University of Stuttgart
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FD Seminar

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Quasihereditary Algebras with Exact Borel Subalgebras

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(1) Motivation and background

$\mathfrak{g} \rightsquigarrow$ fin. dim. complex semisimple Lie algebra

Fix a decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$. Then $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$ is a Borel subalgebra of $\mathfrak{g}$ and

Poincaré-Birkhoff-Witt Thm $\Rightarrow \begin{cases} U(\mathfrak{b}) \text{ is a subalgebra of } U(\mathfrak{g}) \\ U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} - \text{ is exact} \end{cases}$

$\mathbb{C}_\lambda \rightsquigarrow$ 1-dim $U(\mathfrak{b})$-module on which $U(\mathfrak{n}^+)$ acts as zero and $U(\mathfrak{h})$ acts by the character $\lambda \in \mathfrak{h}^*$

$U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda = \Delta_\lambda \rightsquigarrow$ Verma module with highest weight λ

The Verma modules Δ_λ are the "building blocks" of the BGG category $\mathcal{O}$.

- ★ $\mathcal{O} \subseteq U(\mathfrak{g})$-Mod and $\Delta_\lambda \in \mathcal{O}$
- ★ $\mathcal{O}$ is a highest weight category
- ★ $\mathcal{O}$ decomposes as a direct sum of blocks $\bigoplus_\chi \mathcal{O}_\chi$
- ★ $\mathcal{O}_\chi \simeq A_\chi$-Mod with A_χ a quasihereditary algebra
- ★ The Verma modules in $\mathcal{O}_\chi$ correspond to the standard A_χ-modules

DO QUASIHEREDITARY ALGEBRAS HAVE SOME SORT OF BOREL SUBALGEBRAS?

$k \rightsquigarrow$ fixed field with $k = \overline{k}$

Let A be a fin. dim. k-algebra with isoclasses of simples indexed by Φ.
For $i \in \Phi$, set

$L_i \rightsquigarrow$ simple A-module w/ label i | $P_i \rightsquigarrow$ projective cover of L_i | $Q_i \rightsquigarrow$ injective hull of L_i

If $(\Phi, \trianglelefteq)$ is a poset

$$\begin{cases} \Delta_i \rightsquigarrow \text{largest quotient of } P_i \text{ with composition factors } L_j,\ j \trianglelefteq i \quad \text{(standard)} \\ \nabla_i \rightsquigarrow \quad '' \quad \text{submodule of } Q_i \quad '' \quad '' \quad '' \quad '' \quad '' \quad \text{(costandard)} \end{cases}$$

Def The algebra A is quasihereditary w.r.t. the poset $(\Phi, \trianglelefteq)$ if:

(1) $[\Delta_i : L_i] = 1$

(2) P_i is filtered by standards $\qquad \forall i \in \Phi$

(3) $\mathrm{Ext}^1_A(\Delta_i, \Delta_j) \neq 0 \Rightarrow i \triangleleft j$

Def [Koenig '95] A subalgebra B of a quasihereditary algebra $(A, \Phi, \trianglelefteq)$ is an exact Borel subalgebra if:

(1) $A \otimes_B - : B\text{-Mod} \longrightarrow A\text{-Mod}$ is exact

(2) the simples over B are in bijection with Φ and $(B, \Phi, \trianglelefteq)$ is quasihereditary with simple standard modules

(3) $A \otimes_B L_i^B = \Delta_i^A \quad , \forall i \in \Phi$

Ex Let $A = K\left(\overset{2}{\circ} \longrightarrow \overset{1}{\circ}\right)$ with labelling poset $(\{1,2\}, \leq)$. Then $P_1 = 1 = \Delta_1$ and $P_2 = \begin{smallmatrix} 2 \\ | \\ 1 \end{smallmatrix} = \Delta_2$. Note that $B = K\left(\overset{2}{\circ} \quad \overset{1}{\circ}\right) \hookrightarrow A$.

$A \otimes_B -$ preserves projs and B semisimple $\Rightarrow$ B exact Borel subalg.

NOT ALL QUASIHEREDITARY ALGS HAVE AN EXACT BOREL SUBALG! ⚠ BUT...

Thm [Koenig-Külshammer-Ovsienko'14] If $(A, \Phi, \trianglelefteq)$ is quasihereditary there exists $(A', \Phi, \trianglelefteq)$ quasihereditary s.t.

(1) $(A', \Phi, \trianglelefteq)$ has ~~an~~ regular exact Borel subalgebra

(2) $(A', \Phi, \trianglelefteq)$ is equivalent to $(A, \Phi, \trianglelefteq)$

cat. of modules filtered by standards

Def Qh algs $(A, \Phi, \trianglelefteq)$ and $(A', \Psi, \trianglelefteq)$ are equivalent if $\mathcal{F}(\Delta_A) \simeq \mathcal{F}(\Delta_{A'})$.

