Stability and tilts on triangulated categories

Jason Lo
California State University, Northridge (CSUN), US

FD Seminar
Thursday, January 21, 2021

I. Equivalences of triangulated categories

Whenever we have an exact equivalence of triangulated categories

$$\Phi : \mathcal{D} \to \mathcal{U},$$

a natural question is:

How do objects and structures behave under Φ?

Examples of Φ:

- representation theory: tilting equivalances
- algebraic geometry: Fourier-Mukai transforms

Examples of 'objects and structures': t-structures, semistable objects, stability, moduli spaces, ...

Theorem 1 (Chindris)

Let A be a bound quiver algebra, T a basic tilting A-module, and θ an integral weight of A which is 'well-positioned' with respect to T. Let

$$F = \begin{cases} \mathrm{Hom}_A(T,-) & \text{if } \theta\text{-semistable } A\text{-modules are 'torsion'} \\ \mathrm{Ext}^1_A(T,-) & \text{if } \theta\text{-semistable } A\text{-modules are 'torsion-free'} \end{cases}.$$

Let $B = \mathrm{End}_A(T)^{op}$ and $u : K(A) \to K(B)$ the isometry induced by the tilting module T. Then F defines an equivalence of categories

$$\mathrm{mod}(A)^{ss}_{\theta} \cong \mathrm{mod}(B)^{ss}_{\theta'}$$

where $\theta' = |\theta \circ u^{-1}|$.

In the theorem:

- We have a derived equivalence

$$\mathrm{RHom}_A(T,-) : D^b(A-\mathrm{mod}) \to D^b(B-\mathrm{mod}).$$

- Given a weight $\theta : K(A-\mathrm{mod}) \to \mathbb{Z}$, an A-module M is θ-*semistable* if
  - $\theta(M) = 0$.
  - $\theta(M') \leq 0$ for all A-submodule M' of M.

$\to$ Chindris proceeded to use his theorem to construct singular moduli spaces of modules over a wild tilted algebra.

Theorem 2 (Atiyah, Tu, Polishchuk, Hein-Ploog)

Let X be an elliptic curve. The Fourier-Mukai transform

$$\Phi : D^b(\mathrm{Coh}(X)) \to D^b(\mathrm{Coh}(X))$$

with normalised Poincaré line bundle as the kernel takes a semistable sheaf on X to a semistable sheaf on X (up to a shift).

Semistability for coherent sheaves on X is determined by the slope function

$$\mu(E) := \frac{\deg E}{\mathrm{rank}\, E} \qquad \text{for } E \in \mathrm{Coh}(X).$$

A coherent sheaf E on C is called (slope-)semistable if

- $\mu(E') \leq \mu(E)$ for all subsheaves E' of E.

Are these two results related?

	Theorem 1	Theorem 2
context	modules	sheaves
equivalence	tilting equivalence	Fourier-Mukai transform
homological dim	1	1
stability	θ-stability	slope stability

II. Ingredients for connecting Theorem 1 and Theorem 2

Ingredient 1: reformulate slope stability for sheaves

Recall: slope stability for sheaves uses $\mu(E) = \frac{\deg E}{\operatorname{rank} E}$ on $\mathrm{Coh}(X)$.

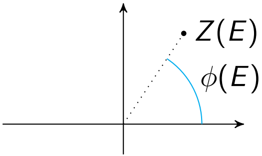
More generally, suppose $\mathcal{A}$ is an abelian category with additive functions $C_0, C_1 : \mathcal{A} \to \mathbb{Z}$ such that, for any $E \in \mathcal{A}$,

- $C_0(E) \geq 0$.
- If $C_0(E) = 0$ then $C_1(E) \geq 0$.

Then $\mu(E) := \frac{C_1(E)}{C_0(E)}$ defines μ-stability for objects in $\mathcal{A}$.

Equivalently, we can use the 'phase' function ϕ from

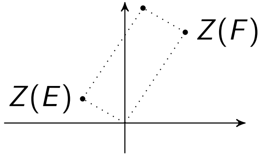
$$Z : K(\mathcal{A}) \to \mathbb{C} : E \mapsto -C_1(E) + iC_0(E).$$



Then an object $E \in \mathcal{A}$ is μ-semistable iff $\phi(E') \leq \phi(E)$ for all subobjects E'.

