}^n_{\mathcal{M}}(M, X) = 0$ for all $X \in \mathcal{X}$.
- An object $M \in \mathcal{M}$ is τ_n-rigid if $\operatorname{Hom}_A(M, \tau_n M) = 0$.
- Let $M \in \mathcal{M}$ and P be a projective A-module. We say that the pair (M, P) is τ_n-rigid if M is τ_n-rigid and $\operatorname{Hom}_A(P, M) = 0$.

Ext^n-projective modules in n-torsion classes

Ext^n-projective modules in n-torsion classes

Theorem (Auslander-Smalø)

Let $\mathcal{T}$ be a torsion class in $\operatorname{mod} A$ and let $M \in \mathcal{T}$. Then M is Ext-projective in $\mathcal{T}$ if and only if $\operatorname{Hom}_A(T, \tau M) = 0$ for all $T \in \mathcal{T}$. In particular M is τ-rigid.

Ext^n-projective modules in n-torsion classes

Theorem (Auslander-Smalø)

Let $\mathcal{T}$ be a torsion class in $\operatorname{mod} A$ and let $M \in \mathcal{T}$. Then M is Ext-projective in $\mathcal{T}$ if and only if $\mathrm{Hom}_A(T, \tau M) = 0$ for all $T \in \mathcal{T}$. In particular M is τ-rigid.

Theorem (August-Haugland-Jacobsen-Kvamme-Palu-T.)

Let $\mathcal{U}$ be an n-torsion class in $\mathcal{M}$ and let $M \in \mathcal{U}$. Then M is Ext^n-projective in $\mathcal{U}$ if and only if $\mathrm{Hom}_A(U, \tau_n M) = 0$ for all $U \in \mathcal{U}$. In particular M is τ_n-rigid.

Generating functorially finite n-torsion classes

- From now on $\mathcal{U}$ is a functorially finite n-torsion class of $\mathcal{M}$.

Generating functorially finite n-torsion classes

- From now on $\mathcal{U}$ is a functorially finite n-torsion class of $\mathcal{M}$.
- $A_A \in \mathcal{M} \subset \operatorname{mod} A$.

Generating functorially finite n-torsion classes

- From now on $\mathcal{U}$ is a functorially finite n-torsion class of $\mathcal{M}$.
- $A_A \in \mathcal{M} \subset \operatorname{mod} A$.

Proposition (Auslander-Smalø)

Let $\mathcal{T}$ be a functorially finite torsion class in $\operatorname{mod} A$ and let $T_A \in \mathcal{T}$ be the minimal left $\mathcal{T}$-approximation of A. Then T_A is τ-rigid and $\mathcal{T} = \operatorname{Fac} T_A$.

Generating functorially finite n-torsion classes

- From now on $\mathcal{U}$ is a functorially finite n-torsion class of $\mathcal{M}$.
- $A_A \in \mathcal{M} \subset \operatorname{mod} A$.

Proposition (Auslander-Smalø)

Let $\mathcal{T}$ be a functorially finite torsion class in $\operatorname{mod} A$ and let $T_A \in \mathcal{T}$ be the minimal left $\mathcal{T}$-approximation of A. Then T_A is τ-rigid and $\mathcal{T} = \operatorname{Fac} T_A$.

Proposition (August-Haugland-Jacobsen-Kvamme-Palu-T.)

Let $\mathcal{U}$ be a functorially finite n-torsion class of $\mathcal{M}$ and let U_A^0 be the minimal left $\mathcal{U}$-approximation of A. Then U_A^0 is τ_n-rigid and $\mathcal{U} = \operatorname{Fac} U_A^0 \cap \mathcal{M}$.

Generating functorially finite n-torsion classes

- From now on $\mathcal{U}$ is a functorially finite n-torsion class of $\mathcal{M}$.
- $A_A \in \mathcal{M} \subset \operatorname{mod} A$.

Proposition (Auslander-Smalø)

Let $\mathcal{T}$ be a functorially finite torsion class in $\operatorname{mod} A$ and let $T_A \in \mathcal{T}$ be the minimal left $\mathcal{T}$-approximation of A. Then T_A is τ-rigid and $\mathcal{T} = \operatorname{Fac} T_A$.

Proposition (August-Haugland-Jacobsen-Kvamme-Palu-T.)