$\Downarrow$ Morita eqv.

Def An exact Borel subalgebra B of a quasihereditary alg $(A, \Phi, \trianglelefteq)$ is regular if $\mathrm{Ext}^n_B(L^B_i, L^B_j) \simeq \mathrm{Ext}^n_A(A \otimes_B L^B_i, A \otimes_B L^B_j)$, $\forall i,j \in \Phi, n \geq 1$.

The KKO result does not explicitly describe a quasihereditary alg equivalent to $(A, \Phi, \trianglelefteq)$ that has an exact Borel subalgebra.

$$\mathrm{Ext}^*_A\left(\bigoplus_{i\in\Phi}\Delta_i, \bigoplus_{i\in\Phi}\Delta_i\right) =: \mathcal{A}$$

A_∞-algebra

$$H^0(\mathrm{tw}\,\mathrm{mod}\,\mathcal{A}) \xrightarrow{\sim} \mathrm{mod}\,\mathcal{B}$$

B-coring / bocs

($\approx$ B-coalgebra)

exact Borel subalg

$$B \hookrightarrow R_{\mathcal{B}}$$

qh alg eqv to $(A, \Phi, \trianglelefteq)$

Right alg of $\mathcal{B}$

The KKO result does not explicitly describe a quasihereditary alg equivalent to $(A, \Phi, \trianglelefteq)$ that has an exact Borel subalgebra.

GIVEN $(A, \Phi, \trianglelefteq)$, HOW TO FIND AN EQUIVALENT QH ALG WITH AN EXACT BOREL SUBALG?

WHICH ALGEBRAS IN $[(A, \Phi, \trianglelefteq)]$ HAVE A REGULAR EXACT BOREL SUBALG?

HOW TO DETECT IF $(A, \Phi, \trianglelefteq)$ HAS A REGULAR EXACT BOREL SUBALG?

IF B IS A REGULAR EXACT BOREL SUBALG OF AN ALG IN $[(A, \Phi, \trianglelefteq)]$ CAN WE DETERMINE INFO ABOUT B WITHOUT KNOWING B?

WHEN DOES A BASIC QH ALG HAVE A REGULAR EXACT BOREL SUBALG?

(2) ALL quasihereditary algs w/
a regular exact Borel subalg

STRATEGY: Find a matrix to solve all our problems

Given $(A, \Phi, \trianglelefteq)$ quasihereditary,

★ $[\Delta_i : L_j]$ ★ $[\nabla_i : L_j]$ ★ $\dim(\mathrm{Hom}_A(\Delta_i, \Delta_j))$	$\forall i,j \in \Phi$	$\xrightarrow{\text{INPUT}}$	**Algorithm**	$\xrightarrow{\text{OUTPUT}}$ ★ Special matrix $V_{[(A,\Phi,\trianglelefteq)]} = (v_{ij})_{i,j\in\Phi}$

Description

Define a sequence of vectors $(v_i)_{i\in\Phi}$ in the free $\mathbb{Z}$-module $\mathbb{Z}^{\Phi}$ via

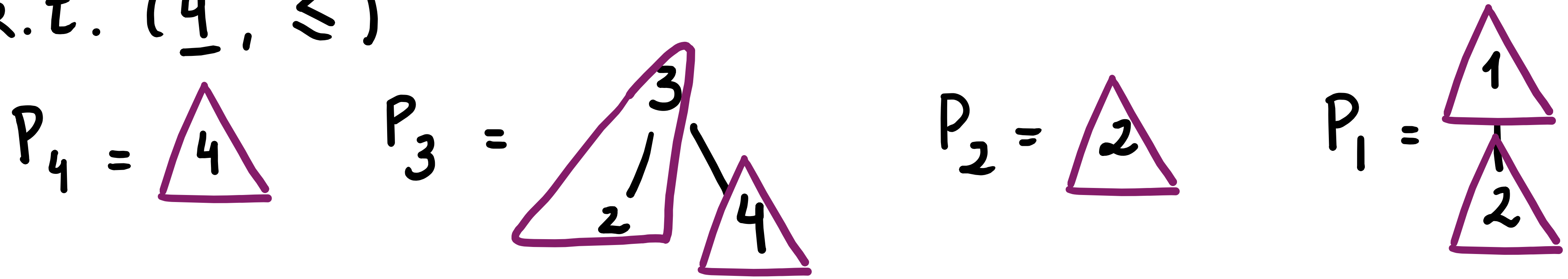
$$v_i = \underbrace{e_i}_{\text{Standard basis of } \mathbb{Z}^{\Phi}} + \sum_{\substack{j,k\in\Phi \\ k \trianglelefteq j \triangleleft i}} [\nabla_j : L_k] \dim(\mathrm{Hom}_A(\Delta_j, \Delta_i))\, v_k - \sum_{\substack{j\in\Phi \\ j \triangleleft i}} [\Delta_i : L_j]\, v_j$$