Any time we have an abelian category $\mathcal{A}$ and a group homomorphism $Z : K(\mathcal{A}) \to \mathbb{C}$ as above, if we fix an object $F \in \mathcal{A}$ with $Z(F) \neq 0$, we can construct a weight function $\theta_F : K(\mathcal{A}) \to \mathbb{R}$ by setting

$$\theta_F(E) = \begin{vmatrix} \Re Z(F) & \Im Z(F) \\ \Re Z(E) & \Im Z(E) \end{vmatrix} = \text{ area of parallelogram}$$



Lemma 3

For objects $E \in \mathcal{A}$ with $\mathrm{Hom}_{\mathcal{A}}(Z^{-1}(0), E) = 0$,

$$E \text{ is } \theta_F\text{-semistable} \Leftrightarrow \begin{cases} \phi(E) = \phi(F) \\ E \text{ is } \mu\text{-semistable.} \end{cases}$$

Ingredient 2: tilting

In general, given the heart $\mathcal{A}$ of a t-structure on a triangulated category $\mathcal{D}$, if $(\mathcal{T}, \mathcal{F})$ is a torsion pair in $\mathcal{A}$ then

$$\mathcal{A}^\dagger := \langle \mathcal{F}[1], \mathcal{T} \rangle$$

is the heart of a t-structure on $\mathcal{D}$, referred to as the *tilt* of $\mathcal{A}$ at $(\mathcal{T}, \mathcal{F})$.

Note that

$$\mathcal{A}^\dagger \subset \langle \mathcal{A}[1], \mathcal{A} \rangle.$$

Conversely, any two hearts $\mathcal{A}^\dagger, \mathcal{A}$ satisfying such a relation are related by a tilt.

Given an exact equivalence $\Phi : D^b(\mathcal{A}_1) \to D^b(\mathcal{A}_2)$ such that

$$\Phi(\mathcal{A}_1) \subset \langle \mathcal{A}_2, \mathcal{A}_2[-1] \rangle,$$

the heart $\Phi(\mathcal{A}_1)[1]$ is a *tilt* of $\mathcal{A}_2$.
This means that there is a torsion pair $(\mathcal{T}, \mathcal{F})$ in $\mathcal{A}_2$ such that

$$\Phi(\mathcal{A}_1)[1] = \langle \mathcal{F}[1], \mathcal{T} \rangle$$

in $D^b(\mathcal{A}_2)$.

The equivalences in Theorems 1 and 2 both fall into this setting with $\mathcal{A}_1, \mathcal{A}_2$ being hearts of standard t-structures.

III. Configurations of equivalences, t-structures, and stability conditions

Definition 4 (Configuration I)

- $\mathcal{D}$ triangulated category, $\mathcal{A}$ heart of bounded t-structure
- $(\mathcal{T}, \mathcal{F})$ torsion pair in $\mathcal{A}$
- $(R, \preceq)$ totally ordered abelian group
- $S, S' : K(\mathcal{D}) \to R$ group homomorphisms with sign compatibility

$$(*) \qquad \operatorname{sgn} S(E) = \operatorname{sgn} S'(E) \text{ for any } E \in \mathcal{T} \text{ or } \mathcal{F}.$$

$\mathcal{A}^\dagger := \langle \mathcal{F}[1], \mathcal{T} \rangle$.

We say S satisfies *refinement-i* with respect to $(\mathcal{T}, \mathcal{F})$ if:

- Every nonzero S-semistable object in $\mathcal{A}$ lies in $\mathcal{A} \cap (\mathcal{A}^\dagger[-i])$
- Every nonzero $G \in \mathcal{A} \cap (\mathcal{A}^\dagger[i-1])$ satisfies $(-1)^i \operatorname{sgn} S(G) = -1$.

Refinement-i mimics Chindris' 'well-positioned'.

Theorem 5

Assume Configuration I. Suppose S satisfies refinement-i for $i = 0$ or 1. Then for any object $E \in \mathcal{D}$,

$$E \text{ is } S\text{-semistable in } \mathcal{A} \quad \Leftrightarrow \quad E[i] \text{ is } (-1)^i S'\text{-semistable in } \mathcal{A}^\dagger.$$

