Cover Relations in the Lattice of Torsion Classes: Dynamics and Completability

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Outline

Goal of the Talk

I am a combinatorialist who likes to study certain lattice-posets called *semidistributive lattices*.
I want to tell you a story that begins with purely combinatorial work from my thesis, and ends in the world of torsion classes.

Set Up

- Let Λ be a finite dimensional, basic algebra over an arbitrary field K.
- Denote by $\text{mod}\Lambda$ the category of finitely generated (right) modules.
- All subcategories are assumed full and closed under isomorphisms.
- $(-)[1]$ is the shift functor.
- $S \in \text{mod}\Lambda$ or $\mathcal{D}^b(\text{mod}\Lambda)$ is called a *brick* if $\mathrm{End}(S)$ is a division algebra. A collection of Hom-orthogonal bricks is a *semibrick*.

Torsion Classes

Let $\mathcal{T}, \mathcal{F}$ be (full, closed under isomorphism) subcategories of modΛ. Then the pair $(\mathcal{T}, \mathcal{F})$ is called a *torsion pair* if each of the following holds:

1. $\mathrm{Hom}_\Lambda(M, N) = 0$ for all $M \in \mathcal{T}$ and $N \in \mathcal{F}$.
2. $\mathrm{Hom}_\Lambda(M, -)|_\mathcal{F} = 0$ implies that $M \in \mathcal{T}$.
3. $\mathrm{Hom}_\Lambda(-, N)|_\mathcal{T} = 0$ implies that $N \in \mathcal{F}$.

For a torsion pair $(\mathcal{T}, \mathcal{F})$, we say that $\mathcal{T}$ is a *torsion class*, and $\mathcal{F}$ is a *torsion free class*.

Running Example

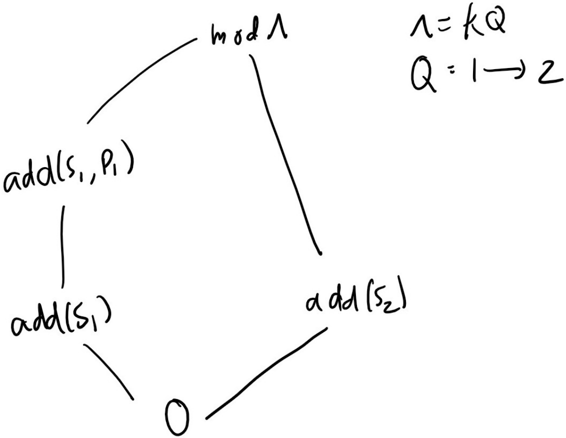
Equivalently, a *torsion class* $\mathcal{T}$ is a class of modules that is closed under quotients, isomorphisms, and extensions. Consider the set of modules over the path algebra with quiver $Q = 1 \to 2$.

- S_1 - Simple (no submodules or quotients)
- S_2 - Simple (no submodules or quotients)
- P_1 - Projective modules which is an extension of S_1 and S_2.

$$S_2 \hookrightarrow P_1 \twoheadrightarrow S_1$$

Lattice of Torsion classes

We study the lattice (poset) of torsion classes also denoted torsΛ in which $\mathcal{S} \leq \mathcal{T}$ whenever $\mathcal{S} \subseteq \mathcal{T}$.



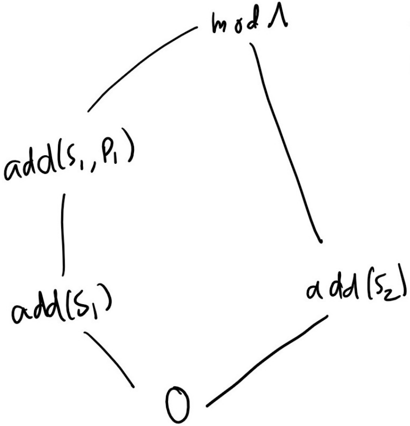
Semidistributive lattices

Definition

A lattice L is a poset such that for each pair of elements u and w

- the smallest upper bound or *join* $u \vee w$ exists and
- the greatest lower bound or *meet* $u \wedge w$ exists.