Let $\mathcal{U}$ be a functorially finite n-torsion class of $\mathcal{M}$ and let U_A^0 be the minimal left $\mathcal{U}$-approximation of A. Then U_A^0 is τ_n-rigid and $\mathcal{U} = \operatorname{Fac} U_A^0 \cap \mathcal{M}$.

Not every τ_n-rigid module generates an n-torsion class.

- Let $\mathcal{U}$ be a functorially finite n-torsion class in $\mathcal{M} \subset \operatorname{mod} A$.

- Let $\mathcal{U}$ be a functorially finite n-torsion class in $\mathcal{M} \subset \operatorname{mod} A$.

$$A \to U_A^0 \to U_A^1 \to \ldots \to U_A^n \to 0$$

- Let $\mathcal{U}$ be a functorially finite n-torsion class in $\mathcal{M} \subset \operatorname{mod} A$.

$$A \to U_A^0 \to U_A^1 \to \ldots \to U_A^n \to 0$$

- Let U_A be the minimal add-generator of $\operatorname{add}\left\{\bigoplus_{i=0}^{n} U_A^i\right\}$.

- Let $\mathcal{U}$ be a functorially finite n-torsion class in $\mathcal{M} \subset \operatorname{mod} A$.

$$A \to U_A^0 \to U_A^1 \to \ldots \to U_A^n \to 0$$

- Let U_A be the minimal add-generator of $\mathrm{add}\left\{\bigoplus_{i=0}^{n} U_A^i\right\}$.
- Remark: Note that $U_A \in \mathcal{U}$ because $\mathcal{U}$ is closed under n-cokernels.

- Let $\mathcal{U}$ be a functorially finite n-torsion class in $\mathcal{M} \subset \operatorname{mod} A$.

$$A \to U_A^0 \to U_A^1 \to \ldots \to U_A^n \to 0$$

- Let U_A be the minimal add-generator of $\mathrm{add}\left\{\bigoplus_{i=0}^{n} U_A^i\right\}$.
- Remark: Note that $U_A \in \mathcal{U}$ because $\mathcal{U}$ is closed under n-cokernels.

Theorem (August-Haugland-Jacobsen-Kvamme-Palu-T.)

Let $\mathcal{U}$ be a functorially finite n-torsion class in $\mathcal{M}$.

- Let $\mathcal{U}$ be a functorially finite n-torsion class in $\mathcal{M} \subset \operatorname{mod} A$.

$$A \to U_A^0 \to U_A^1 \to \ldots \to U_A^n \to 0$$

- Let U_A be the minimal add-generator of $\mathrm{add}\left\{ \bigoplus_{i=0}^{n} U_A^i \right\}$.
- Remark: Note that $U_A \in \mathcal{U}$ because $\mathcal{U}$ is closed under n-cokernels.

Theorem (August-Haugland-Jacobsen-Kvamme-Palu-T.)

Let $\mathcal{U}$ be a functorially finite n-torsion class in $\mathcal{M}$. Then U_A is an Ext^n-projective object of $\mathcal{U}$.

- Let $\mathcal{U}$ be a functorially finite n-torsion class in $\mathcal{M} \subset \operatorname{mod} A$.

$$A \to U_A^0 \to U_A^1 \to \ldots \to U_A^n \to 0$$

- Let U_A be the minimal add-generator of $\operatorname{add}\left\{\bigoplus_{i=0}^{n} U_A^i\right\}$.
- Remark: Note that $U_A \in \mathcal{U}$ because $\mathcal{U}$ is closed under n-cokernels.

Theorem (August-Haugland-Jacobsen-Kvamme-Palu-T.)

Let $\mathcal{U}$ be a functorially finite n-torsion class in $\mathcal{M}$. Then U_A is an Ext^n-projective object of $\mathcal{U}$. Moreover, if $U' \in \mathcal{U}$ is an Ext^n-projective object, then $\operatorname{add} U' \subset \operatorname{add} U_A$.

- Let $\mathcal{U}$ be a functorially finite n-torsion class in $\mathcal{M} \subset \operatorname{mod} A$.

$$A \to U_A^0 \to U_A^1 \to \ldots \to U_A^n \to 0$$

- Let U_A be the minimal add-generator of $\operatorname{add}\left\{\bigoplus_{i=0}^{n} U_A^i\right\}$.
- Remark: Note that $U_A \in \mathcal{U}$ because $\mathcal{U}$ is closed under n-cokernels.