Proof of $i = 0$ and $\Rightarrow$. Suppose E is S-semistable in $\mathcal{A}$ ($\Rightarrow S(E) = 0$). Refinement-0 means $E \in \mathcal{T}$ and every $0 \neq G \in \mathcal{F}$ has $S(G) \prec 0$. So objects in $\mathcal{F}$ have the "right" sign: $S(E) = 0 \succ S(G)$ while $\mathrm{Hom}_{\mathcal{A}}(E, G) = 0$.
Take $\mathcal{A}^\dagger$-ses $0 \to M \to E \to N \to 0$ then $\mathcal{A}$-les:

$$0 \to \mathcal{H}_{\mathcal{A}}^{-1}(N) \to M \overset{\alpha}{\to} E \to \mathcal{H}_{\mathcal{A}}^{0}(N) \to 0.$$

$\mathcal{H}_{\mathcal{A}}^{-1}(M) = 0$ so $M \in \mathcal{T}$. S-semistability of E gives $S(\mathrm{im}\,\alpha) \preceq 0$.
If $\mathcal{H}_{\mathcal{A}}^{-1}(N) \neq 0$, then refinement-0 on $\mathcal{H}_{\mathcal{A}}^{-1}(N)$ gives $S(M) \preceq 0$.
If $\mathcal{H}_{\mathcal{A}}^{-1}(N) = 0$, then S-semistability of E gives $S(M) \preceq 0$.
Sign compatibility in Configuration I now gives $S'(M) \preceq 0$. $\square$

Definition 6 (Configuration II)

$(R, \preceq)$ totally ordered abelian group.

(a) $\Phi : \mathcal{D} \to \mathcal{U}$ and $\Psi : \mathcal{U} \to \mathcal{D}$ exact equivalences of triangulated categories satisfying $\Psi\Phi \cong \mathrm{id}_{\mathcal{D}}[-1]$ and $\Phi\Psi \cong \mathrm{id}_{\mathcal{U}}[-1]$.

(b) $\mathcal{A}, \mathcal{B}$ are hearts of t-structures on $\mathcal{D}, \mathcal{U}$, respectively, such that $\Phi\mathcal{A} \subset \langle \mathcal{B}, \mathcal{B}[-1] \rangle$.

(c) $S_{\mathcal{A}} : K(\mathcal{D}) \to R$ and $S_{\mathcal{B}} : K(\mathcal{U}) \to R$ are group homomorphisms such that

$$\operatorname{sgn} S_{\mathcal{A}}(E) = \operatorname{sgn} S_{\mathcal{B}}(\Phi E)$$

for any $E \in \mathcal{A}$ that is either $\Phi_{\mathcal{B}}$-WIT$_0$ or $\Phi_{\mathcal{B}}$-WIT$_1$.

Can convert between Configuration I and Configuration II.

Theorem 7

Assume Configuration II. Suppose the weight function $S_{\mathcal{A}}$ satisfies refinement-i with respect to the torsion pair $(W_{0,\Phi,\mathcal{A},\mathcal{B}}, W_{1,\Phi,\mathcal{A},\mathcal{B}})$. Then for any $E \in \mathcal{D}$,

E is $S_{\mathcal{A}}$-semistable in $\mathcal{A}$ $\Leftrightarrow$ $(\Phi[i])(E)$ is $(-1)^i S_{\mathcal{B}}$-semistable in $\mathcal{B}$.

Recovers Theorem 1 when

- $\mathcal{D} = D^b(A-\mathrm{mod})$ for bound quiver algebra A
- $\mathcal{U} = D^b(B-\mathrm{mod})$ where $B = \mathrm{End}(T)^{op}$, for T basic tilting module over A
- $\mathcal{A} = A-\mathrm{mod}$ and $\mathcal{B} = B-\mathrm{mod}$
- $\Phi = \mathrm{RHom}(T, -)$
- $\Psi = (T \overset{L}{\otimes} -)[-1]$
- $S_{\mathcal{A}} = \theta, S_{\mathcal{B}} = \theta \circ u^{-1}$

What about Theorem 2?

We saw that slope stability for sheaves on an elliptic curve X could be reformulated using

$$Z : K(\mathrm{Coh}(X)) \to \mathbb{C} : -\deg(E) + i\,\mathrm{rank}(E).$$

Stability in algebraic geometry is often defined via group homomorphisms such as

- $Z : K(\mathcal{D}) \to \mathbb{C}$ (Bridgeland stability or 'weak' stability such as slope stability on surfaces), or
- $Z : K(\mathcal{D}) \to \mathbb{C}((\frac{1}{v}))^c$ (Bayer's polynomial stability, extended)

Definition 8 (Configuration III)

(a) $\Phi : \mathcal{D} \to \mathcal{U}$ and $\Psi : \mathcal{U} \to \mathcal{D}$ exact equivalences of triangulated categories satisfying $\Psi\Phi \cong \mathrm{id}_{\mathcal{D}}[-1]$ and $\Phi\Psi \cong \mathrm{id}_{\mathcal{U}}[-1]$.