Semidistributive lattices



$\Lambda = kQ$

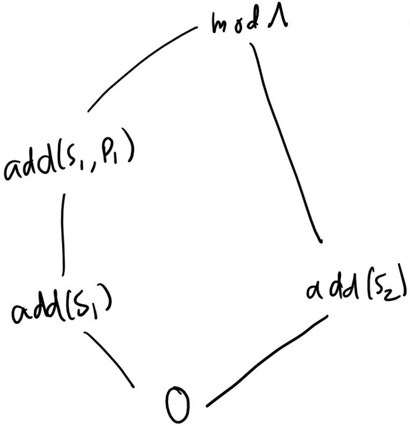
$Q : 1 \longrightarrow 2$

Cover relations

Definition

An element y *covers* x if $y > x$ and there is no z such that $y > z > x$. In this case we also say that x is *covered by* y, and we use the notation $y \gtrdot x$. The pair (x, y) is called a *cover relation*.

Cover relations



$\Lambda = kQ$

$Q: 1 \longrightarrow 2$

Semidistributive Lattices

Definition

A semidistributive lattice L satisfies a weakening of the distributive law. For any x, y, and z in L:

$$\text{If } x \vee y = x \vee z, \text{ then } x \vee (y \wedge z) = x \vee y$$

$$\text{If } x \wedge y = x \wedge z, \text{ then } x \wedge (y \vee z) = x \wedge y$$

Important Examples

- the Tamari lattices and c-Cambrian lattices
- the weak order for any finite Coxeter group W
- the lattice of torsion classes*

Semidistributive lattices are special

Each element of a finite semidistributive lattice can be factored uniquely as the join of certain irreducible elements.

Definition

- An element $j \in L$ is *join-irreducible* if $j = \bigvee A$ implies $j \in A$, where A is finite.
- An element is *completely join-irreducible* if j is covers a unique element, which we write as j_*.
- When lattice is finite these notions coincide.

Semidistributive lattices are special

Definition: A unique join factorization

The *canonical join representation* of an element x is the unique "lowest" irredundant expression $x = \bigvee A$.

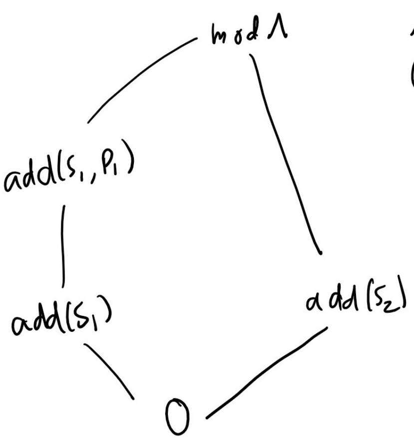
One can define an analogous "factorization" in terms of the meet operation called the *canonical meet representation*.

Theorem

A finite lattice L is semidistributive provided that each element has

- a canonical join representation and
- a canonical meet representation.

Running Example



$\Lambda = kQ$

$Q : 1 \longrightarrow 2$

Running Example

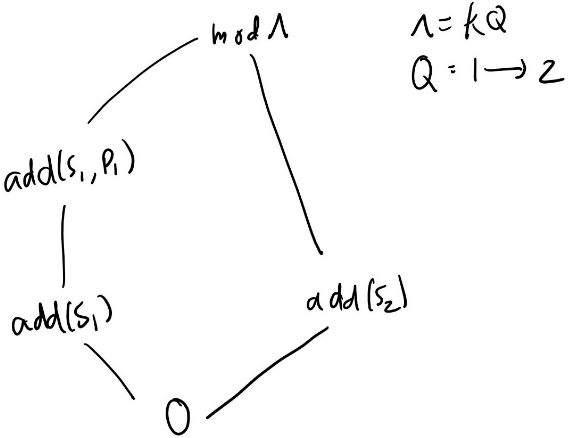
Facts and Observations

- In a finite lattice, each canonical join representation consists of only completely join-irreducible elements.
- For torsion classes, there is a bijection between completely join-irreducible torsion classes and bricks:

$$M \mapsto \mathsf{Filt}(\mathsf{Gen}(M))$$

- Not all subsets of completely join-irreducible elements give rise to a canonical join representation.