Theorem (August-Haugland-Jacobsen-Kvamme-Palu-T.)

Let $\mathcal{U}$ be a functorially finite n-torsion class in $\mathcal{M}$. Then U_A is an Ext^n-projective object of $\mathcal{U}$. Moreover, if $U' \in \mathcal{U}$ is an Ext^n-projective object, then $\operatorname{add} U' \subset \operatorname{add} U_A$. In particular U_A^i is an Ext^n-projective object in $\mathcal{U}$ for all $0 \leq i \leq n$.

- Let $\mathcal{U}$ and U_A as above. We define $P_{\mathcal{U}}$ to be the minimal add-generator of

$$\begin{aligned}\mathcal{P}_{\mathcal{U}} &= \{P : P \text{ is projective and } \mathrm{Hom}_A(P,U)=0 \text{ with } U\in\mathcal{U}\}\\ &= \{P : P \text{ is projective and } \mathrm{Hom}_A(P,U_A)=0\}\,.\end{aligned}$$

- Let $\mathcal{U}$ and U_A as above. We define $P_{\mathcal{U}}$ to be the minimal add-generator of

$$\begin{aligned}\mathcal{P}_{\mathcal{U}} &= \{P : P \text{ is projective and } \mathrm{Hom}_A(P, U) = 0 \text{ with } U \in \mathcal{U}\} \\ &= \{P : P \text{ is projective and } \mathrm{Hom}_A(P, U_A) = 0\}\,.\end{aligned}$$

- $|M|$ is the number of isomorphism classes of indecomposable direct summands of M.

- Let $\mathcal{U}$ and U_A as above. We define $P_{\mathcal{U}}$ to be the minimal add-generator of

$$\begin{aligned}\mathcal{P}_{\mathcal{U}} &= \{P : P \text{ is projective and } \operatorname{Hom}_A(P, U) = 0 \text{ with } U \in \mathcal{U}\} \\ &= \{P : P \text{ is projective and } \operatorname{Hom}_A(P, U_A) = 0\}\,.\end{aligned}$$

- $|M|$ is the number of isomorphism classes of indecomposable direct summands of M.

Theorem (August-Haugland-Jacobsen-Kvamme-Palu-T.)

Let $\mathcal{U}$ be a functorially finite n-torsion class of $\mathcal{M} \subset \operatorname{mod} A$. Then $(U_A, P_{\mathcal{U}})$ is a τ_n-rigid pair and $|U_A| + |P_{\mathcal{U}}| = |A|$.

- Let $\mathcal{U}$ and U_A as above. We define $P_{\mathcal{U}}$ to be the minimal add-generator of

$$\begin{aligned}\mathcal{P}_{\mathcal{U}} &= \{P : P \text{ is projective and } \operatorname{Hom}_A(P,U) = 0 \text{ with } U \in \mathcal{U}\} \\ &= \{P : P \text{ is projective and } \operatorname{Hom}_A(P,U_A) = 0\}\,.\end{aligned}$$

- $|M|$ is the number of isomorphism classes of indecomposable direct summands of M.

Theorem (August-Haugland-Jacobsen-Kvamme-Palu-T.)

Let $\mathcal{U}$ be a functorially finite n-torsion class of $\mathcal{M} \subset \operatorname{mod} A$. Then $(U_A, P_{\mathcal{U}})$ is a τ_n-rigid pair and $|U_A| + |P_{\mathcal{U}}| = |A|$.

- Remark: Martinez-Mendoza have similar results studying τ_n-rigid modules in $\operatorname{mod} A$, regardless of the existence of the n-cluster tilting subcategory $\mathcal{M} \subset \operatorname{mod} A$.

From τ-tilting theory to τ_n-tilting theory

From τ-tilting theory to τ_n-tilting theory

- A τ-rigid pair (T, P) is said to be τ-tilting if $|T| + |P| = |A|$.

From τ-tilting theory to τ_n-tilting theory

- A τ-rigid pair (T, P) is said to be τ-tilting if $|T| + |P| = |A|$.

Theorem (Adachi-Iyama-Reiten)

A torsion class $\mathcal{T}$ is functorially finite if and only if there exists a τ-tilting pair (T, P) such that $\mathcal{T} = \operatorname{Fac} T$.

From τ-tilting theory to τ_n-tilting theory

- A τ-rigid pair (T, P) is said to be τ-tilting if $|T| + |P| = |A|$.