(b) $\mathcal{A}, \mathcal{B}$ are hearts of t-structures on $\mathcal{D}, \mathcal{U}$, respectively, such that $\Phi\mathcal{A} \subset \langle \mathcal{B}, \mathcal{B}[-1] \rangle$.

(c) There exist weak polynomial stability functions $Z_{\mathcal{A}} : K(\mathcal{D}) \to \mathbb{C}((\frac{1}{v}))^c$ and $Z_{\mathcal{B}} : K(\mathcal{U}) \to \mathbb{C}((\frac{1}{v}))^c$ on $\mathcal{A}, \mathcal{B}$, respectively, and some $T \in \mathrm{GL}^{l,+}(2, \mathbb{R}((\frac{1}{v}))^c)$ such that

$$\begin{array}{ccc} K(\mathcal{D}) & \xrightarrow{\Phi^K} & K(\mathcal{U}) \\ \downarrow{\scriptstyle Z_{\mathcal{A}}} & & \downarrow{\scriptstyle Z_{\mathcal{B}}} \\ \mathbb{C}((\frac{1}{v}))^c & \xrightarrow{T} & \mathbb{C}((\frac{1}{v}))^c \end{array}$$

commutes.

Note: In all of Configurations I, II and III, Harder-Narasimhan property is *not* needed.

Theorem 9

Assume Configuration III. Then for any $E \in \mathcal{A}$,

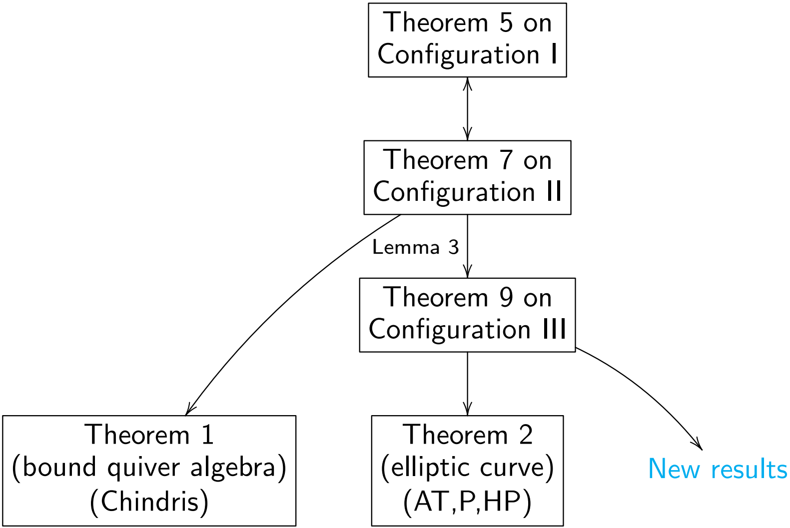
E is $Z_{\mathcal{A}}$-semistable in $\mathcal{A} \Leftrightarrow \Phi E$ (up to shift) is $Z_{\mathcal{B}}$-semistable in $\mathcal{B}$.

Recovers Theorem 2 when

- $\mathcal{D} = \mathcal{U} = D^b(\mathrm{Coh}(X))$ where X is elliptic curve
- $\mathcal{A} = \mathcal{B} = \mathrm{Coh}(X)$
- Φ is Fourier-Mukai transform with Poincaré line bundle as kernel, Ψ is 'dual' functor
- $Z_{\mathcal{A}} = Z_{\mathcal{B}}$ is $K(\mathrm{Coh}(X)) \to \mathbb{C} : -\deg(E) + i\,\mathrm{rank}(E)$

This seems to suggest, that given the Poincaré bundle, preservation of stability under the FMT is largely a "homological" result.

Family tree:



Theorem 5 on Configuration I
Theorem 7 on Configuration II
Lemma 3
Theorem 9 on Configuration III
Theorem 1 (bound quiver algebra) (Chindris)
Theorem 2 (elliptic curve) (AT,P,HP)
New results

In closing,

IV. Applications in algebraic geometry

New results on Weierstraß elliptic surfaces X:

$\Phi : D^b(\mathrm{Coh}(X)) \to D^b(\mathrm{Coh}(X))$ is relative Fourier-Mukai transform with normalised relative Poincaré sheaf as kernel.