Running Example



A Pairwise Property

Theorem [B. 2016]

Let L be a finite semidistributive lattice and let $\mathcal{D}$ be a set of completely join-irreducible elements in L. Then there exists an element $x \in L$ such that $x = \bigvee \mathcal{D}$ is the CJR of x if and only if there exists an element $x_{s,t}$ such that $x_{s,t} = s \vee t$ is the CJR of $x_{s,t}$ for each pair $s, t \in \mathcal{D}$.

Theorem[B., Carroll, Zhu]

Let $\mathcal{D}$ be a set of bricks of Λ. Then $\bigvee_{M \in \mathcal{D}} \mathrm{Filt}(\mathrm{Gen}(M))$ is the CJR for some torsion class if and only if $\mathcal{D}$ is a semibrick.

Part II (Joint with G. Todorov and S. Zhu)

We study a certain map called κ which was key in proving that CJR's are defined by a pairwise condition.

Part III (Joint with E. J. Hanson)

We study a pairwise condition for 2-term simple minded collections.

Part II

The "kappa" map is a map which takes completely join-irreducible elements to completely meet-irreducible elements.

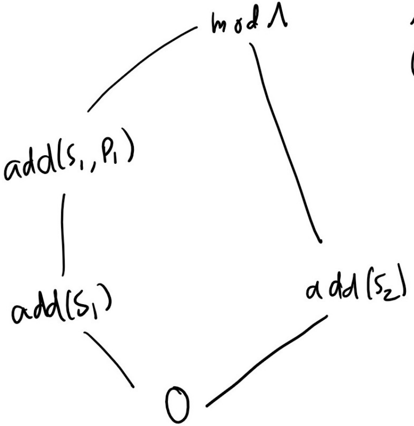
Main Definition

Let j be a (completely) join-irreducible element of a lattice L, and let j_* be the unique element covered by j. Define $\kappa(j)$ to be:

$$\kappa(j) := \text{unique } \max\{x \in L : j_* \leq x \text{ and } j \not\leq x\},$$

when such an element exists.

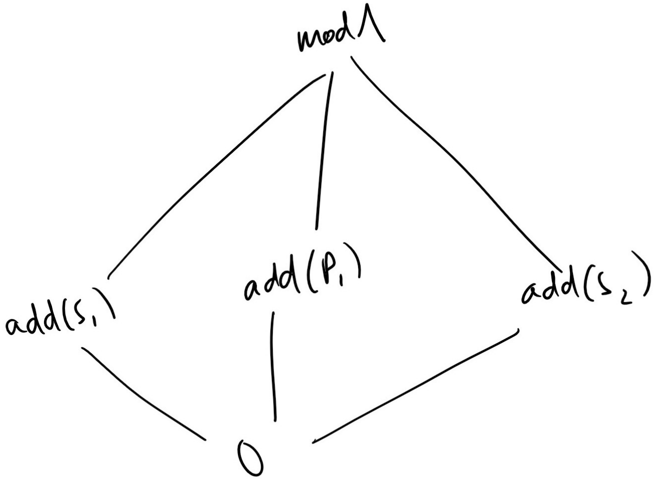
Running Example



$\Lambda = kQ$

$Q : 1 \longrightarrow 2$

Running Example



When does kappa exist?

Fact/Observation

- If L is a finite lattice, then κ is well-defined and is a bijection if and only if L is semidistributive.
- *kappa* helps connect the unique factorization in terms of the join (CJR) to the unique factorization in terms of the meet.

Notation

In the next slide CJI stands for the set of completely join-irreducible elements (i.e. the domain of κ), and CMI stands for the set of completely meet-irreducible elements (i.e. the codomain).

kappa for torsion classes

Main Theorem A [B., Todorov, Zhu]

Let Λ be a finite dimensional algebra, and let M be a Λ-brick.

- Each completely join-irreducible torsion class has the form $\mathsf{Filt}(\mathsf{Gen}(M))$, where M is a brick.
- $\kappa : \mathsf{CJI}(\mathsf{tors}\Lambda) \to \mathsf{CMI}(\mathsf{tors}\Lambda)$ is a bijection with

$$\kappa(\mathsf{Filt}(\mathsf{Gen}(M))) = {}^{\perp}M$$

where ${}^{\perp}M$ denotes the set $\{X \in \mathsf{mod}\,\Lambda | \mathrm{Hom}_{\Lambda}(X, M) = 0\}$.