DEF

$$v_{ij} = j^{\text{th}} \text{ coordinate of } v_i$$

$$\boxed{v_i = e_i + \sum_{\substack{j\in\Phi,\, k\in\Phi\setminus\Phi_{i-}\\ k\trianglelefteq j\triangleleft i}} (P_k:\Delta_j)\,\dim\left(\mathrm{Hom}_A(\Delta_j,\Delta_i)\right) v_k - \sum_{\substack{j\in\Phi\setminus\Phi_{i-}\\ j\triangleleft i}} [\Delta_i:L_j]\, v_j}$$

Ex Let $A = K\left(\overset{1}{\circ}\longrightarrow\overset{2}{\circ}\longleftarrow\overset{3}{\circ}\longrightarrow\overset{4}{\circ}\right)$. A is quasihereditary w.r.t. $(\underline{4}, \leq)$



$v_1 = e_1$, $v_2 = e_2$, $v_3 = e_3 + (P_1:\Delta_1)\dim(\mathrm{Hom}_A(\Delta_1,\Delta_3))v_1$, $v_4 = e_4$

$+ (P_1:\Delta_2)\dim(\mathrm{Hom}_A(\Delta_2,\Delta_3))v_1$

$- [\Delta_3:L_1]\,v_1$

$= e_3 + e_1$

$$\boxed{V_{[(A,\underline{4},\leq)]} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}$$

$V_{[(A,\Phi,\trianglelefteq)]} = (v_{ij})_{i,j\in\Phi}$ is lower triangular with ones on the diagonal and zeros on the lower diagonal, and $v_{ij} \in \mathbb{N}_0$.

$$\boxed{\ell_{[(A,\Phi,\trianglelefteq)]} = (\ell_i)_{i\in\Phi} \text{ as } \ell_i \overset{\text{DEF}}{=} \sum_{j\in\Phi} v_{ij}}$$

Thm A [C'20] Let $(A,\Phi,\trianglelefteq)$ be quasihereditary.

(1) $\forall\, (K_i)_{i\in\Phi} \in \mathbb{N}^{\Phi}$ $\mathrm{End}_A\left(\bigoplus_{i\in\Phi} P_i^{m_i}\right)^{op}$ with $m_i = \sum_{j\in\Phi} v_{ij}K_j$ is (up to iso) the only algebra in $[(A,\Phi,\trianglelefteq)]$ containing a regular exact Borel subalg w/ $\dim L_i^B = K_i$.

(2) $(R_{[(A,\Phi,\trianglelefteq)]}, \Phi, \trianglelefteq)$ with $R_{[(A,\Phi,\trianglelefteq)]} = \mathrm{End}_A\left(\bigoplus_{i\in\Phi} P_i^{\ell_i}\right)^{op}$ is the unique (up to iso) algebra in $[(A,\Phi,\trianglelefteq)]$ containing a basic regular exact Borel subalg.

Thm B [C'20] Let $(A, \Phi, \trianglelefteq)$ be quasihereditary and consider $V_{[(A,\Phi,\trianglelefteq)]} = (v_{ij})_{i,j\in\Phi}$ and $\ell_{[(A,\Phi,\trianglelefteq)]} = (\ell_i)_{i\in\Phi}$.

(1) $(A, \Phi, \trianglelefteq)$ has a regular exact Borel subalg iff the unique solution of $V_{[(A,\Phi,\trianglelefteq)]}\, x = (\dim L_i^A)_{i\in\Phi}$ is a vector with entries in $\mathbb{N}$.

(2) If $(A, \Phi, \trianglelefteq)$ has a regular exact Borel subalg B, then $(\dim L_i^B)_{i\in\Phi}$ is the unique solution of $V_{[(A,\Phi,\trianglelefteq)]}\, x = (\dim L_i^A)_{i\in\Phi}$.

(3) The dimension of the simples over a regular exact Borel subalg of $(A, \Phi, \trianglelefteq)$ is univocally determined by $[\Delta_i : L_j]$, $[\nabla_i : L_j]$, $\dim(\mathrm{Hom}_A(\Delta_i, \Delta_j))$, $\dim L_i^A$.