Theorem (Adachi-Iyama-Reiten)

A torsion class $\mathcal{T}$ is functorially finite if and only if there exists a τ-tilting pair (T, P) such that $\mathcal{T} = \operatorname{Fac} T$.

- Recall that $\mathcal{U} = \mathcal{M} \cap \mathcal{T}$ for some torsion class $\mathcal{T}$ of $\operatorname{mod} A$.

From τ-tilting theory to τ_n-tilting theory

- A τ-rigid pair (T, P) is said to be τ-tilting if $|T| + |P| = |A|$.

Theorem (Adachi-Iyama-Reiten)

A torsion class $\mathcal{T}$ is functorially finite if and only if there exists a τ-tilting pair (T, P) such that $\mathcal{T} = \operatorname{Fac} T$.

- Recall that $\mathcal{U} = \mathcal{M} \cap \mathcal{T}$ for some torsion class $\mathcal{T}$ of $\operatorname{mod} A$.

Lemma (August-Haugland-Jacobsen-Kvamme-Palu-T.)

Let $\mathcal{U} = \mathcal{T} \cap \mathcal{M}$ be an n-torsion class of $\mathcal{M}$. If $\mathcal{T}$ is functorially finite then $\mathcal{U}$ is also functorially finite.

From τ-tilting theory to τ_n-tilting theory

- Let $\mathcal{U}$ be an n-torsion class such that $\mathcal{U} = \mathcal{M} \cap \operatorname{Fac} T$ for some τ-tilting pair (T, P) and consider the $\mathcal{M}$-coresolution of T.

From τ-tilting theory to τ_n-tilting theory

- Let $\mathcal{U}$ be an n-torsion class such that $\mathcal{U} = \mathcal{M} \cap \operatorname{Fac} T$ for some τ-tilting pair (T, P) and consider the $\mathcal{M}$-coresolution of T.

$$0 \to T \to U_T^0 \to U_T^1 \to \ldots \to U_T^{n-1} \to 0$$

From τ-tilting theory to τ_n-tilting theory

- Let $\mathcal{U}$ be an n-torsion class such that $\mathcal{U} = \mathcal{M} \cap \operatorname{Fac} T$ for some τ-tilting pair (T, P) and consider the $\mathcal{M}$-coresolution of T.

$$0 \to T \to U_T^0 \to U_T^1 \to \ldots \to U_T^{n-1} \to 0$$

- Define U_T be the minimal add-generator of $\operatorname{add}\left\{\bigoplus_{i=0}^{n-1} U_T^i\right\}$.

From τ-tilting theory to τ_n-tilting theory

- Let $\mathcal{U}$ be an n-torsion class such that $\mathcal{U} = \mathcal{M} \cap \operatorname{Fac} T$ for some τ-tilting pair (T, P) and consider the $\mathcal{M}$-coresolution of T.

$$0 \to T \to U_T^0 \to U_T^1 \to \ldots \to U_T^{n-1} \to 0$$

- Define U_T be the minimal add-generator of $\operatorname{add} \left\{ \bigoplus_{i=0}^{n-1} U_T^i \right\}$.
- Remark: $U_T \in \mathcal{U}$.

From τ-tilting theory to τ_n-tilting theory

- Let $\mathcal{U}$ be an n-torsion class such that $\mathcal{U} = \mathcal{M} \cap \operatorname{Fac} T$ for some τ-tilting pair (T, P) and consider the $\mathcal{M}$-coresolution of T.

$$0 \to T \to U_T^0 \to U_T^1 \to \ldots \to U_T^{n-1} \to 0$$

- Define U_T be the minimal add-generator of $\mathrm{add}\left\{\bigoplus_{i=0}^{n-1} U_T^i\right\}$.
- Remark: $U_T \in \mathcal{U}$.

Theorem (August-Haugland-Jacobsen-Kvamme-Palu-T.)

Let $\mathcal{U}$ be an n-torsion class such that $\mathcal{U} = \mathcal{M} \cap \operatorname{Fac} T$ for some τ-tilting pair (T, P). Then the pair (U_T, P) is a τ_n-rigid pair such that $\mathrm{add}\, U_T = \mathrm{add}\, U_A$ and $\mathrm{add}\, P = \mathrm{add}\, P_{\mathcal{U}}$.

Thank you very much! ☺