Remark

The kappa-map is well defined for *finite* semidistributive lattices, but the lattice of torsion classes is rarely finite. What makes this result interesting is that we show that κ is well-defined even when the lattice of torsion classes is infinite.

Extending the kappa map

Definition

Let L be a finite semidistributive lattice. Let x be an element which has a canonical join representation such that $\kappa(j)$ is defined for each $j \in \mathrm{CJR}(x)$. Define

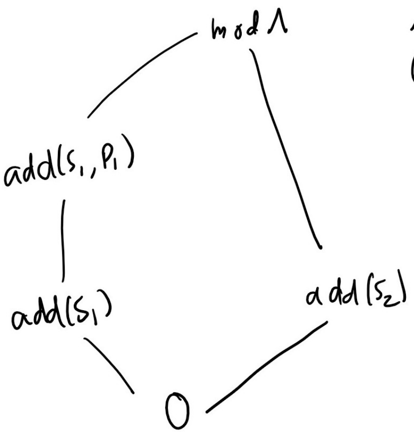
$$\bar{\kappa}(x) = \bigwedge \{\kappa(j) : j \in \mathrm{CJR}(x)\}.$$

Corollary[B., Todorov, Zhu]

Let Λ be a finite dimensional algebra. Let be a torsion class which has a canonical join representation of the following form: $\mathrm{CJR}(\mathcal{T}) = \bigvee_{\alpha \in A} \mathsf{Filt}(\mathsf{Gen}(M_\alpha))$, where M_α are Λ-bricks. Then $\bar{\kappa}(\mathcal{T})$ is defined and is of the form:

$$\bar{\kappa}(\mathcal{T}) = \bigcap_{\alpha \in A} {}^{\perp} M_\alpha.$$

Running Example



$\Lambda = kQ$

$Q: 1 \longrightarrow 2$

Iterative Compositions of κ

Theorem B

Let torsΛ be finite, and let r be the number of vertices in the corresponding quiver Q. For any $\mathcal{T} \in$ torsΛ let $|\mathcal{T}| := |\mathrm{CJR}(\mathcal{T})|$ denote the number of canonical joinands of $\mathcal{T}$. Then for any $\bar{\kappa}$-orbit $\mathcal{O}$ we have

$$\frac{1}{|\mathcal{O}|} \sum_{\mathcal{T} \in \mathcal{O}} |\mathcal{T}| = r/2$$

Iterative Compositions of κ

Theorem C

Recall that each join-irreducible torsion class is $\mathsf{Filt}(\mathsf{Gen}(M))$, where M is a brick. When Λ is hereditary, then applying $\bar{\kappa}$ twice corresponds to applying the (inverse of the) Auslander-Reiten translation to S.

$$\bar{\kappa}^2(\mathsf{Filt}(\mathsf{Gen}(M))) = \mathsf{Filt}(\mathsf{Gen}(\bar{\tau}^{-1}M)).$$

Here $\bar{\tau}^{-1}M = \tau^{-1}M$ for non-injective modules M and $\bar{\tau}^{-1}I(S) = P(S)$ where $I(S)$ and $P(S)$ are the injective envelope and projective cover of the same simple S.

Part III

Now we return to thinking about pairwise conditions. From here on out we will restrict to Λ τ-tilting finite, so that $\text{tors}\Lambda$ is finite.

Part III: A Pairwise condition

- Recall that CJR's are determined by a pairwise condition: A collection $\mathcal{D}$ of bricks gives rise to a CJR iff $\mathcal{D}$ is a semibrick.
- The same statement is true for canonical meet representations: A collection $\mathcal{U}$ of bricks gives rise to a CMR iff $\mathcal{U}$ is a semibrick.
- What about when we look at $\mathcal{D} \sqcup \mathcal{U}$ together?