(4) $(A, \Phi, \trianglelefteq)$ has a basic regular exact Borel subalg iff $\dim L_i^A = \ell_i$ $\forall i \in \Phi$

Ex Let $A = K(\overset{1}{\circ} \longrightarrow \overset{2}{\circ} \longleftarrow \overset{3}{\circ} \longrightarrow \overset{4}{\circ})$. A is quasihereditary w.r.t. $(\underline{4}, \leq)$ and

$$V_{[(A,\underline{4},\leq)]} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$V_{[(A,\underline{4},\leq)]}\, x = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \iff x = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \overset{\text{Thm B}}{\Longrightarrow} A \text{ has no Reg exact Borel subalg!}$$

$$\ell_1 = 1,\ \ell_2 = 1,\ \ell_3 = 2,\ \ell_4 = 1 \overset{\text{Thm A}}{\Longrightarrow} \left(\mathrm{End}_A(P_1 \oplus P_2 \oplus P_3^2 \oplus P_4)^{op}, \underline{4}, \leq\right) \text{ is}$$

the only qh alg eqv to $(A, \underline{4}, \leq)$ w/ a basic reg exact Borel subalg

(3) Cartan matrix of a regular exact Borel subalg

Thm C [C'20] Let $(A, \Phi, \trianglelefteq)$ be quasihereditary. Assume that B is a reg exact Borel subalg of some alg in $[(A, \Phi, \trianglelefteq)]$. Then

$$\begin{matrix}\text{Cartan matrix} \\ \text{of } B\end{matrix} = \left(\left([\nabla_i : L_j] \right)_{i,j \in \Phi} \times V_{[(A,\Phi,\trianglelefteq)]} \right)^T .$$

Equivalence classes of quasihereditary algebras	$\xleftrightarrow{1:1}$	Isomorphism classes of directed regular corings $\mathcal{B} = (B, W, \mu, \varepsilon)$ w/ B basic

? $[R_{\mathcal{B}}] \longleftarrow\!\shortmid \mathcal{B}/_{\cong}$

(work in progress by Külshammer-Miemietz)

Cor [C'20] If $\mathcal{B} = (B, W, \mu, \varepsilon)$ and $\mathcal{B}' = (B', W', \mu', \varepsilon')$ are directed regular corings with B and B' basic and $R_{\mathcal{B}}$ and $R_{\mathcal{B}'}$ eqv, then $\dim B = \dim B'$, $\dim W = \dim W'$, $R_{\mathcal{B}} \cong R_{\mathcal{B}'}$, and B and B' have same Cartan m.

(4) Basic qh algebras w/ a regular exact Borel subalg

Thm D [C'20] Let $(A, \Phi, \trianglelefteq)$ be quasihereditary. TFAE:

(1) every qh alg in $[(A, \Phi, \trianglelefteq)]$ has a regular exact Borel subalg

(2) the sequence $\ell_{[(A,\Phi,\trianglelefteq)]}$ is constant and $=1$

(3) $V_{[(A,\Phi,\trianglelefteq)]}$ is the identity matrix

(4) $\forall i \in \Phi$, Δ_i is a right $\mathcal{F}(\Delta)$-approximation of L_i

(5) $\operatorname{Rad}\Delta_i \in \mathcal{F}(\nabla)$, $\forall i \in \Phi$

(6) $\forall i \in \Phi$, the map $\operatorname{Ext}^1_A(X, \pi_i): \operatorname{Ext}^1_A(X, \Delta_i) \longrightarrow \operatorname{Ext}^1_A(X, L_i)$, where $\pi_i : \Delta_i \twoheadrightarrow L_i$, is an isomorphism for every $X \in \mathcal{F}(\Delta)$

If A is basic, then $(A, \Phi, \trianglelefteq)$ contains a regular exact Borel subalg iff it contains a basic Reg exact Borel subalg iff (1)-(6) hold.

* qh w/ simple standards
* qh w/ simple costandard
* Ringel dual of dual extension alg of incidence algebra of trees

Thm D [C'20] Let $(A, \Phi, \trianglelefteq)$ be quasihereditary. TFAE:

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If A is basic, then $(A, \Phi, \trianglelefteq)$ contains a regular exact Borel subalg iff it contains a basic reg exact Borel subalg iff (1)-(6) hold.

(work in progress w/ J. Külshammer)

* All results can be adapted to homological exact Borel subalg

B Borel subalg of A homological $\overset{\text{DEF}}{\Longleftrightarrow}$ $\operatorname{Ext}^1_B(L^B_i, L^B_j) \twoheadrightarrow \operatorname{Ext}^1_A(A\otimes_B L^B_i, A\otimes_B L^B_j)$ and $\operatorname{Ext}^n_B(L^B_i, L^B_j) \simeq \operatorname{Ext}^n_A(A\otimes_B L^B_i, A\otimes_B L^B_j)$, $\forall n \geq 2, \forall i,j$

* The special matrix $V_{[(A,\Phi,\trianglelefteq)]}$ is well-behaved w.r.t. good subalg and quotients

$$V_{[(A,\Phi,\trianglelefteq)]} = \left[\begin{array}{c|c} V_{[A/AeA]} & 0 \\ \hline * & V_{[eAe]} \end{array}\right], \quad e \in A \text{ idempotent supported in coideal } \Psi \text{ of } (\Phi, \trianglelefteq)$$

Thank you for your attention!