Part III: A Pairwise condition

Main Question

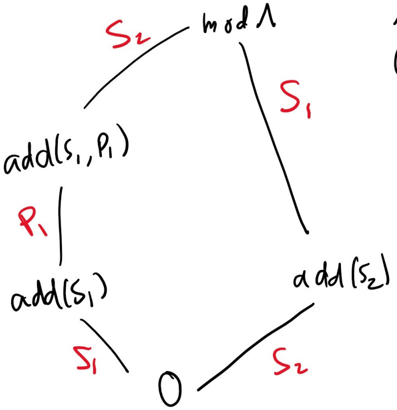
Suppose that $\mathcal{D}$ is a semibrick and $\mathcal{U}$ is a semibrick.

- Then we know that $\bigvee\{\mathrm{Filt}(\mathrm{Gen}(S)) : S \in \mathcal{D}\}$ is the CJR for some torsion class $\mathcal{T}$.
- We also know that $\bigcap\{{}^{\perp}T : T \in \mathcal{U}\}$ is the CMR for some torsion class $\mathcal{T}'$.
- Can we tell whether $\mathcal{T} = \mathcal{T}'$ just by checking a condition for pairs $S \in \mathcal{D}$ and $T \in \mathcal{U}$?

Brick Labeling

- We say that a brick S *labels* an upper cover relation $\mathcal{T} \lessdot \mathcal{T}'$ in the lattice torsΛ provided that $\mathcal{T}' = \mathsf{Filt}(\mathcal{T} \cup S)$.
- That is, $\mathcal{T}'$ is the closure of $\mathcal{T} \cup \{S\}$ under iterative extensions.
- The brick S is called a *minimal extending module* following [BCZ19].

Brick Labeling



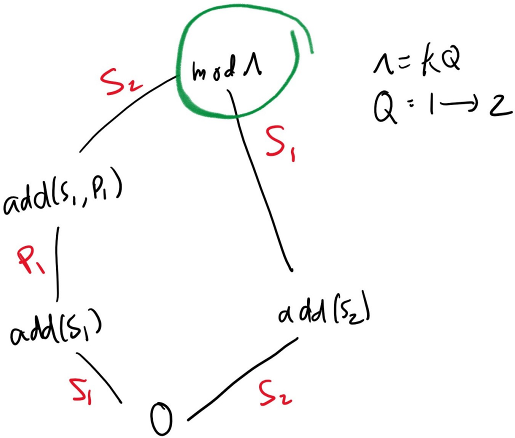
$\Lambda = kQ$

$Q : 1 \longrightarrow 2$

Brick Labeling

- The set of bricks that label the lower cover relations for a torsion class $\mathcal{T}$ is precisely the set of bricks in its CJR. We denote this set of bricks with $\mathcal{D}$ (for "down").
- The set of bricks that label the upper cover relations for a torsion class $\mathcal{T}$ is precisely the set of bricks in its CMR. We denote this set of bricks with $\mathcal{U}$ (for "up").

Brick Labeling



Brick Labeling

Reframe our main question

Given semibricks $\mathcal{D} \sqcup \mathcal{U}$, when does there exist a torsion class $\mathcal{T}$ such that $\mathcal{D}$ labels the lower cover relations and $\mathcal{U}$ labels the upper cover relations. Can we check a condition on pairs $S \in \mathcal{D}$ and $T \in \mathcal{U}$?

Equivalently...

Given semibricks $\mathcal{D} \sqcup \mathcal{U}[1]$, when is $\mathcal{D} \sqcup \mathcal{U}[1]$ a 2-term simple minded collection? Can we check a condition on pairs $S \in \mathcal{D}$ and $T[1] \in \mathcal{U}[1]$?

Observations and Key Definitions

Necessary Conditions

Let $\mathcal{D}$ and $\mathcal{U}$ be semibricks. If $\mathcal{D} \sqcup \mathcal{U}$ label the cover relations of some torsion class $\mathcal{T}$ then...

1. $\mathrm{Hom}(\mathcal{D}, \mathcal{U}) = 0$
2. $\mathrm{Ext}(\mathcal{D}, \mathcal{U}) = 0$

Observations and Key Definitions

Let $\mathcal{D}$ and $\mathcal{U}$ be semibricks, and let $\mathcal{X} = \mathcal{D} \sqcup \mathcal{U}[1]$.

1. $\mathcal{X}$ is called a *semibrick pair* if $\mathrm{Hom}(\mathcal{D}, \mathcal{U}) = 0 = \mathrm{Ext}(\mathcal{D}, \mathcal{U})$.
2. If in addition the smallest triangulated subcategory of $\mathcal{D}^b(\mathrm{mod}\Lambda)$ containing $\mathcal{X}$ is $\mathcal{D}^b(\mathrm{mod}\Lambda)$, then $\mathcal{X}$ is called a *2-term simple minded collection*.

Restate Main Question

Given a semibrick pair $\mathcal{X} = \mathcal{D} \sqcup \mathcal{U}[1]$, can we determine whether $\mathcal{X}$ is a 2-term simple minded collection by checking some conditions for pairs $S \in \mathcal{D}$ and $T[1] \in \mathcal{U}[1]$?

Observations and Key Definitions

We have the following main definition.

Definition

Let $\mathcal{D} \sqcup \mathcal{U}[1]$ be a semibrick pair.

1. We say that $\mathcal{D} \sqcup \mathcal{U}[1]$ is *completable* provided that there exists a 2-term simple minded collection that contains it.
2. We say that $\mathcal{D} \sqcup \mathcal{U}[1]$ is *pairwise completable* provided that for all $S \in \mathcal{D}$ and $T \in \mathcal{U}$ there exists a 2-term simple minded collection containing S and $T[1]$.
3. We say that Λ has the *pairwise completability property* provided that each pairwise completable semibrick pair is completable.

Remark

Let $\mathrm{rk}(\Lambda)$ be the number of simple modules in Λ, up to isomorphism. Each 2-term simple minded collection has $\mathrm{rk}(\Lambda)$-many elements.

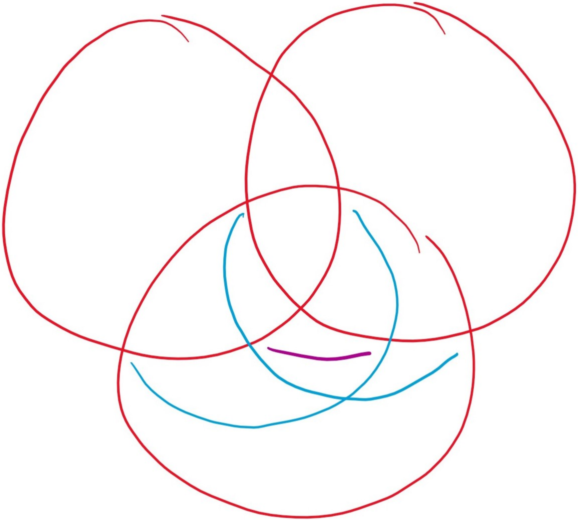
Observations about completability

- If we want to check a pairwise condition, we have to phrase our question in terms of *completability*. A pair of modules S and $T[1]$ will be a simple minded collection only if $\mathrm{rk}(\Lambda) = 2$.
- If $\mathcal{D} \sqcup \mathcal{U}[1]$ is completable, then it is pairwise completable.
- We are interested in the converse.
- These notions coincide trivially when $\mathrm{rk}(\Lambda) = 2$.

Motivation

- Our motivation comes from the study of *picture groups* and *picture spaces*.
- The picture group of an algebra was first defined by Igusa-Todorov-Weyman [ITW] in the (representation finite) hereditary case and later generalized to τ-tilting finite algebras by the second author and Igusa [HI].
- It is a finitely presented group whose relations encode the structure of the lattice of torsion classes.

Motivation



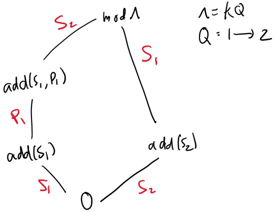
Motivation

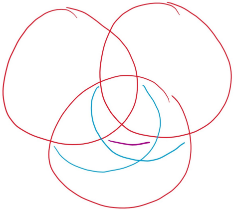
- The corresponding picture space is the classifying space of the *(τ)-cluster morphism category* of the algebra.
- The second author and Igusa have shown that the picture group and picture space have isomorphic (co-)homology when Λ has the pairwise completability property (plus one technical condition outlined in [HI]).

Favorable Evidence

- Hereditary algebras [IT] and Nakayama algebras [HI] have the pairwise completability property.
- In [GM20], 2-term simple minded collections were classified using a combinatorial model for certain special Nakayama algebras called *tiling algebras*.
- Not only do tiling algebras have the pairwise completability property, but this pairwise condition can be described in terms of a (non)crossing condition for certain arcs in a disc.

Main Tools



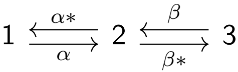


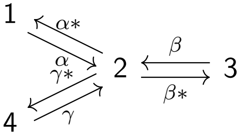
"Definition"

- Given a semibrick pair $\mathcal{X} = (\mathcal{D}, \mathcal{U}[1])$ and a brick $S \in \mathcal{D}$ the *left mutation* of $\mathcal{X}$ at S is a new semibrick pair $\mathcal{X}'$.
- When $\mathcal{X}$ is completable, left mutation corresponds to moving down by a cover relation in the lattice of torsion classes.

Preprojective algebras

- Consider a Dynkin diagram W of type A, D, or E.
- Let Q be the quiver obtained by replacing each edge of W with a 2-cycle.
- The *preprojective algebra* of type W is the algebra $\Pi_W := KQ/I$, where I is generated by the sums of all 2-cycles sharing a source/target.





$A_3 : \alpha\alpha^*, \beta\beta^*, \alpha^*\alpha + \beta^*\beta \qquad D_4 : \alpha\alpha^*, \beta\beta^*, \gamma\gamma^*, \alpha^*\alpha + \beta^*\beta + \gamma^*\gamma$

Preprojective algebras

Theorem[B.Hanson]

Let W be a Dynkin diagram of type A, D, or E. Then Π_W has the pairwise 2-simple minded completability property if and only if $W = A_n$ with $n \leq 3$.

Idea of the proof:

1. Show directly that Π_W has the property if $W = A_n$ with $n \leq 3$ (or reference our later result!)
2. Reduce to the cases $W = A_4$ and $W = D_4$.
3. Substitute the algebra RA_4 (which has *all* 2-cycles as relations) for Π_{A_4}. This is a string algebra and has the same torsion lattice as Π_{A_4}[BCZ19, Miz14].
4. Use the relationship between completability and *mutation* [HI21] to find counterexamples for RA_4 and Π_{D_4}.

Counterexample for RA_4

$$1 \underset{\longrightarrow}{\overset{\longleftarrow}{}} 2 \underset{\longrightarrow}{\overset{\longleftarrow}{}} 3 \underset{\longrightarrow}{\overset{\longleftarrow}{}} 4$$

The semibrick pair $\mathcal{X} = \begin{array}{c}2\\3\\4\end{array} \sqcup 4[1] \sqcup \begin{array}{c}3\\2\\1\end{array}[1]$ is pairwise completable but not completable.

- Reason 1: Suppose a brick S or $S[1]$ could be added to $\mathcal{X}$. Then using the vanishing conditions on Hom-sets in the definition of a 2-SMC, every possibility for the socle of S can be eliminated by checking few cases.
- Reason 2: Mutating at $\begin{array}{c}2\\3\\4\end{array}$ yields $\begin{array}{c}2\\3\end{array} \sqcup \begin{array}{c}2\\3\\4\end{array}[1] \sqcup \begin{array}{c}3\\2\\1\end{array}[1]$ and, the map $\begin{array}{c}3\\2\\1\end{array} \to \begin{array}{c}2\\3\end{array}$ is neither mono nor epi.

Other known cases

Other known results about the pairwise completability property are as follows:

Theorem

[H.-Igusa [HI21]] A (τ-tilting finite) gentle algebra whose quiver contains no loops or 2-cycles has the pairwise 2-simple minded completability property if and only if its quiver contains no vertex of degree 3 or 4.

A Rank 3 Pattern Emerges

The counterexamples to the pairwise 2-simple minded completability property in the our current and in [HI21] come from semibrick pairs $\mathcal{D} \sqcup \mathcal{U}[1]$ satisfying $|\mathcal{D}| + |\mathcal{U}| = 3 < \mathrm{rk}(\Lambda)$. Our next results offer an explanation as to why this is the case.

Theorem

Let Λ be any τ-tilting finite algebra. Then the following are equivalent.

1. Λ has the pairwise 2-simple minded completability property.
2. Every pairwise completable semibrick pair $\mathcal{D} \sqcup \mathcal{U}[1]$ which satisfies $|\mathcal{D}| + |\mathcal{U}| = 3$ is completable.

The importance of Rank 3

Theorem

Let Λ be any τ-tilting finite algebra. Then the following are equivalent.

1. Λ has the pairwise 2-simple minded completability property.
2. Every pairwise completable semibrick pair $\mathcal{D} \sqcup \mathcal{U}[1]$ which satisfies $|\mathcal{D}| + |\mathcal{U}| = 3$ is completable.

Theorem

Let Λ be a τ-tilting finite algebra with $\mathrm{rk}(\Lambda) \leq 3$. Then Λ has the pairwise 2-simple minded completability property.

“Full-size” semibrick pairs

- The key to the rank 3 case was that if $\mathrm{rk}(\Lambda) = 3$, then any semibrick of size 3 is a 2-SMC.

Conjecture

Let Λ be a τ-tilting finite algebra of rank n. Then any semibrick pair $\mathcal{D} \sqcup \mathcal{U}[1]$ with $|\mathcal{D}| + |\mathcal{U}| = n$ is a 2-SMC.

- The converse is proven in [KY14].
- This conjecture would imply that $\mathrm{rk}(\Lambda)$ is an upper bound on the size of a semibrick pair (when Λ is τ-tilting finite).
- This is (very) false in the τ-tilting infinite case:
  - Over a tame hereditary algebra, any finite collection of homogeneous bricks is a semibrick.
  - Tame hereditary algebras can even have pairwise completable semibrick pairs of size $\mathrm{rk}(\Lambda)$ which are not completable.

Evidence

Theorem

Let $n \in \mathbb{N}$ and let $\mathcal{D} \sqcup \mathcal{U}[1]$ be a semibrick pair for Π_{A_n} with $|\mathcal{D}| + |\mathcal{U}| = n$. Then $\mathcal{D} \sqcup \mathcal{U}[1]$ is a 2-SMC.

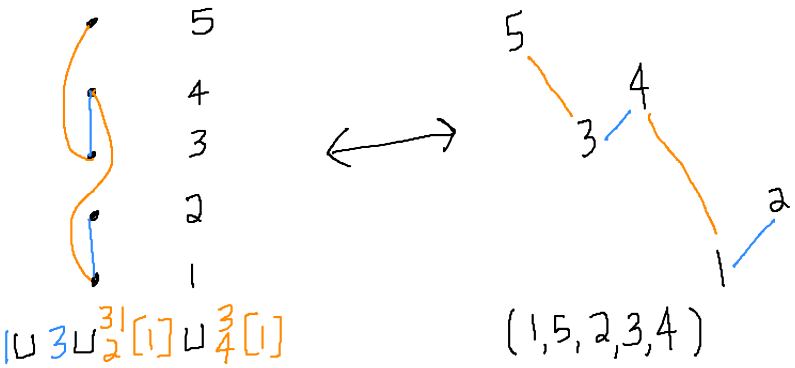
Idea of the proof:

1. As before, we work over RA_n instead of Π_{A_n}.
2. The torsion lattice is isomorphic to the weak order on the Coxeter group A_n (the group of permutations on $n+1$ letters) [BCZ19].
3. The *canonical join representations* (the bricks in $\mathcal{D}$) and the *canonical meet representations* (the bricks in $\mathcal{U}$) are separately encoded by *arc diagrams* [Rea15, BCZ19].

(continued on next slide)

Proof cont.

4. We define *2-colored arc diagrams* to encode both sets of bricks simultaneously and show a collection of n arcs always defines a permutation in A_n (and hence a 2-SMC).



Thank you!!

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